A confluence of algebraic topology and numerical analysis

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October 28, 2013
Introduction and motivation

Hilbert complexes and their discretization

Finite element differential forms and de Rham subcomplexes

Consistency of the combinatorial codifferential
Introduction and motivation
Find nonzero $u \in \mathring{H}^1$ such that \[(\text{grad } u, \text{grad } v)_{L^2} = \lambda (u, v)_{L^2} \quad \forall v \in \mathring{H}^1.\]

To solve, we use Galerkin’s method, with $V_h \subset \mathring{H}^1$ a finite element space, to reduce to a matrix eigenvalue problem.
Eigenvalues of the 1-form Laplacian

\[ (d^*d + dd^*)u = (\text{curl curl} - \text{grad div})u = \lambda u, \quad u \cdot n = 0, \text{curl } u = 0 \text{ on bdry} \]

Find nonzero \( u \in \Lambda^1 \) such that

\[ (du, dv) + (d^*u, d^*v) = \lambda (u, v) \quad \forall v \in \Lambda^1 \]

\[ \lambda_1 = 1.94 \]
\[ \lambda_2 = 2.02 \]
\[ \lambda_3 = 2.26 \]

\[ \lambda_1 = 0 \]
\[ \lambda_1 = 0.617 \]
\[ \lambda_2 = 0.658 \]
The $d^*d$ eigenvalue problem with $P_1$ elements

Find nonzero $u$ such that \((\text{curl } u, \text{curl } v) = \lambda(u, v)\) \quad \forall v$

For $\Omega = (0, \pi) \times (0, \pi)$, \(\lambda = m^2 + n^2\), \(m, n > 0\)

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Simulation of radar scattering off building

10,114,695,855 FEEC elements ($\approx$ 1 cm resolution), 12,000 time steps of 13 picoseconds

Finite element exterior calculus

DNA, Differential complexes and numerical stability, plenary ICM 2002


Numerical analysis antecedents


Topology antecedents

Whitney 1957; Sullivan 1978; Dodziuk, Patodi 1976

Recent developments

Christiansen, Hirani, Schöberl and many more
Hilbert complexes and their discretization
Hilbert space setting

We view the exterior derivative $d$ as a closed unbounded operator $L^2 \Lambda^k \to L^2 \Lambda^{k+1}$ with domain

$$H\Lambda^k = \{ u \in L^2 \Lambda^k \mid du \in L^2 \Lambda^{k+1} \}.$$

Inspired by the de Rham complex,

$$0 \to H\Lambda^0 \xrightarrow{d} H\Lambda^1 \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^n \to 0$$

we define a closed Hilbert complex:

- Hilbert spaces $W^0, W^1, \ldots, W^n$;
- Densely defined closed operators $W^k \xrightarrow{d^k} W^{k+1}$ with domains $V^k \subset W^k$ and closed ranges, satisfying:
  - $d^{k-1} \circ d^k = 0$ ($B^k \subset Z^k \subset V^k$)

The domains $V^k$ are then H-spaces with the graph norm $\|v\|^2_V = \|v\|^2 + \|dv\|^2$, and form a complex in the category of H-spaces

$$0 \to V^0 \xrightarrow{d} V^1 \xrightarrow{d} \cdots \xrightarrow{d} V^n \to 0$$

with associated cohomology spaces $Z^k / B^k$. 
Implications from closed Hilbert complexes

Adjoint complex: \( d_k^* \) is densely-defined, closed, w/ closed range

\[
0 \leftarrow V_0^* \xleftarrow{d^*} V_1^* \xleftarrow{d^*} \cdots \xleftarrow{d^*} V_n^* \leftarrow 0
\]

Abstract Hodge Laplacian: \( dd^* + d^*d : \mathcal{W}^k \rightarrow \mathcal{W}^k \)

Harmonic forms: \( \mathcal{Z}^k / \mathcal{B}^k \equiv \mathcal{Z}^k \cap \mathcal{B}^k \perp = \ker(dd^* + d^*d) := \mathcal{H}^k \)

Hodge decomposition: \( \mathcal{W}^k = \mathcal{B}^*_k \oplus \mathcal{H}^k \oplus \mathcal{B}^k \)

Poincaré inequality: \( \exists c \text{ such that } \|u\| \leq c\|du\| \quad \forall u \in \mathcal{Z}^k \perp \cap \mathcal{V}^k \)
Mixed formulation of the abstract Hodge Laplacian

Given $f \in W^k$, find $\sigma \in V^{k-1}, u \in V^k, p \in \mathcal{H}^k$ such that

$$
\langle \sigma, \tau \rangle - \langle u, d\tau \rangle = 0 \quad \forall \tau \in V^{k-1}
$$

$$
\langle d\sigma, v \rangle + \langle du, dv \rangle + \langle p, v \rangle = \langle f, v \rangle \quad \forall v \in V^k
$$

$$
\langle u, q \rangle = 0 \quad \forall q \in \mathcal{H}^k
$$

$$
\sigma = d^*u, \quad d\sigma + d^*du = f \pmod{\mathcal{H}}, \quad u \perp \mathcal{H}
$$

This problem is well-posed. Idea of proof:

Need to control $\|\sigma\|_V + \|u\|_V + \|p\|$ by a bounded choice of $\tau, v, q$. $\tau = \sigma \implies \|\sigma\|, v = d\sigma \implies \|d\sigma\|, v = p \implies \|p\|, v = u \implies \|du\|$. How to control $\|u\|$?

Hodge decomp.: $u = d\eta + s + z$, $\eta \in V^{k-1}, s \in \mathcal{H}^k, z \in (\mathcal{Z}^k)^\perp$

$\tau = \eta \implies \|d\eta\|$ and $q = s \implies \|s\|$. To bound $\|z\|$ we use Poincaré’s inequality:

$$
\|z\| \leq c\|dz\| = c\|du\| \quad \text{(which is under control)}
$$
Discretization

Galerkin’s method using f.d. subspaces $V^k_h \subset V^k$ reduces this to a finite system of linear equations:

Find $\sigma_h \in V^{k-1}_h$, $u_h \in V^k_h$, $p_h \in \mathcal{H}_h^k$ such that

\[
\langle \sigma_h, \tau \rangle - \langle u_h, d\tau \rangle = 0 \quad \forall \tau \in V^{k-1}_h
\]
\[
\langle d\sigma_h, v \rangle + \langle du_h, dv \rangle + \langle p, v \rangle = \langle f, v \rangle \quad \forall v \in V^k_h
\]
\[
\langle u_h, q \rangle = 0 \quad \forall q \in \mathcal{H}_h^k
\]

$Z^k_h = \mathcal{N}(d|_{V^k_h})$, $\mathcal{B}^k_h = dV^{k-1}_h$, $\mathcal{H}_h^k = Z^k_h \cap \mathcal{B}^k_h \perp$.

We are interested in the convergence of $(\sigma_h, u_h, p_h)$ to $(\sigma, u, p)$.

The fundamental paradigm of numerical analysis tells us that convergence depends on

- **Consistency:** the extent to which the exact solution satisfies the discrete problem.
- **Stability:** the well-posedness of the discrete problem.

These properties are determined by the choice of the Galerkin subspaces $V^k_h$. 
Ensuring consistency and stability

For consistency we must require that

\[ E(\nu) := \inf_{\nu_h \in V_k} \| \nu - \nu_h \|_V \to 0 \quad \text{as } h \to 0 \quad \forall \nu \in V \]  \hspace{1cm} (A)

The consistency error is then bounded by \( E(\sigma) + E(u) + E(p) \) plus a term involving \( \text{gap}(S_h^k, S_h^{k+1}) \).

To obtain a stable discretization there are \textit{two more key assumptions}.

\textbf{Subcomplex assumption (SC):} \quad d(V^k_h) \subset V^{k+1}_h

The subcomplex \( \cdots \to V^k_h \xrightarrow{d} V^{k+1}_h \xrightarrow{d} \cdots \) is itself an H-complex so we have (discrete) harmonic forms \( S_h^k \), Hodge decomposition, and Poincaré inequality with constant \( c_{P,h} \).

\textbf{Bounded Cochain Projection assumption (BCP):} \quad \exists \pi^k_h : V^k \to V^k_h

\[ \cdots \to V^k \xrightarrow{d^k} V^{k+1} \to \cdots \]

\[ \downarrow \pi^k_h \hspace{1cm} \downarrow \pi^{k+1}_h \]

\[ \cdots \to V^k_h \xrightarrow{d^k} V^{k+1}_h \to \cdots \]

- \( \pi^k_h \) is bounded, uniformly in \( h \)
- \( \pi^k_h \) preserves \( V^k_h \)
- \( \pi^{k+1}_h d^k = d^k \pi^k_h \)
Theorem

Let \((V^k, d^k)\) be a Hilbert complex and \(V^k_h\) finite dimensional subspaces satisfying A, SC, and BCP. Then

- \(\pi_h\) induces an isomorphism on cohomology for \(h\) small.
- \(\text{gap}(\mathcal{S}^k, \mathcal{S}^k_h) \to 0\).
- The Galerkin method is consistent.
- The discrete Poincaré inequality \(\|\omega\| \leq c\|d\omega\|, \ \omega \in \mathcal{Z}_h^k\), holds with \(c\) independent of \(h\).
- The Galerkin method is stable.
- The Galerkin method is convergent with the error estimate:

\[
\|\sigma - \sigma_h\|_V + \|u - u_h\|_V + \|p - p_h\|_V \leq c\left[ E(\sigma) + E(u) + E(p) + \varepsilon \right]
\]

where \(\varepsilon \leq E(P_B u) \times \sup_{r \in \mathcal{S}^k, \|r\| = 1} E(r)\).

- Once we construct specific finite element spaces satisfying the hypotheses, this theorem gives us concrete rates of convergence.
- It is the starting point for more refined estimates, and estimates for the eigenvalue problem.
Finite element differential forms and de Rham subcomplexes
Finite element spaces

For the de Rham complex, the question becomes: How can we construct FE subspaces of $H\Lambda^k$ satisfying SP and BCP?

A finite element space is constructed from three ingredients:

- A triangulation $\mathcal{T}$ consisting of polygonal elements $T$.

- For each $T$, a space of shape functions $V(T)$, typically polynomial. E.g., $V(T) = P_3(T)$.

- For each $T$, a set of DOFs: a basis for $V(T)^*$, with each element associated to a face of $T$.

$V_h$ is defined as functions piecewise in $V(T)$ with DOFs single-valued on faces. Interelement continuity is not specified a priori, but inferred:

$$V_h = \{ u \in H^1 \mid \text{tr}_T u \in \mathcal{P}_3(T) \quad \forall T \in \mathcal{T} \}$$
Finite elements for the de Rham complex

FEEC reveals that there are precisely two natural families:

\[ \mathcal{P}_r \Lambda^k(T) \quad \text{and} \quad \mathcal{P}_{r-1} \Lambda^k(T) \]

Shape fns for \( \mathcal{P}_r \Lambda^k \): polynomial \( k \)-forms of degree \( \leq r \).

Shape fns for \( \mathcal{P}_{r-1} \Lambda^k \) defined via Koszul differential \( \kappa : \Lambda^{k+1} \to \Lambda^k \):

\[
\kappa(dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_{k+1}}) = x_{i_1} dx_{i_2} \wedge \cdots \wedge dx_{i_{k+1}} - x_{i_2} dx_{i_1} \wedge dx_{i_3} \wedge \cdots \wedge dx_{i_{k+1}} + \cdots
\]

\[ \mathcal{P}_{r-1} \Lambda^k(T) = \mathcal{P}_{r-1} \Lambda^k(T) + \kappa \mathcal{P}_{r-1} \Lambda^{k+1}(T) \]

The Koszul differential satisfies the homotopy formula

\[(d\kappa + \kappa d)u = (r + k)u, \quad u \in \mathcal{P}_r \Lambda^k \text{ homogeneous,} \]

which is key to establishing almost all the properties of \( \mathcal{P}_r \Lambda^k \) and \( \mathcal{P}_{r-1} \Lambda^k \).
Degrees of freedom

DOFs for $\mathcal{P}_r \Lambda^k(T)$: to a subsimplex $f$ of dimension $d \geq k$ we associate

$$\omega \mapsto \int_f \text{tr}_f \omega \wedge \eta, \quad \eta \in \mathcal{P}_{r+k-d} \Lambda^{d-k}(f)$$

**Theorem**

*These DOFs are unisolvent and the resulting finite element space satisfies*

$$\mathcal{P}_r \Lambda^k(T) = \{ \omega \in H \Lambda^k(\Omega) : \omega|_T \in \mathcal{P}_r \Lambda^k(T) \quad \forall T \in \mathcal{T} \}$$

DOFs for $\mathcal{P}_r^- \Lambda^k(T)$ (Hiptmair ’99):

$$\omega \mapsto \int_f \text{tr}_f \omega \wedge \eta, \quad \eta \in \mathcal{P}_{r+k-d-1} \Lambda^{d-k}(f)$$

+ similar theorem…
\[ \mathcal{P}_r \Lambda^k \]

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\( n = 2 \)

\( r = 1 \)
- \text{Lagrange '75}

\( r = 2 \)
- \text{Raviart-Thomas '75}

\( r = 3 \)

\( n = 3 \)

\( r = 1 \)

\( r = 2 \)
- \text{Nedelec edge elts '80}

\( r = 3 \)
- \text{Nedelec face elts '80}

\text{Whitney '57}

\text{DG}
$\mathcal{P}_r \Lambda^k$

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Sullivan '78

Lagrange

DG

BDM '85

Nedelec edge elts, 2nd kind '86

Nedelec face elts, 2nd kind '86
Finite element de Rham subcomplexes

We don’t only want spaces, we also need them to fit together into discrete de Rham complexes with BCP.

- One such FEdR subcomplex uses $P_r^\Lambda_k$ spaces of constant degree $r$:

$$0 \to P_r^\Lambda_0(T) \xrightarrow{d} P_r^\Lambda_1(T) \xrightarrow{d} \cdots \xrightarrow{d} P_r^\Lambda_n(T) \to 0$$

- Another uses $P_r\Lambda_k$ spaces with decreasing degree:

$$0 \to P_r\Lambda_0(T) \xrightarrow{d} P_{r-1}\Lambda_1(T) \xrightarrow{d} \cdots \xrightarrow{d} P_{r-n}\Lambda_n(T) \to 0$$
We also need to know that there exist bounded cochain projections into these de Rham subcomplexes (although the projections are not implemented).

The DOFs for any finite element space determine a projection operator into the space. The DOFs defining $P_r \Lambda^k(T_h)$ and $P_- \Lambda^k(T_h)$ were defined so as to ensure that the corresponding projection operators $\Pi_h$ from $\mathcal{C}\Lambda^k$ onto $V^k_h$ commute with $d$ (Stokes theorem).

However, the DOFs, and so $\Pi_h$, are not bounded on $H\Lambda^k$ (much less uniformly bounded wrt $h$). A technical construction modifies $\Pi_h$ to obtain a cochain projection which is uniformly bounded.
Characterization of the $\mathcal{P}_r \Lambda^k$ and $\mathcal{P}_r^- \Lambda^k$ spaces

**Theorem**

The following spaces of polynomial differential k-forms are invariant under all affine transformations of $\mathbb{R}^n$:

- $\mathcal{P}_r \Lambda^k$, $r \geq 0$,
- $\mathcal{P}_r^- \Lambda^k$, $r \geq 1$,
- $\{ u \in \mathcal{P}_r \Lambda^k \mid du \in \mathcal{P}_s \Lambda^k \}$, $r \geq 1$, $s < r - 1$

Moreover, these are the only affine invariant proper subspaces.

The proof is based on the representation theory of $GL(n)$. 
Cubical finite elements

Starting from the simple 1-D degree \( r \) finite element de Rham complex:

\[
0 \rightarrow \mathcal{P}_r \Lambda^0(\mathcal{T}_h) \xrightarrow{d} \mathcal{P}_{r-1} \Lambda^1(\mathcal{T}_h) \rightarrow 0
\]

we may apply a tensor product construction repeatedly, resulting in a finite element de Rham complex on a mesh of \( n \)-cubes:

**Shape fns:**

\[
Q^-_r \Lambda^k(I^n) = \bigoplus_{\sigma \in \Sigma(k,n)} \left[ \bigotimes_{i=1}^n \mathcal{P}_{r-\delta_{i,\sigma}}(I) \right] dx^{\sigma_1} \wedge \cdots \wedge dx^{\sigma_k}
\]

**DOFs:**

\[
u \mapsto \int f \text{tr}_f \nu(x) \wedge q(x), \quad q \in Q^-_{r-1} \Lambda^{d-k}(f)
\]

\[
0 \rightarrow Q^-_r \Lambda^0(\mathcal{T}_h^\otimes n) \xrightarrow{d} Q^-_r \Lambda^1(\mathcal{T}_h^\otimes n) \xrightarrow{d} \cdots \xrightarrow{d} Q^-_r \Lambda^n(\mathcal{T}_h^\otimes n) \rightarrow 0
\]

This family may be viewed as the cubical analogue of the \( \mathcal{P}_r^- \Lambda^k \) family.
\( Q_r^{\Lambda^k} \)

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The $S_r \Lambda^k$ family

With Awanou we recently found a 2nd family of de Rham subcomplexes based on cubical meshes. The spaces $S_r \Lambda^k$ were mostly unknown before. They are significantly smaller than the $Q_r^{-} \Lambda^k$ spaces for the same accuracy.

$$0 \to S_r \Lambda^0(T_h) \xrightarrow{d} S_{r-1} \Lambda^1(T_h) \xrightarrow{d} \cdots \xrightarrow{d} S_{r-n} \Lambda^n(T_h) \to 0$$

DOFs: $u \mapsto \int_f \text{tr}_f u \wedge q, \quad q \in P_{r-2d} \Lambda^{d-k}(f)$

Shape fns:

$$S_r \Lambda^k(I^n) = P_r \Lambda^k(I^n) \oplus \bigoplus_{\ell \geq 1} [\kappa H_{r+\ell-1} \Lambda^{k+1}(I^n) \oplus d\kappa H_{r+\ell} \Lambda^k(I^n)]_{\text{deg}=r+\ell}$$

$$H_{r,\ell} \Lambda^k(I^n) = \text{span of monomials } x_1^{\alpha_1} \cdots x_n^{\alpha_n} dx_{\sigma_1} \wedge \cdots \wedge dx_{\sigma_k}, \quad |\alpha| = r, \text{ linear in } \geq \ell \text{ variables not counting the } x_{\sigma_i}$$
Periodic Table of the Finite Elements

How to read the scheme (???)
Bla bla bla
Name of the bla bla
The FEniCS codestring
Consistency of the combinatorial codifferential
The combinatorial codifferential

Whitney forms \( P_1^{-} \Lambda^k(T_h) \cong C^k(T_h) \) (simplicial cochains)

[Therefore the isomorphism on cohomology given above, gives a proof of de Rham’s theorem.]

The restriction of the exterior derivative \( d_h : P_1^{-} \Lambda^k \to P_1^{-} \Lambda^{k+1} \) then corresponds to the natural combinatorial differential \( C^k \partial^* \to C^{k+1} \). The adjoint \( d_h^* : P_1^{-} \Lambda^{k+1} \to P_1^{-} \Lambda^k \) then can be taken to define a combinatorial codifferential \( C^{k+1} \to C^k \).

In 1976, Dodziuk and Patodi studied the combinatorial Laplacian \( d_h d_h^* + d_h^* d_h \) acting on cochains and proved that its eigenvalues converged to those of the \( k \)-form Laplacian with rate \( O(h | \log h |) \).

Their discretization method is equivalent to the FE method above, but the viewpoint is different, with the FEEC viewpoint avoiding \( d_h^* \) and leading to a sharper rate \( O(h^2) \). D-P conjectured that unlike \( d_h \), \( d_h^* \) did not satisfy the consistency result

\[
\lim_{h \to 0} \| d_h^* \Pi_h u - d^* u \| = 0
\]

and pointed out that as a source of difficulties.
Consistency of $d^*_h$

In 1991 Smits showed for 1-forms in 2D and uniformly refined mesh sequences $d^*_h$ is consistent. This left open several questions:

- Is $d^*_h$ consistent for 1-forms in $n$-D on uniformly refined meshes?
- Is $d^*_h$ consistent for 1-forms in 2D (or $n$-D) on other meshes?
- Is it consistent for $k$-forms in $n$-D, $1 < k < n$, on uniform or other meshes?

In work with Falk, Guzmán, and Tsogtgerel (to appear in Trans. AMS), we related this to the theory of superconvergence of finite element methods, and answered these questions.
Generalization of Smits theorem to $n$-D

Definition: A triangulation $\mathcal{T}_h$ of a domain in $\mathbb{R}^n$ is uniform if there is a basis for $\mathbb{R}^n$ such that

1. Every simplex contains edges parallel to each of the basis elements.
2. The union of simplices containing such an interior edge is invariant under reflection through the midpoint of the edge.

Theorem

Assume that the family of triangulations $\mathcal{T}_h$ of a domain in $\mathbb{R}^n$ is shape regular, quasiuniform, and piecewise uniform. Then

$$\lim \|d^*_h \Pi_h u - d^* u\| = 0$$

for every smooth 1-form in the domain of $d^*$. 
Inconsistency of $d_h^*$ on criss-cross triangulations

Let $T_h$ be a criss-cross triangulation of $(-1, 1) \times (-1, 1)$ and $u = (1 - x^2)dx$. Then

$$d^* u = 2x, \quad d_h^* u(x, y) = \begin{cases} 
-h, & x = -1, \\
0, & -1 < x < 1, \text{ x a multiple of } h, \\
h, & x = 1, \\
-6 + 2h, & x = -1 + h/2, \\
6x, & -1 + h/2 < x < 1 - h/2, \text{ x an odd multiple of } h/2, \\
6 - 2h, & x = 1 - h/2.
\end{cases}$$

and so $\lim \|d_h^* \Pi_h u - d^* u\| \neq 0$. 
Inconsistency of $d_h^*$ for 2-forms in 3D

Numerical computation demonstrates clearly that for 2-forms in 3D, $d_h^*$ is inconsistent even for uniform meshes.

<table>
<thead>
<tr>
<th>triangles</th>
<th>$|d_h^<em>\Pi_h u - d^</em> u|_{L^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>48</td>
<td>1.59</td>
</tr>
<tr>
<td>384</td>
<td>1.18</td>
</tr>
<tr>
<td>3072</td>
<td>1.00</td>
</tr>
<tr>
<td>24,576</td>
<td>0.95</td>
</tr>
<tr>
<td>196,608</td>
<td>3.37</td>
</tr>
</tbody>
</table>

Thus the conjecture of Dodziuk–Patodi is true: $d_h^*$ is not consistent in general.

Fortunately, consistency of $d_h^*$ plays no role in the FEEC approach.