Coloring the Integers with Rainbow Arithmetic Progressions

Michael Young

Iowa State University

Coauthors

Steve Butler
Craig Erickson (Grand View University)
Leslie Hogben
Kirsten Hogenson
Lucas Kramer (Carroll College)
Richard Kramer
Jephian Lin
Ryan Martin
Derrick Stolee
Nathan Warnberg (UW La Crosse)

A k-term arithmetic progression is a finite sequence of k terms of the form $\{a, a+d, a+2d, ..., a+(k-1)d\}$, where k, a, and d are nonnegative integers.

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$$3, 10, 17, 24, 31$$

An r-coloring of a set S is a function $c : S \rightarrow C$, such that $|C| = r$.

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A set S is monochromatic under an r-coloring c, if $c(s_1) = c(s_2)$, for each $s_1, s_2 \in S$.

Van der Waerdens Theorem

Let k and r be positive integers. Then there exists a positive integer w such that every r-coloring of $[w]$ contains a monochromatic k-term arithmetic progression.

Van der Waerdens Theorem

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The smallest integer $w(r, k)$ satisfying the theorem is called the van der Waerden number.

Proving van der Waerden numbers.

To show $w(r, k) = w$, the following two statements must be proven:

- There exists an r-coloring of $[w - 1]$ with no monochromatic k-term APs.
- Every r-coloring of $[w]$ has a monochromatic k-term AP.

$w(2,3) \leq 9$

Case I:

1 2 3 4 5 6 7 8 9

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1 2 3 4 5 6 7 8 9

$w(2,3) \leq 9$

Case II:

1 2 3 4 5 6 7 8 9

$w(2,3) \leq 9$

Case II:

1 2 3 4 5 6 7 8 9

$w(2,3) \leq 9$

Case II:

$$1\ 2\ 3\ {\color{red}4}\ {\color{red}5}\ {\color{blue}6}\ 7\ 8\ 9$$

$w(2,3) \leq 9$

Case II:

1 2 3 4 5 6 7 8 9

$w(2,3) \leq 9$

Case II:

1 2 3 4 5 6 7 8 9

$w(2,3) \leq 9$

Case II:

1 2 3 4 5 6 7 8 9

$w(2, 3) \leq 9$

Case II:

1 2 3 4 5 6 7 8 9

$w(2,3) \leq 9$

Case II:

1 2 3 4 5 6 7 8 9

$w(2, 3) \geq 9$

Extremal Coloring:

$$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8$$

Known van der Waerden numbers.

- $w(r, 1) = 1$.

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- $w(1, k) = k$.

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- $w(r, 1) = 1$.
- $w(1, k) = k$.
- $w(r, 2) = r + 1$.

Known van der Waerden numbers.

- $w(r,1) = 1$.
- $w(1,k) = k$.
- $w(r,2) = r + 1$.
- $w(2,3) = 9, w(3,3) = 27, w(4,3) = 76$.

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- $w(2, 3) = 9, w(3, 3) = 27, w(4, 3) = 76$.
- $w(2, 4) = 35, w(2, 5) = 178, w(2, 6) = 1132$.

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- $w(r, 2) = r + 1$.
- $w(2, 3) = 9, w(3, 3) = 27, w(4, 3) = 76$.
- $w(2, 4) = 35, w(2, 5) = 178, w(2, 6) = 1132$.
- Thats it!

Known van der Waerden numbers.

$r \backslash k$	3	4	5	6	7	8
2	9	35	178	1,132	$> 3,703$	$> 11,495$
3	27	293	$> 2,173$	$> 11,191$	$> 48,811$	$> 238,400$
4	76	$> 1,048$	$> 17,705$	$> 91,331$	$> 420,217$	
5	> 170	$> 2,254$	$> 98,740$	$> 540,025$		
6	> 223	$> 9,778$	$> 98,748$	$> 819,981$		

anti-van der Waerden numbers

An exact r-coloring of a set S is a surjective function $c : S \to C$, such that $|C| = r$.

$$3, 10, 17, 24, 31$$

anti-van der Waerden numbers

An exact r-coloring of a set S is a surjective function $c : S \to C$, such that $|C| = r$.

$$3, 10, 17, 24, 31$$

A set S is rainbow under an r-coloring c, if $c(s_1) \neq c(s_2)$, for each distinct $s_1, s_2 \in S$.

anti-van der Waerden numbers

Given positive integers n and k with $k \leq n$, the anti-van der Waerden number, denoted by $aw(n,k)$, is the least positive integer r such that every exact r-coloring of $[n]$ contains a rainbow k-term AP.

Proving anti-van der Waerden numbers

To show $aw(n, k) = r$, the following two statements must be proven:

- There exists an $(r-1)$-coloring of $[n]$ with no rainbow k-term APs.
- Every r-coloring of $[n]$ has a rainbow k-term AP.

$aw(8, 3) \leq 5$

1 2 3 4 5 6 7 8

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1 2 3 4 5 6 7 8

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$aw(8,3) \geq 5$

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Properties of $aw(n, k)$

- $k \leq aw(n, k) \leq n$.

Properties of $aw(n, k)$

- $k \leq aw(n, k) \leq n$.
- $aw(n, k) = n$ if and only if $k \geq \frac{n}{2} + 1$.

Small anti-van der Waerden numbers

$n\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
3	3																			
4	4																			
5	4	5																		
6	4	6	6																	
7	4	6	7	7																
8	5	6	8	8	8															
9	4	7	8	9	9	9														
10	5	8	9	10	10	10	10													
11	5	8	9	10	11	11	11	11												
12	5	8	10	11	12	12	12	12	12											
13	5	8	11	11	12	13	13	13	13	13										
14	5	8	11	12	13	14	14	14	14	14	14									
15	5	9	11	13	14	14	15	15	15	15	15	15								
16	5	9	12	13	15	15	16	16	16	16	16	16	16							
17	5	9	13	13	15	16	16	17	17	17	17	17	17	17						
18	5	10	14	14	16	17	17	18	18	18	18	18	18	18	18					
19	5	10	14	15	17	17	18	18	19	19	19	19	19	19	19	19				
20	5	10	14	16	17	18	19	19	20	20	20	20	20	20	20	20	20			
21	5	11	14	16	17	19	20	20	20	21	21	21	21	21	21	21	21	21		
22	6	12	14	17	18	20	21	21	21	22	22	22	22	22	22	22	22	22	22	
23	6	12	14	17	19	20	21	22	22	22	23	23	23	23	23	23	23	23	23	23
24	6	12	15	18	20	20	22	23	23	23	24	24	24	24	24	24	24	24	24	24
25	6	12	15	19	21	21	23	23	24	24	24	24	25	25	25	25	25	25	25	
26	6	12																		
27	5																			
28	6																			
29	6																			
30	6																			
31	6																			
32	6																			

$aw(n,k)$, $k=3$

$$a,\ a+d,\ a+2d$$

$aw(n,k)$, $k=3$

$$a,\ a+d,\ a+2d$$

- The first and last term have the same parity.

$aw(n, k)$, $k = 3$

$$a,\ a+d,\ a+2d$$

- The first and last term have the same parity.
- The terms are all the same or all different modulo 3.

Lowerbound for $aw(n, 3)$

Theorem

$aw(\frac{n}{3}, 3) + 1 \leq aw(n, 3)$.

1 2 3 4 5 6 7 8 9....

Lowerbound for $aw(n, 3)$

Theorem

$aw(\frac{n}{3}, 3) + 1 \leq aw(n, 3)$.

1 2 3 4 5 6 7 8 9....

Corollary

$\log_3(n) + 2 \leq aw(n, 3)$.

Upperbound for $aw(n, 3)$

Theorem

$aw(n, 3) \leq aw(\frac{n}{2}, 3) + 1.$

Upperbound for $aw(n, 3)$

Theorem

$aw(n, 3) \leq aw(\frac{n}{2}, 3) + 1.$

Corollary

$aw(n, 3) \leq \log_2(n) + 1.$

Upperbound for $aw(n, 3)$

Theorem

$aw(n, 3) \leq aw(\frac{n}{2}, 3) + 1.$

Corollary

$aw(n, 3) \leq \log_2(n) + 1.$

Conjecture

$aw(n, 3) \leq \log_3(n) + 4.$

Conjecture

$aw(3^m, 3) \leq m + 2.$

Thank You!