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Frozen modes and resonance phenomena in periodic metamaterials

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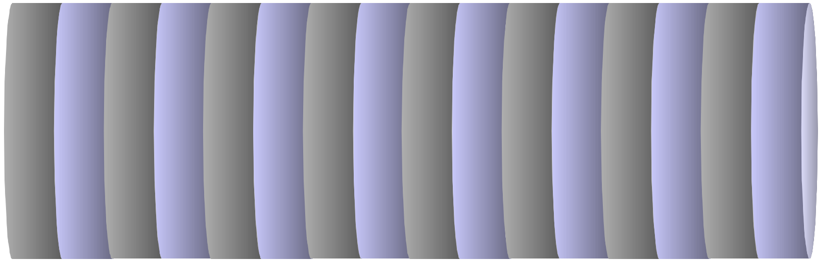
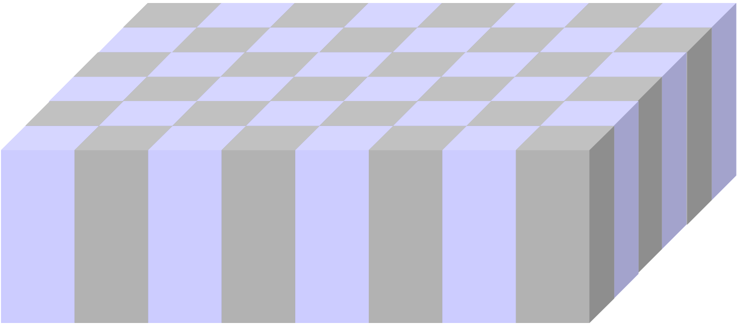
Supported by AFOSR

What are photonic crystals?
Simplest examples of periodic dielectric arrays

1D periodicity



2D periodicity



Each constitutive component is perfectly transparent, while their periodic array may not transmit EM waves of certain frequencies.

What are the differences between "conventional" metamaterials and photonic crystals (PC)?

- At frequency range of interest, all PC should be treated as genuinely heterogeneous media. Equivalent uniform media approach usually does not apply (no effective ε and/or μ can be introduced).

- At frequency range of interest, PC of any dimensionality are essentially anisotropic – even those with cubic symmetry.

The reason for this is that at frequencies where a PC behaves as a PC, the spatial dispersion must be very strong.

At the same time, such phenomena as:

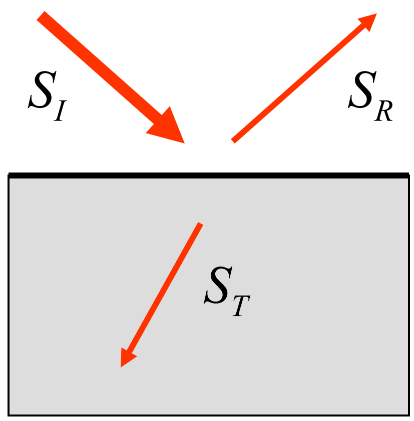
- backward waves (those with opposite phase and group velocities) and

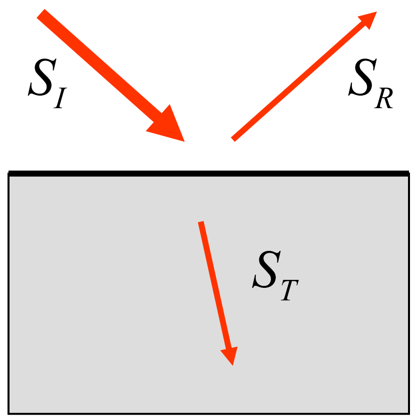
- negative refraction

can be found in almost any PC.

Different scenarios of "negative refraction" in photonic crystals:

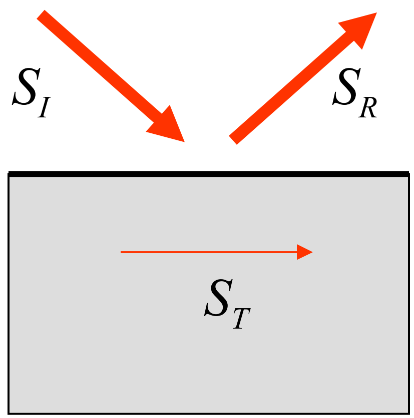
Due to strong structural anisotropy, in either case, the normal component of the transmitted wave phase velocity can be opposite to that of the normal component of its group velocity (possible definition of negative refraction in anisotropic media).



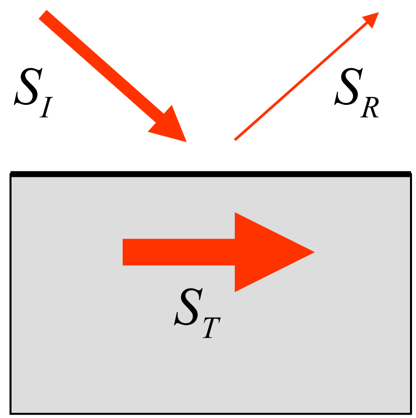


"Zero refraction" scenarios in photonic crystals:

In either case, the normal component of the transmitted wave group velocity vanishes. What happens to the transmitted wave?



a) Total reflection



b) Frozen mode regime

Case (a) is analogous to the phenomenon of total internal reflection (this case is typical of uniform materials).

In case (b), the transmitted wave is a grazing mode with huge amplitude and tangential energy flux. This case is impossible in uniform materials.

From now on, PC will be treated as genuinely heterogeneous periodic media.

Electromagnetic dispersion relation in photonic crystals

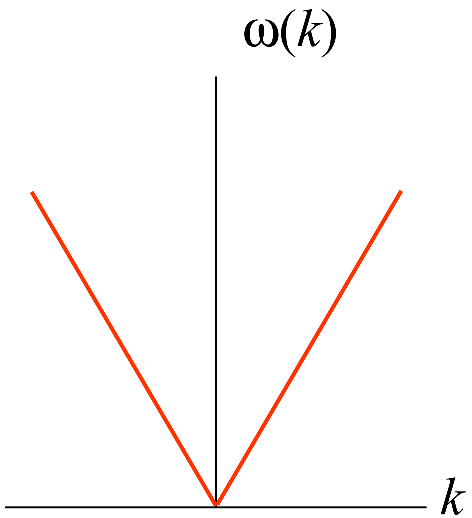
Electromagnetic eigenmodes in periodic media are Bloch waves

$$\Psi\left(\vec{r}+\vec{L}\right)=\Psi\left(\vec{r}\right)\exp\left(i\vec{k}\cdot\vec{L}\right).$$

Propagating Bloch modes: $\vec{k}^{\,*}=\vec{k}$. Evanescent Bloch modes: $\vec{k}\neq\vec{k}^{\,*}$.

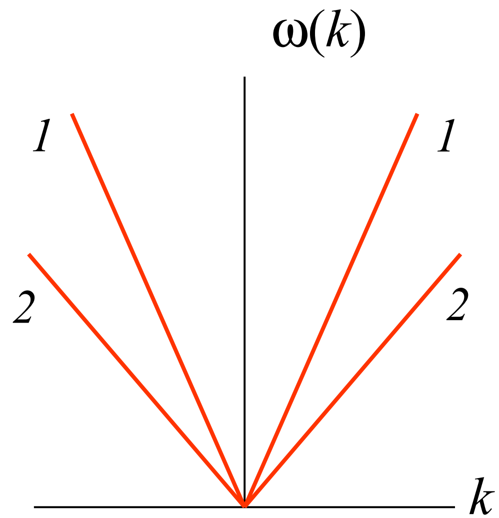
$\omega\leftrightarrow\vec{k}$ - dispersion relation ($\omega-\vec{k}$ diagram).

$\vec{v}_g=\partial\omega/\partial\vec{k}$ - group velocity = energy velocity of propagating mode.



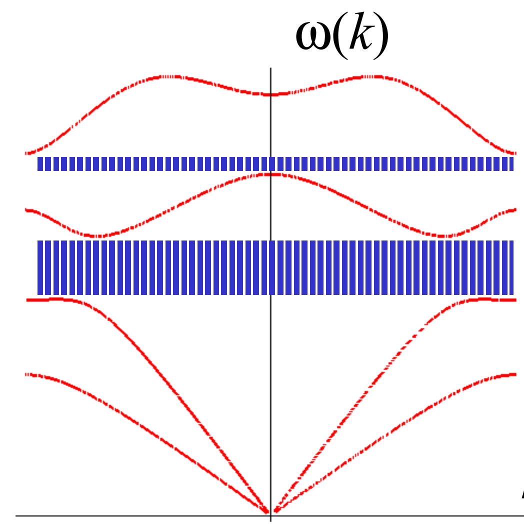
Typical $k-\omega$ diagram of an isotropic non-dispersive medium:

$\omega = v\,k$



Typical $k-\omega$ diagram of a uniform anisotropic medium for a given direction of ***k***.

1 and *2* are two polarizations.

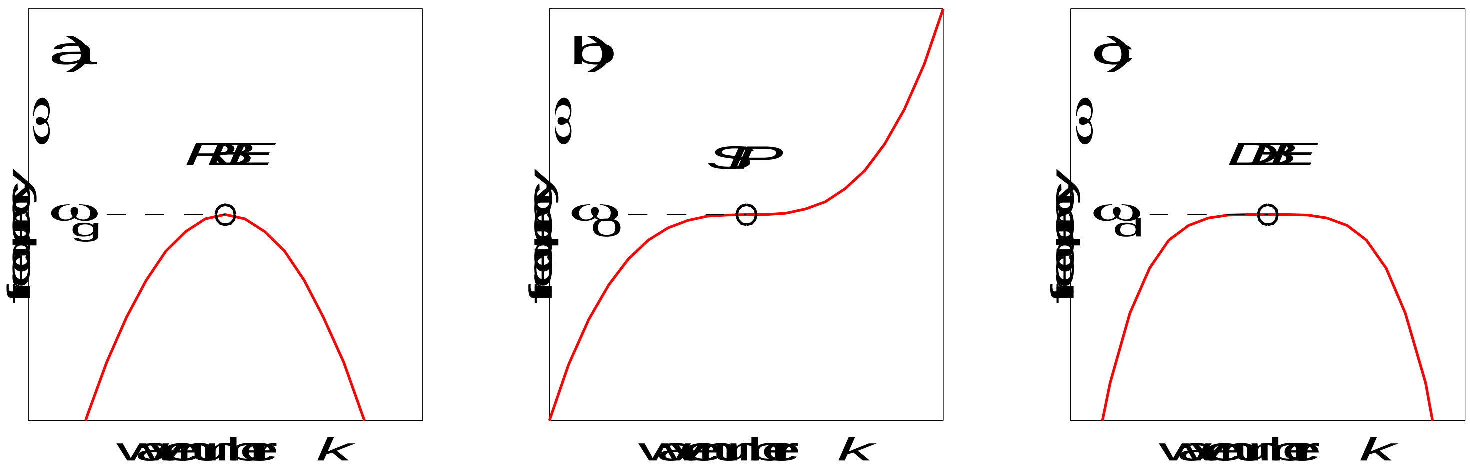


Typical $k-\omega$ diagram of a photonic crystal for a given direction of ***k***.

EM waves with vanishing group velocity in photonic crystals: stationary points of dispersion relation $\omega(k)$

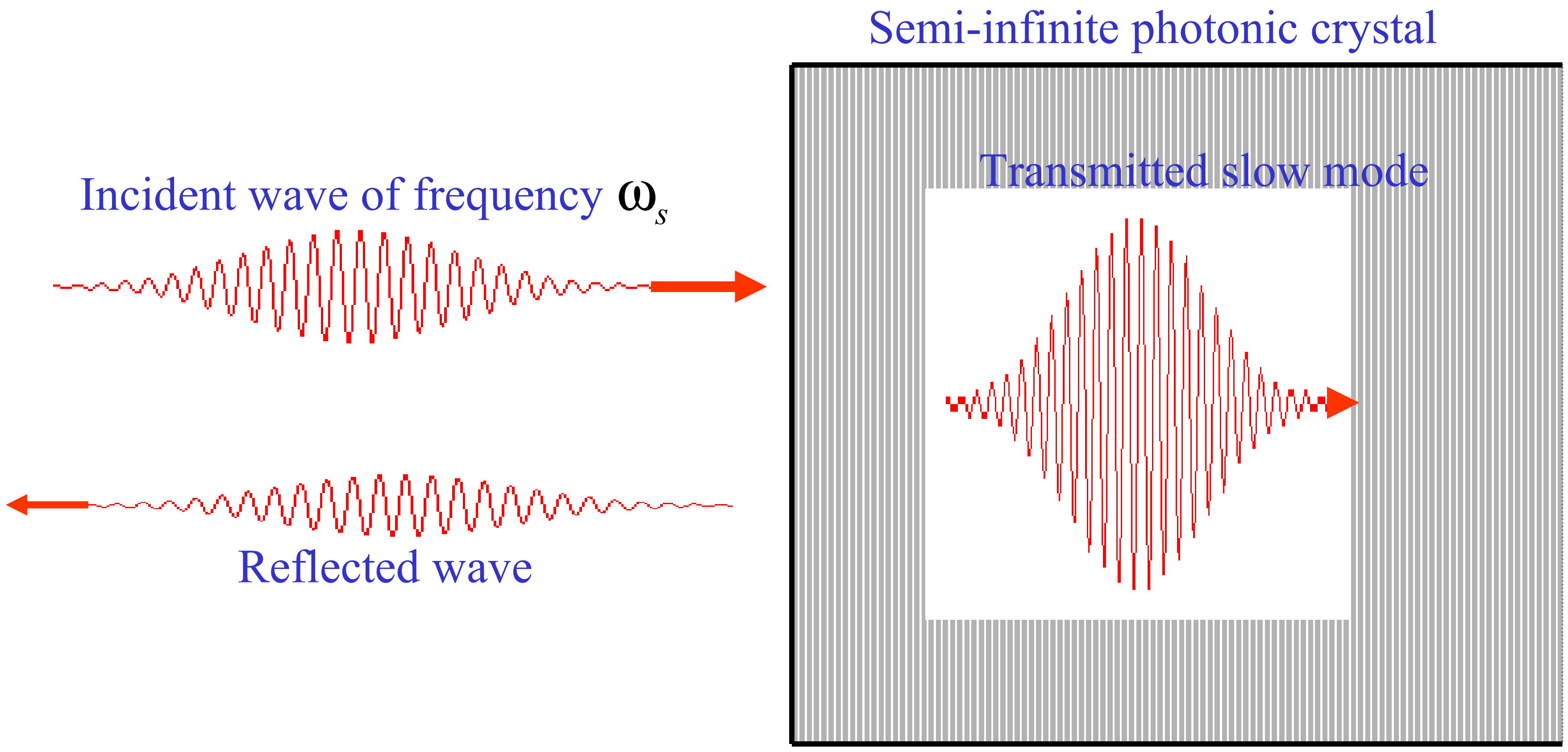
Examples of stationary points:

- Regular band edge (RBE): $\omega - \omega_g \mu \left(k - k_g\right)^2, \quad v_g \mu \left(k - k_g\right) \mu \left(\omega - \omega_g\right)^{1/2}.$
- Stationary inflection point (SIP): $\omega - \omega_0 \mu \left(k - k_0\right)^3, \quad v_g \mu \left(k - k_0\right)^2 \mu \left(\omega - \omega_0\right)^{2/3}.$
- Degenerate band edge (DBE): $\omega - \omega_d \mu \left(k - k_d\right)^4, \quad v_g \mu \left(k - k_d\right)^3 \mu \left(\omega_d - \omega\right)^{3/4}.$



Each stationary point is associated with slow mode with $v_g \sim 0$.
But there are some fundamental differences between these three cases.

A plane wave incident on a semi-infinite photonic crystal



What happens if the incident wave frequency coincides with that of a slow mode with $v_g \sim 0$? Will the incident wave be converted into the slow mode inside the photonic crystal, or will it be reflected back to space?

Energy conservation consideration.

Transmission / reflection coefficients

$$\tau = \frac{S_T}{S_I}, \quad \rho = \frac{S_R}{S_I}, \quad \tau + \rho = 1.$$

where S_I, S_T, S_R are the energy fluxes of the incident, transmitted and reflected waves, respectively.

Slow mode energy flux is $S_T = W \times v_g$, where $v_g \to 0$.

Therefore, as ω approaches ω_s:

$$S_T \to 0 \text{ and } \tau \to 0, \quad \text{unless } W \to \infty.$$

Assuming that the incident wave amplitude is unity, let us see what happens if the slow mode is related to (*1*) regular band edge, (*2*) stationary inflection point or (*3*) degenerate band edge.

1. Regular photonic band edge (generic case)

$$\omega - \omega_g \approx \frac{\omega_b''}{2}\left(k - k_g\right)^2, \quad \omega \leq \omega_g'' < 0.$$

The slow mode mode group velocity v_g is

$$v_g = \frac{\partial \omega}{\partial k} \approx \omega_g''\left(k - k_g\right) \approx \sqrt{2\omega_g''\left(\omega - \omega_g\right)}.$$

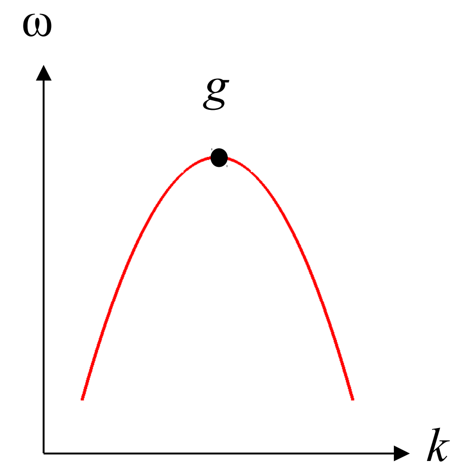
The slow mode energy density at $\omega \approx \omega_g$

$$W \sim 1.$$

The slow mode energy flux vanishes at $\omega \approx \omega_g$

$$S_T = v_g \times W \mu \sqrt{2\omega_g''\left(\omega - \omega_g\right)} \rightarrow 0 \text{ as } \omega \rightarrow \omega_g,$$

implying total reflection of the incident wave.



RBE case

Frozen mode regime at SIP

2. Stationary inflection point (the frozen mode regime)

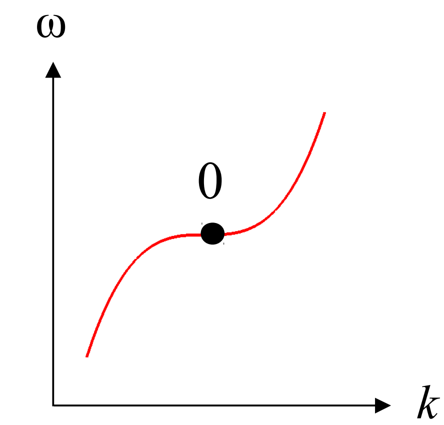
$$\omega - \omega_0 \approx \frac{\omega_0'''}{6}\left(k - k_0\right)^3 .$$

The slow mode group velocity vanishes

$$v_g = \frac{\partial \omega}{\partial k} \approx \frac{\omega_0'''}{2}\left(k - k_0\right)^2 \propto \left(\omega - \omega_0\right)^{2/3},$$

while its energy density diverges as $\omega \to \omega_0$

$$W \propto \left(\omega - \omega_0\right)^{-2/3}.$$



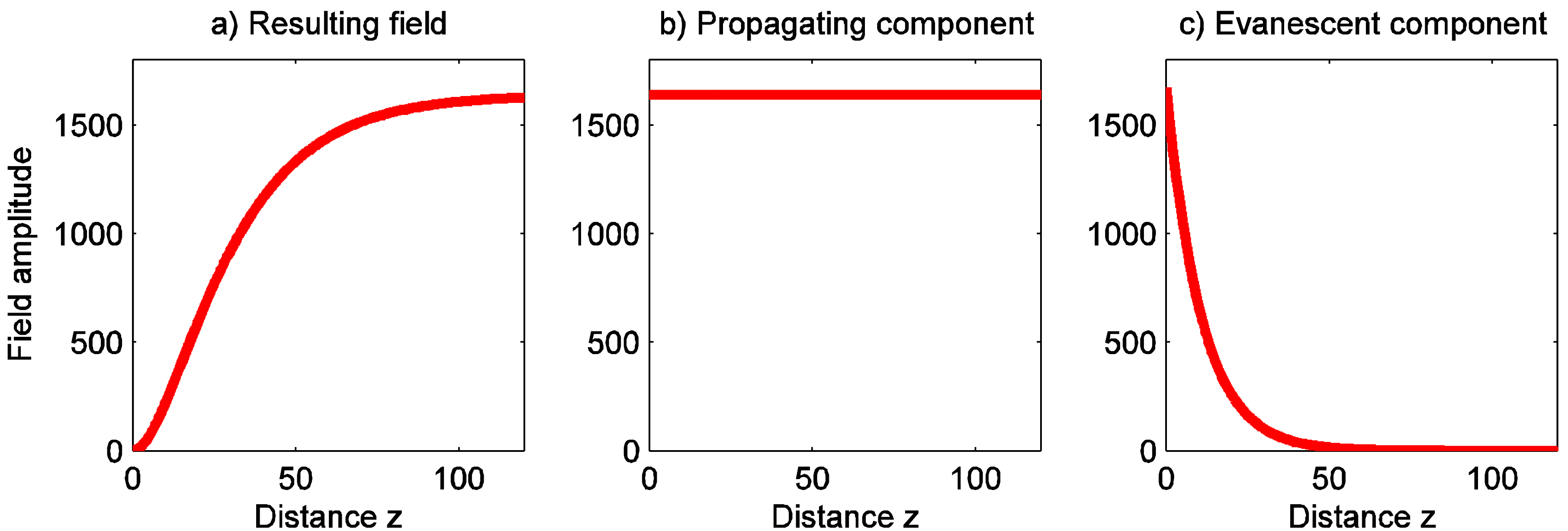
SIP case

The slow mode energy flux remains finite at $\omega \approx \omega_0$

$$S_T = v_g \times W \propto 1,$$

implying conversion of the incident wave to the frozen mode with huge diverging amplitude and nearly zero group velocity.

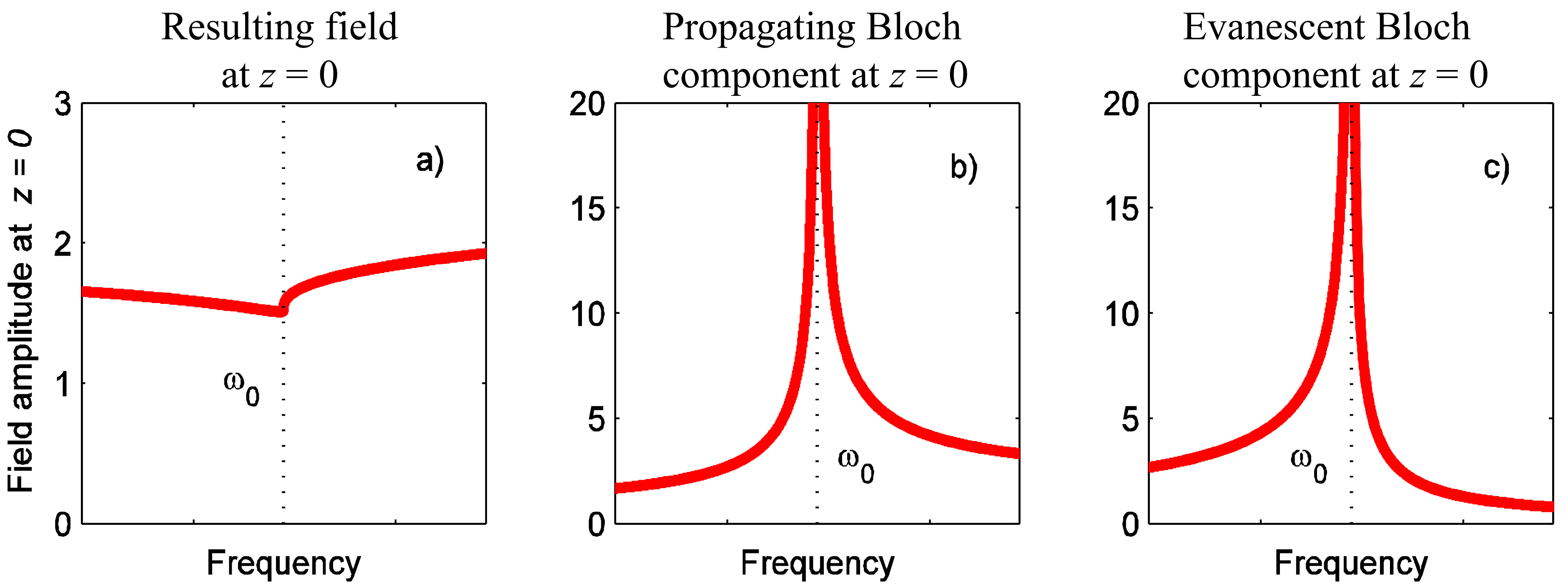
Steady-state frozen mode regime in ideal lossless semi-infinite photonic slab. Frequency is close (but not equal) to that of SIP.



Electromagnetic field and its propagating and evanescent components inside semi-infinite photonic crystal. The amplitude of the incident wave is unity. Frequency ω is close (but not equal) to ω_0 .

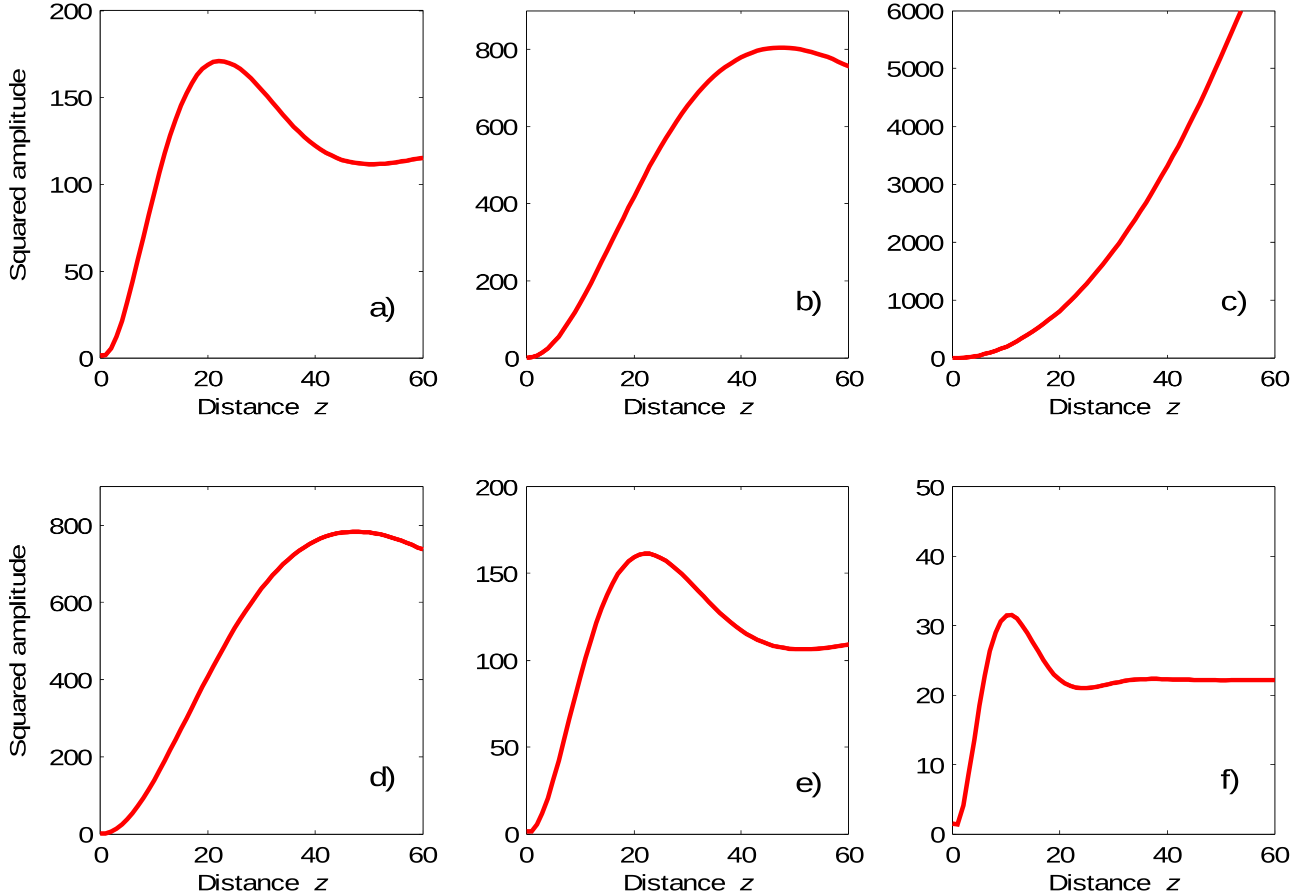
As ω approaches ω_0 , the frozen mode amplitude diverges as $W \sim (\omega - \omega_0)^{-2/3}$.

Destructive interference of propagating and evanescent components of the frozen mode at photonic crystal boundary at $z = 0$.



Frequency dependence of the resulting field amplitude (a) and its propagating and evanescent Bloch components (b and c) at the photonic crystal boundary at $z = 0$. The amplitude of the incident wave is unity.

Frozen mode profile at different frequencies close to SIP [3 - 4]

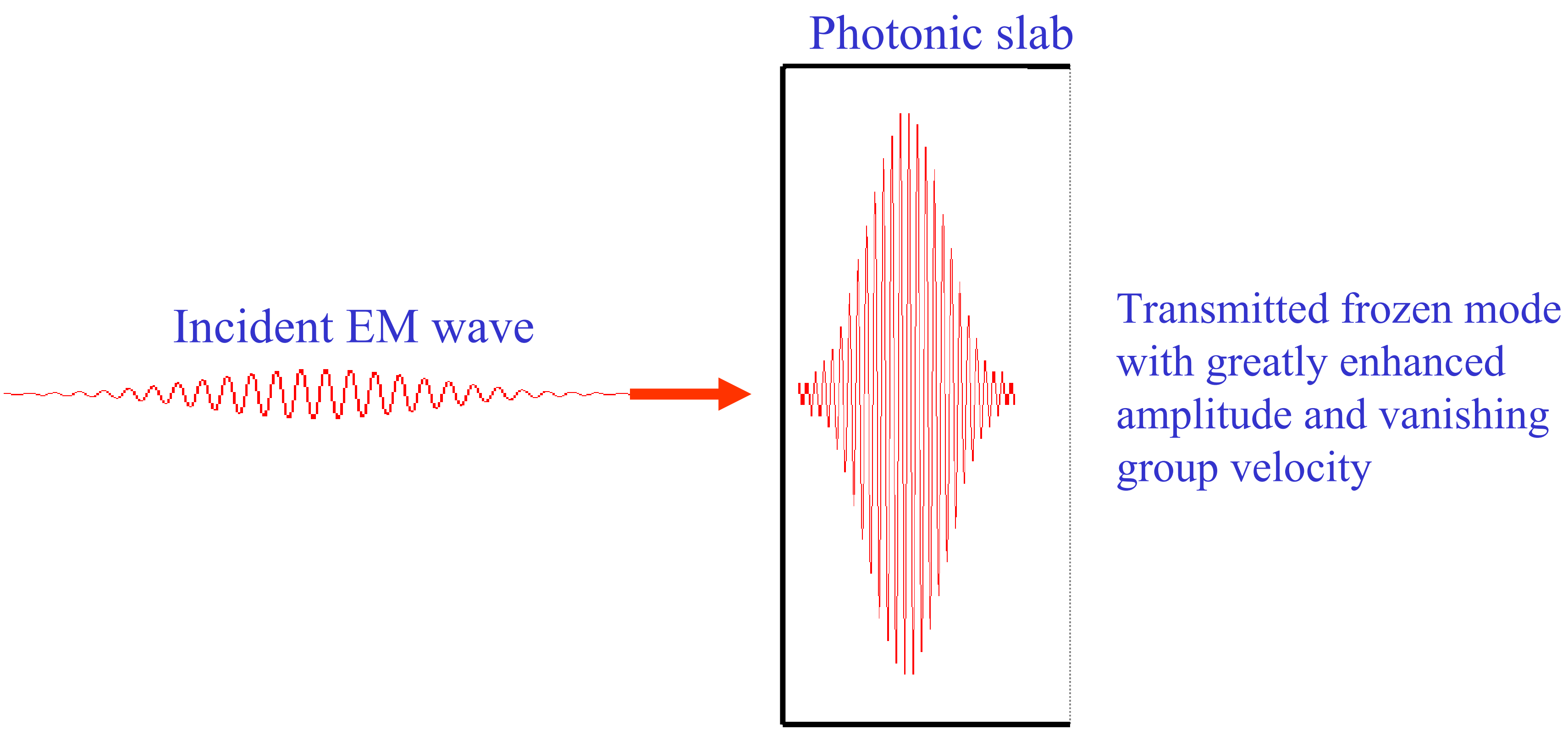


Smoothed profile of the frozen mode at six different frequencies in the vicinity of SIP: (a) $\omega = \omega_0 - 10^{-4}\, c/L$, (b) $\omega = \omega_0 - 10^{-5}\, c/L$, (c) $\omega = \omega_0$, (d) $\omega = \omega_0 + 10^{-5}\, c/L$, (e) $\omega = \omega_0 + 10^{-4}\, c/L$, (f) $\omega = \omega_0 + 10^{-3}\, c/L$. In all cases, the incident wave has the same polarization and unity amplitude. The distance z from the surface of the photonic crystal is expressed in units of L.

Frozen mode regime at SIP. Main features. [3 - 4]

- Both at normal and oblique versions of the frozen mode regime at SIP, the incident wave is completely converted into the frozen mode with greatly enhanced amplitude and vanishing group velocity.

- At oblique frozen mode regime, one can continuously change the operational frequency within certain range, simply by changing the angle of incidence. There is no need to alter the physical parameters of the periodic structure.



Frozen mode regime at DBE

3. Degenerate band edge (the intermidiate case)

$$\omega-\omega_d \approx \frac{\omega_d^{(4)}}{24}\left(k-k_d\right)^4, \quad \omega \le \omega_d < 0.$$

The slow mode group velocity v_g is

$$v_g = \frac{\partial\omega}{\partial k} \approx \frac{\omega_d^{(4)}}{6}\left(k-k_d\right)^3 \propto \left(\omega_d-\omega\right)^{3/4},$$

while its energy density diverges as $\omega \to \omega_d$

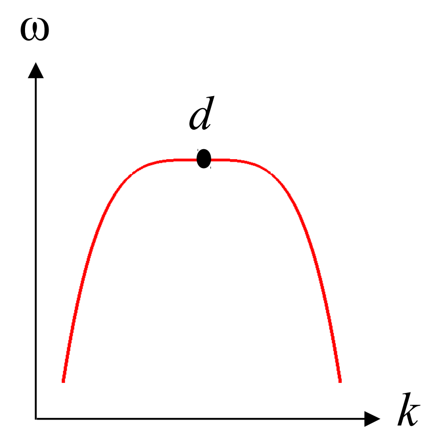
$$W \propto \left(\omega_d-\omega\right)^{-1/2}.$$

The slow mode energy flux vanishes at $\omega \approx \omega_0$

$$S_T = u \times W \propto \left(\omega_d-\omega\right)^{1/4},$$

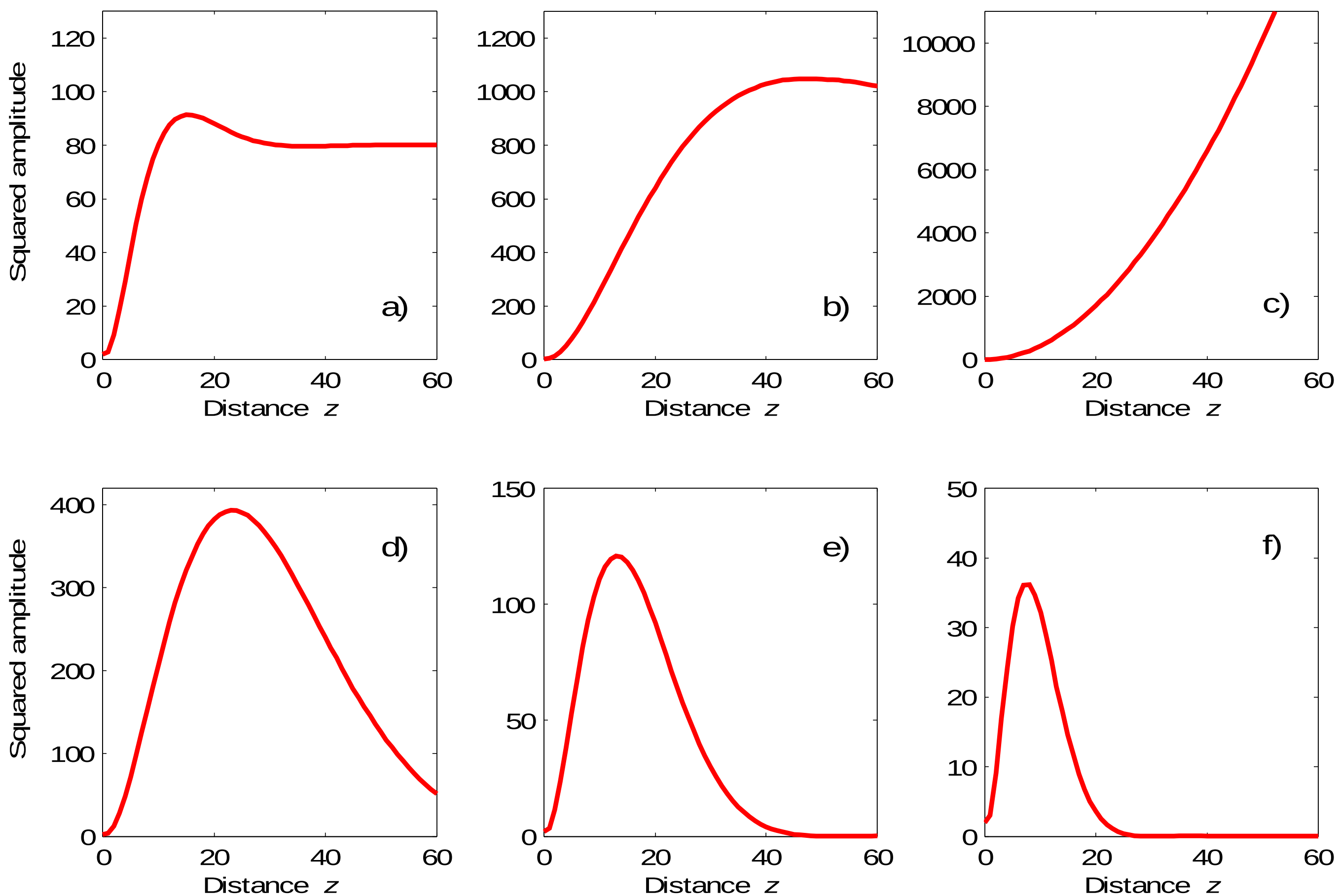
implying total reflection of incident wave.

This case is intermediate between the frozen mode regime at stationary IP and the case of total reflection at a regular BE.



DBE case

Frozen mode profile at different frequencies close to DBE [9]

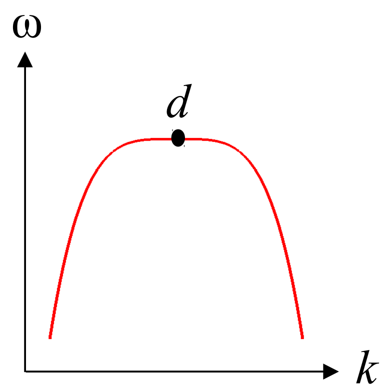


Smoothed profile of the frozen mode at six different frequencies in the vicinity of SIP: (a) $\omega = \omega_d - 10^{-4}\, c/L$, (b) $\omega = \omega_d - 10^{-6}\, c/L$, (c) $\omega = \omega_d$, (d) $\omega = \omega_d + 10^{-6}\, c/L$, (e) $\omega = \omega_d + 10^{-5}\, c/L$, (f) $\omega = \omega_d + 10^{-4}\, c/L$. In all cases, the incident wave has the same polarization and unity amplitude. The distance z from the surface of the photonic crystal is expressed in units of L.

Subsurface waves at band gap frequencies close to DBE

Giant subsurface waves at band gap frequencies close to DBE [9].

This phenomenon can be viewed as a particular case of frozen mode regime.



$$\omega - \omega_d \; \mu \; \left(k - k_d \right)^4$$

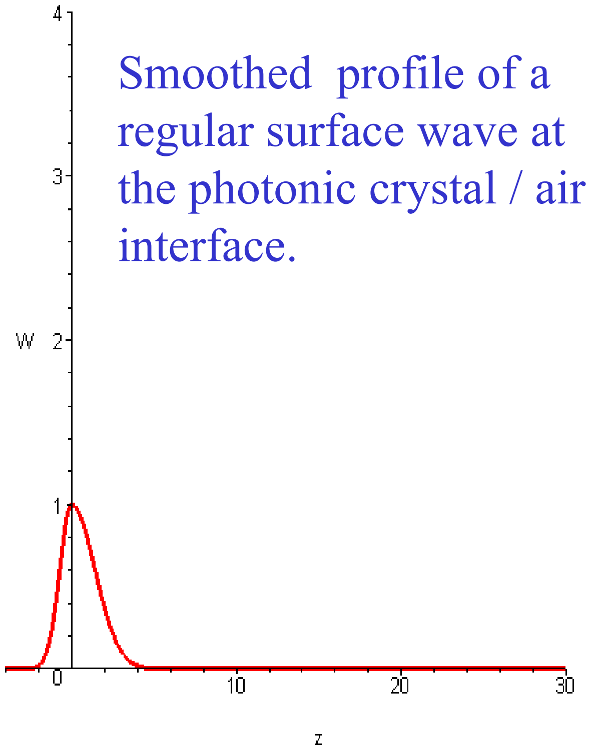
In vacuum (at $z < 0$), the z component of the wave vector $\vec{k}$ is

$$k_z = \sqrt{\omega^2 c^2 - \left(k_x^2 + k_x^2 \right)}$$

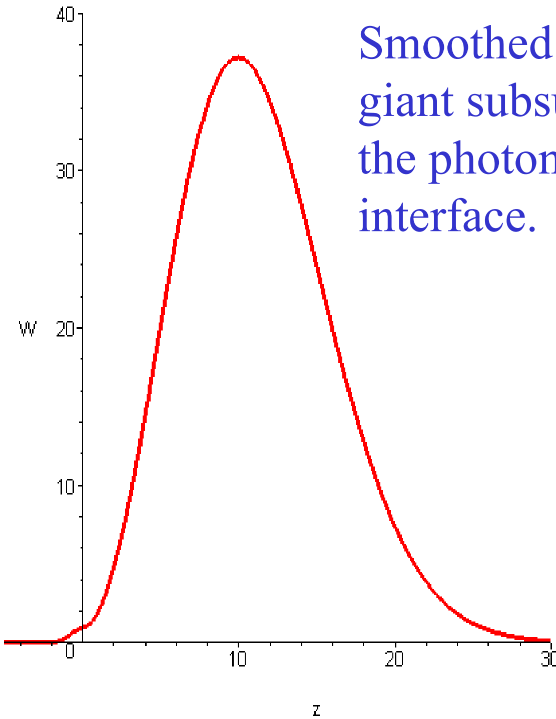
Surface waves corespond to imaginary value of k_z :

$$k_x^2 + k_x^2 > \omega^2 c^2, \quad \text{while} \quad \omega > \omega_d$$

At frequencies close to DBE, the surface wave profile is exactly the same as that of a DBE related frozen mode.



Smoothed profile of a regular surface wave at the photonic crystal / air interface.



Smoothed profile of a giant subsurface wave at the photonic crystal / air interface.

In a subsurface wave, the field amplitude decays exponentially in vacuum (at $z < 0$). But inside the periodic medium (at $z > 0$), the field amplitude increases sharply with the distance z and reaches its maximum value of

$$\Psi_{\max} \mu \left(\omega-\omega_d\right)^{-1/4} \text{ at a distance } Z \mu \left(\omega-\omega_d\right)^{-1/4}.$$

Then it slowly decays as the distance z further increases.

The following features are unique to giant subsurface waves (GSSW):

1. Although the amplitude of GSSW can be very large, it is value at the surface is relatively small. Therefore, GSSW are much better confined and should be much less leaky, compared to regular surface waves.

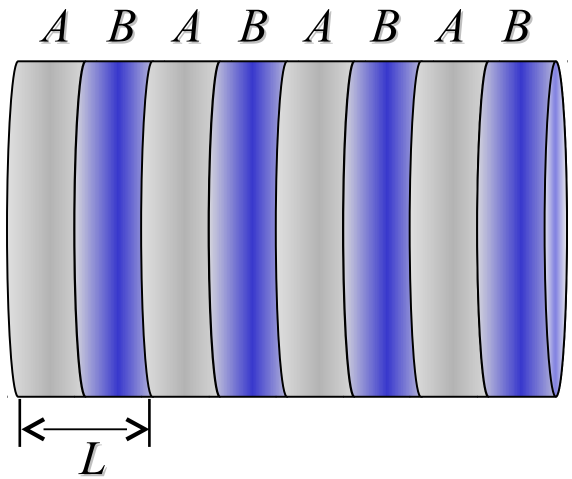
2. At any given frequency, there is only one GSSW propagating in a certain direction along the surface of the photonic slab. This direction is a function of frequency.

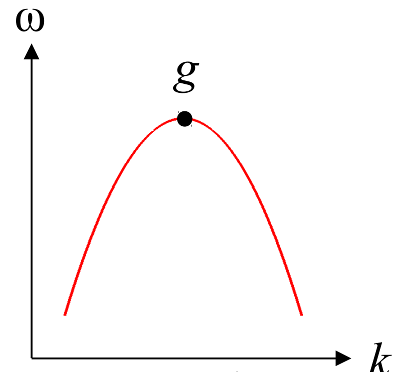
3. If the layered array with DBE has a defect layer, a GSSW can also be realized as a localized mode with the profile similar to that of the frozen mode. Such a mode can accumulate a lot of energy – much more than a regular localized state with the same amplitude at the maximum.

What structures can display band diagrams with RBE, SIP, and DBE?

1. RBE – regular band edge (there is no frozen mode regime here)
2. SIP – stationary inflection point
3. DBE – degenerate band edge

1. RBE occurs in any periodic array of any dimensionality

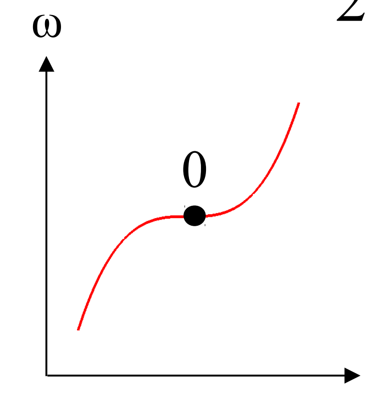




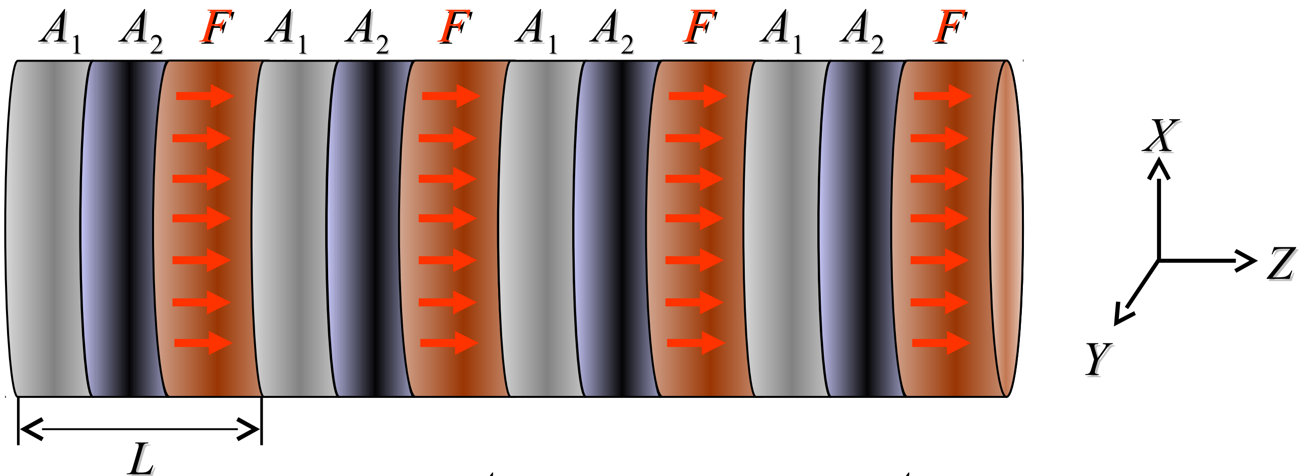
$$\omega - \omega_g \propto \left(k - k_g \right)^2$$

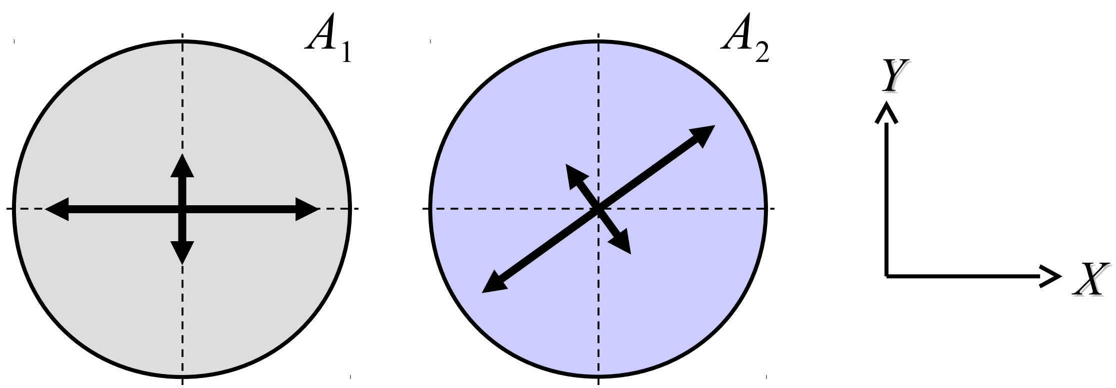
Alternating layers *A* and *B* can be made of any two different low-loss materials – either isotropic, or anisotropic. *L* is a unit cell of the periodic structure – in this case it includes two layers.

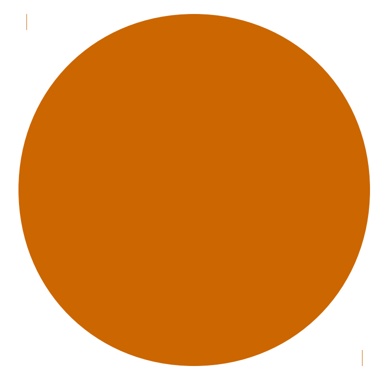
2. A SIP at normal incidence can occur in periodic arrays of magnetic and misaligned anisotropic layers, with 3 layers in a unit cell L [2].



$$\omega - \omega_0 \, \mu \left(k - k_0 \right)^3$$



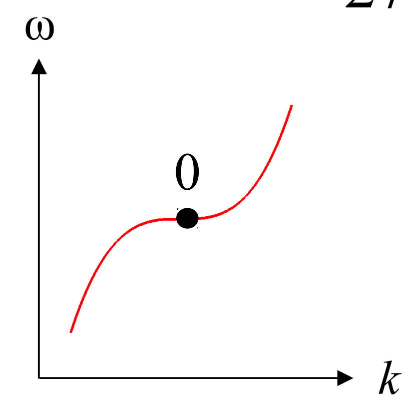




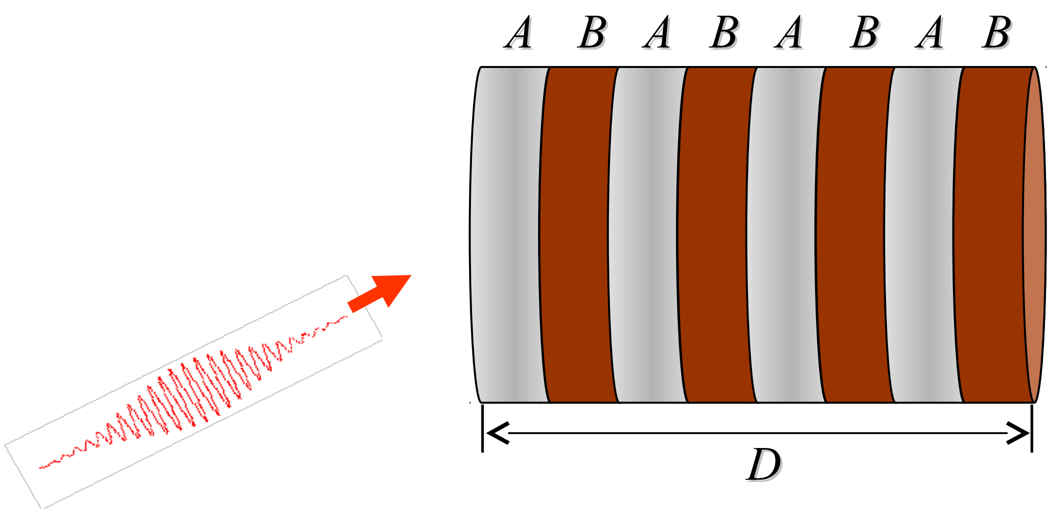
A_1 and A_2 are anisotropic layers with misaligned in-plane anisotropy. The misalignment angle is different from 0 and $\pi/2$.

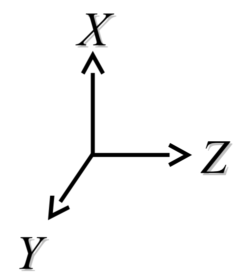
F – layers are magnetized along the z direction.

A SIP at oblique incidence can occur in periodic arrays of two layers, of which one must display off-plane anisotropy [3, 4].



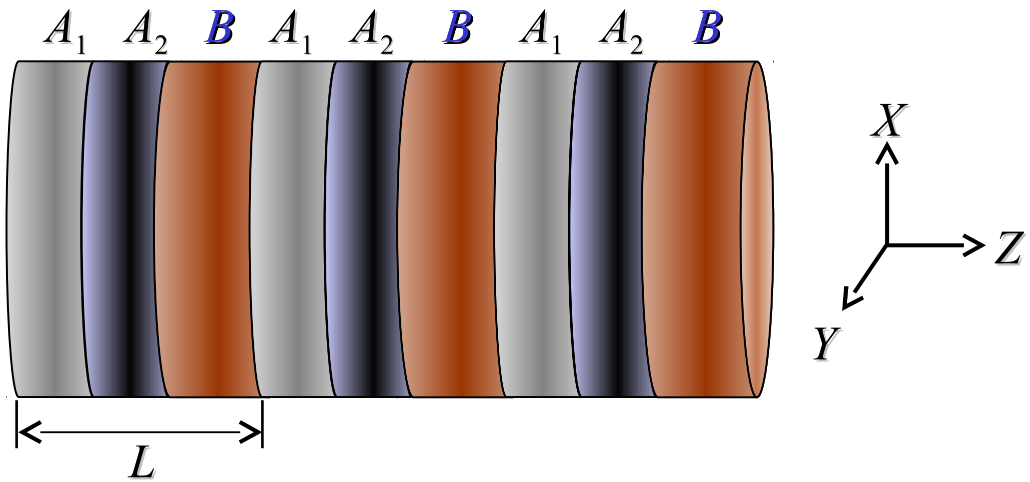
$\omega - \omega_0 \, \mu \left(k - k_0 \right)^3$

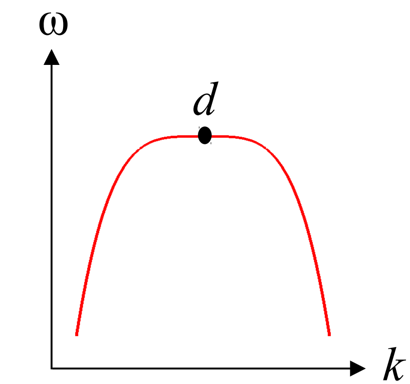




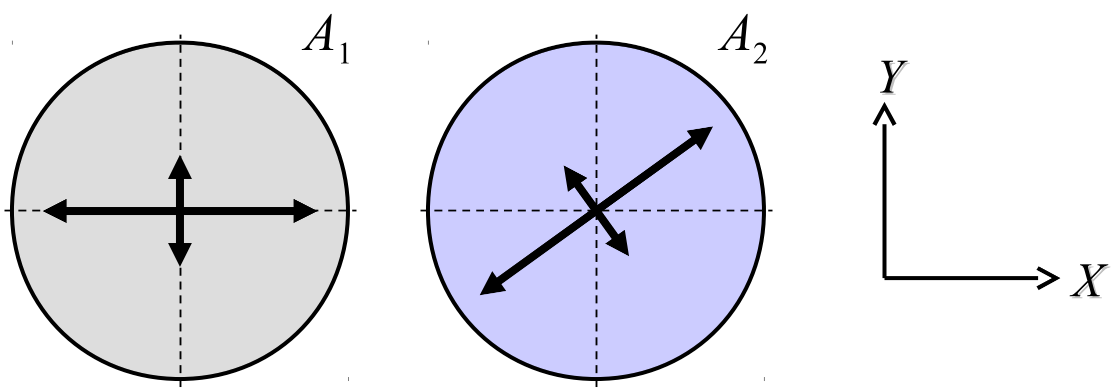
Alternating layers A and B , of which the A layers display off-plane anisotropy ($\varepsilon_{xz} \neq 0$), while the B layers can be isotropic.

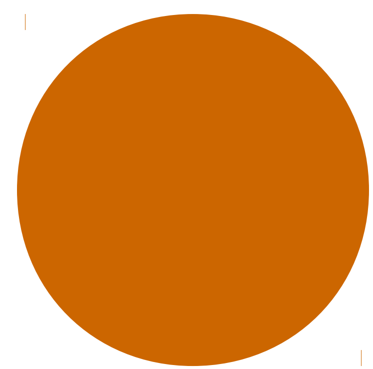
3. A DBE at normal incidence can occur in periodic arrays of with three layers in a unit cell *L*, of which two must display misaligned in-plane anisotropy [6].





$$\omega - \omega_d \ \mu \left(k - k_d \right)^4$$

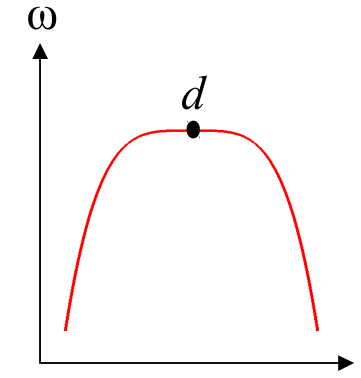




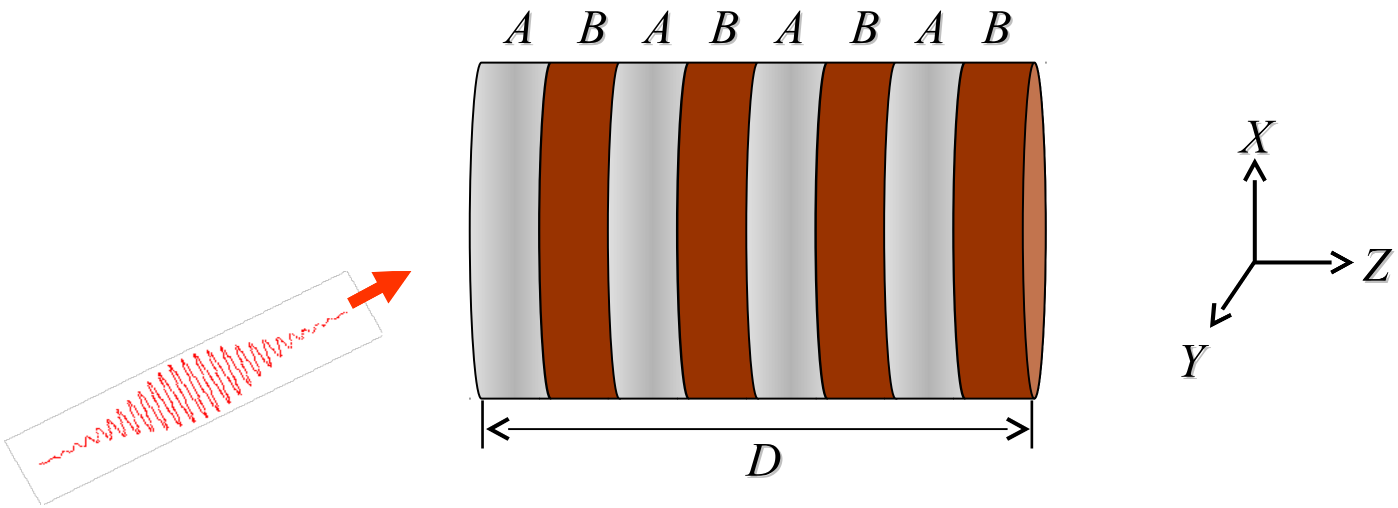
A_1 and A_2 are anisotropic layers with misaligned in-plane anisotropy. The misalignment angle is different from 0 and $\pi/2$.

B – layers can be isotropic.

A DBE at oblique incidence can occur in periodic arrays of two layers, of which one must display in-plane anisotropy [9].

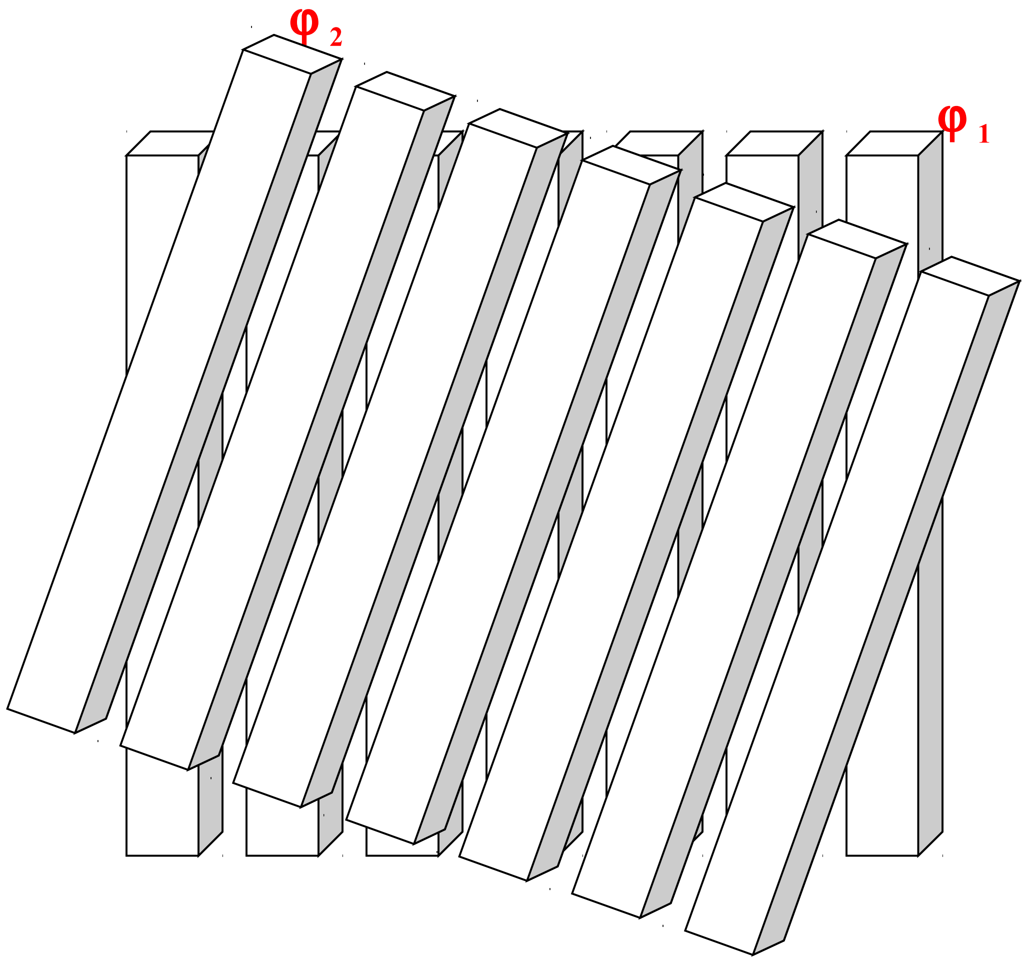


$$\omega-\omega_d \;\mu\; \left(k-k_d\right)^4$$



Alternating layers A and B , of which the A layers must display in-plane anisotropy ($\varepsilon_{xx} \neq \varepsilon_{yy}$, $\varepsilon_{xz} = \varepsilon_{yz} = 0$), while the B layers can be isotropic.

The presence of anisotropic (birefringent) materials may not be necessary in periodic structures with *2D* or *3D* periodicity. For example, one can use periodic arrays of identical rods made of **isotropic** dielectric material to replace **uniform** but **anisotropic** *A* layers. The angles φ_1 and φ_2 now define the orientations of the adjacent arrays of the rods. After the replacement, the entire periodic structure will be a *3D* photonic crystal.



Summary on what structures can support different versions of frozen mode regime.

The layered structures supporting frozen mode regime at oblique incidence are, generally, simpler, compared to those supporting the normal version of frozen mode regime. For instance, at oblique incidence, a unit cell of the periodic array can include as few as two different layers.

The layered structures supporting DBE are, generally, less complicated and more practical, compared to those supporting SIP. The differences are especially pronounced at optical frequencies.

The use of periodic structures with 2D and 3D periodicity can remove the necessity to employ birefringent constitutive components, which is important. The downside, though, is that layered arrays are, generally, easier to build and much easier to analyze.

Publications

[1] A. Figotin and I. Vitebsky. Nonreciprocal magnetic photonic crystals. Phys. Rev. E 63, 066609, (2001)

[2] A. Figotin and I. Vitebskiy. Electromagnetic unidirectionality in magnetic photonic crystals. Phys. Rev. B 67, 165210 (2003).

[3] A. Figotin and I. Vitebskiy. Oblique frozen modes in layered media. Phys. Rev. E 68, 036609 (2003).

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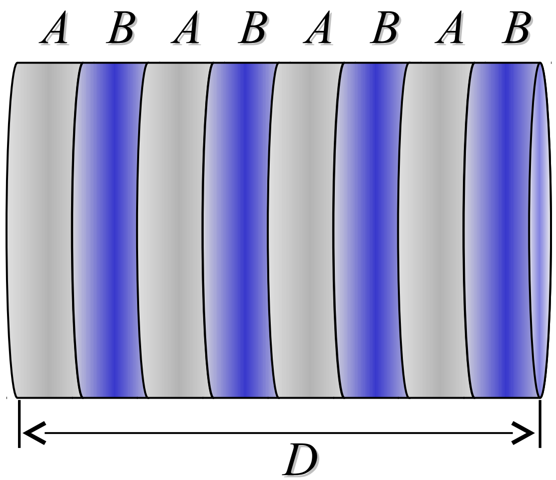
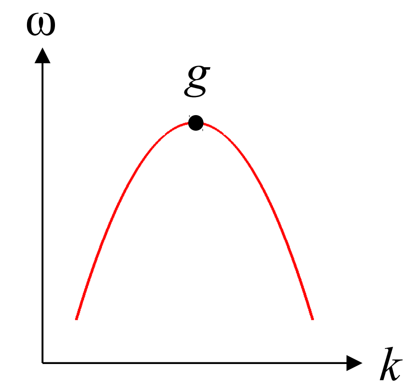
Auxiliary slides

1. Giant transmission band edge resonance

(Fabry-Perrot cavity resonance in the vicinity of DBE)

Standard arrangement: a transmission band-edge resonance below RBE





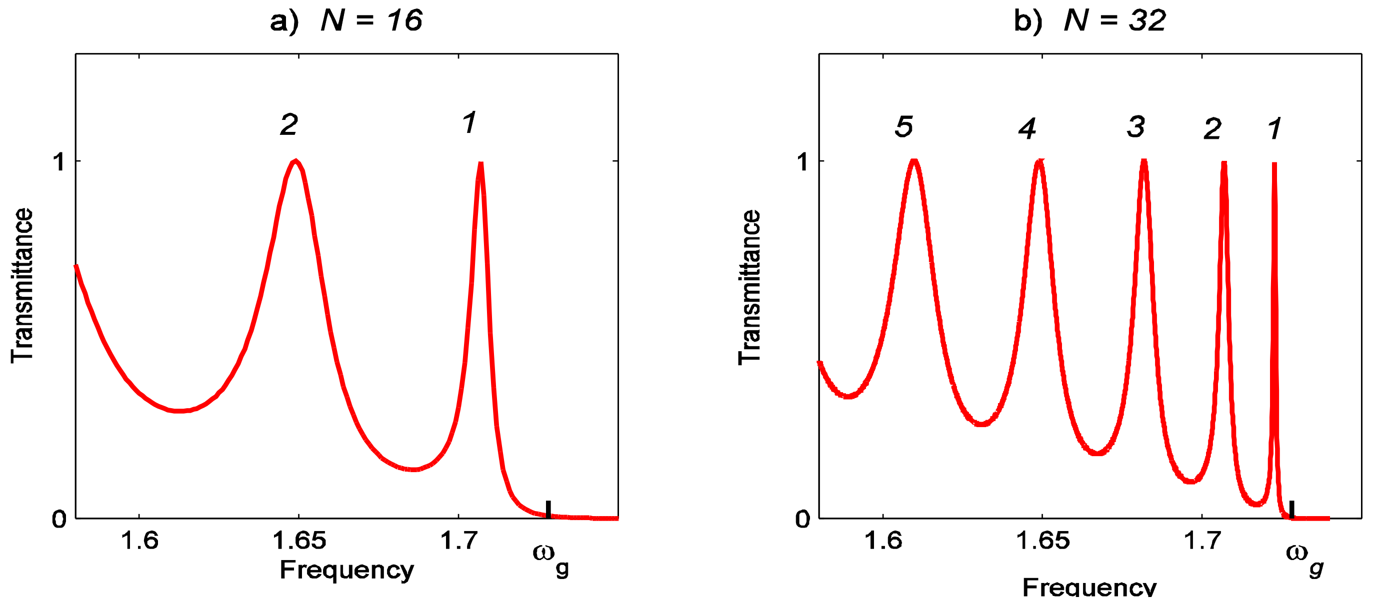
$$\omega - \omega_g \ \mu \left(k - k_g\right)^2$$

Periodic stack is composed of alternating layers *A* and *B*.

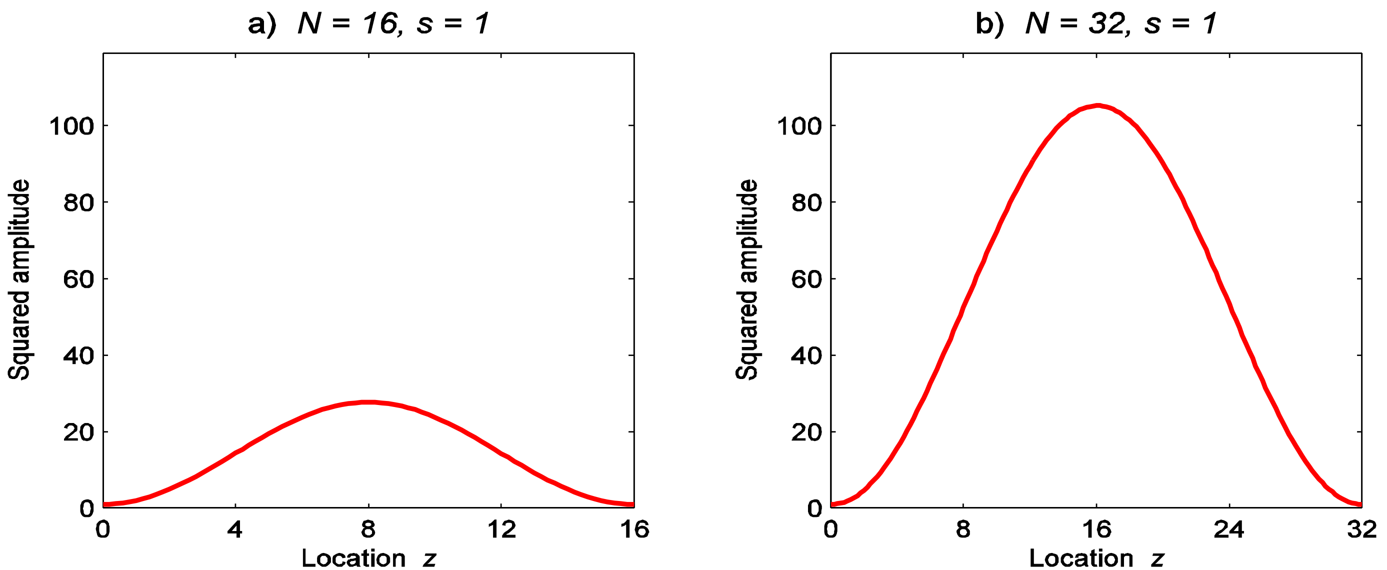
Resonant wave lengths: $\lambda_s \approx Ds/2, \quad s = 1, 2, 3, \ldots$

Resonant wave numbers: $k_s \approx k_g \pm \dfrac{\pi}{NL} s, \quad s = 1, 2, \ldots$

Transmission dispersion and resonant field inside the photonic slab: generic case of RBE

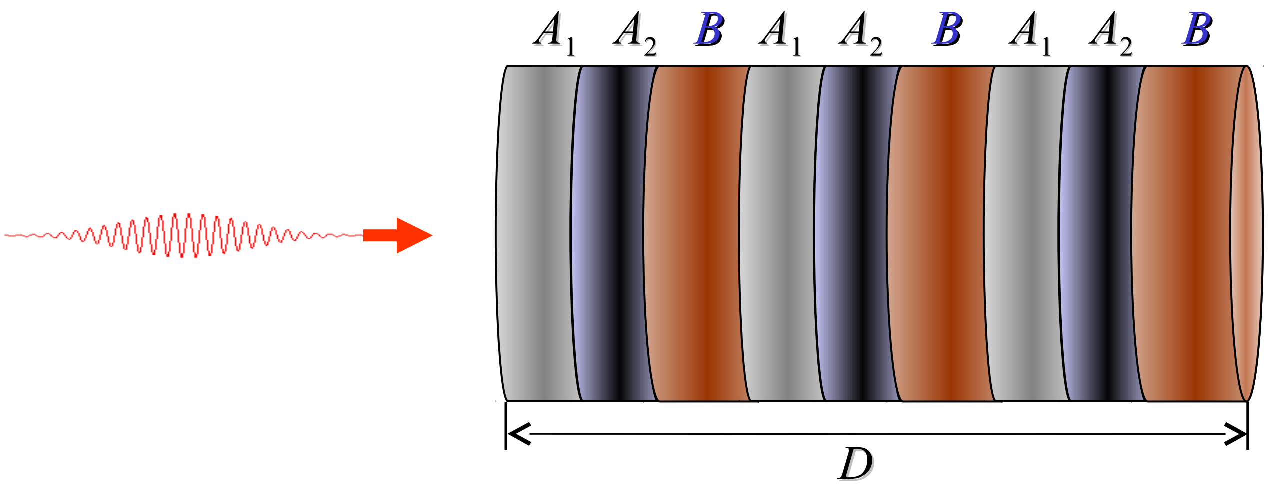


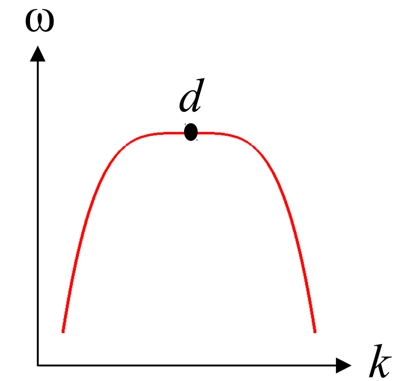
Stack transmission vs. frequency. ω_g is a regular photonic band edge



Smoothed field intensity distribution at the frequency of first transmission resonance

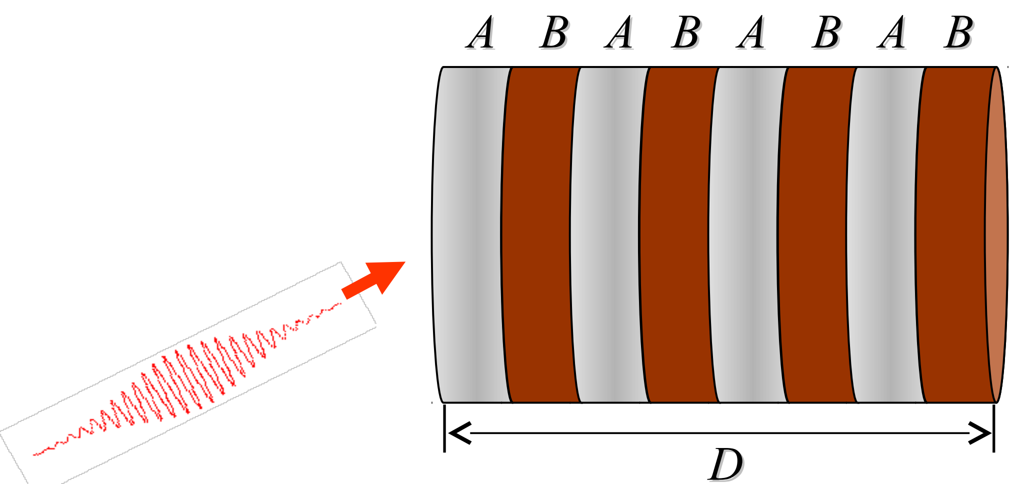
Giant transmission band edge resonance below DBE [6]





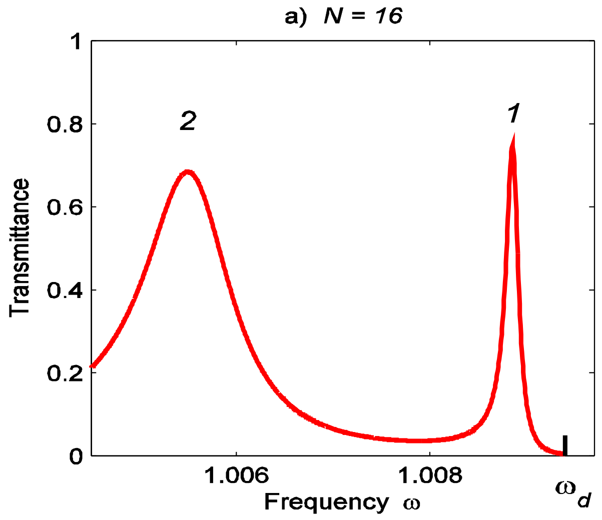
$$\omega-\omega_d \ \mu \left(k-k_d \right)^4$$

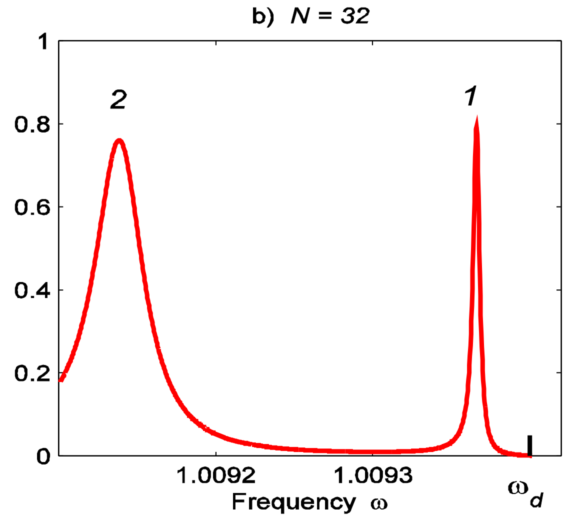
At normal incidence, a unit cell of the stack includes two misaligned anisotropic layers A_1 and A_2, and an isotropic B layer.



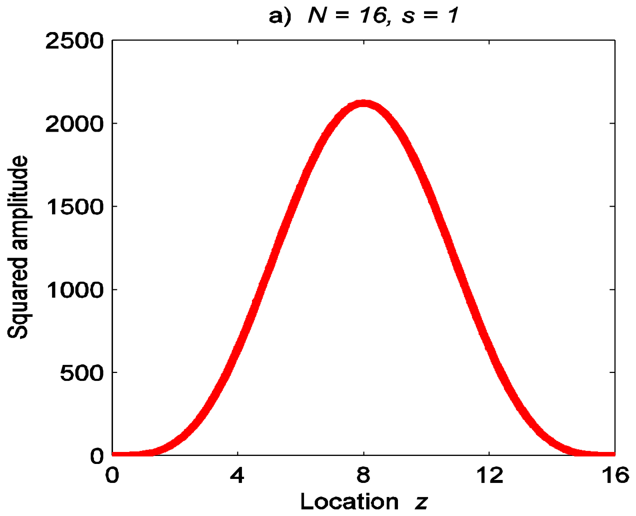
At oblique incidence, the periodic stack involves one anisotropic layer (A) and one isotropic layer (B).

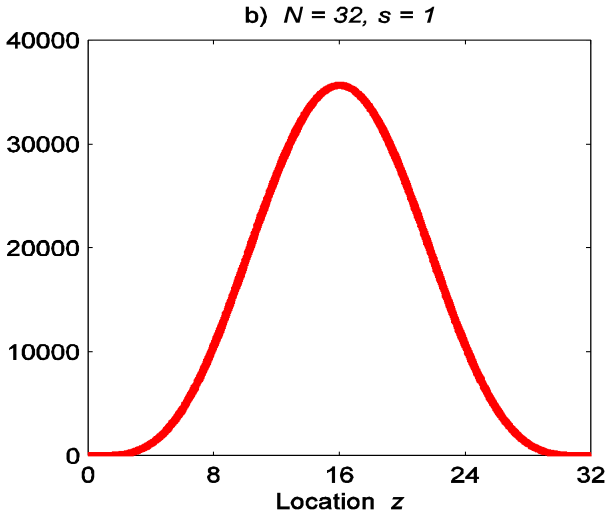
Transmission dispersion and resonant field inside the photonic slab: the case of a degenerate photonic band edge





Stack transmission vs. frequency. ω_d is a degenerate photonic band edge



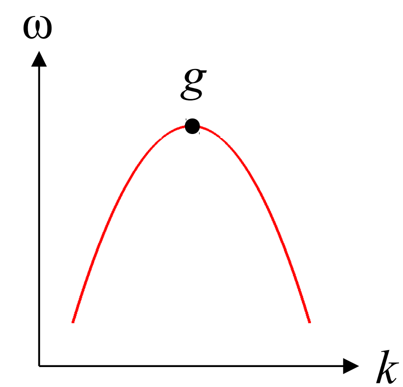


Field intensity distribution at the frequency of first transmission resonance

Regular transmission band-edge resonance vs. giant transmission band-edge resonance [6]

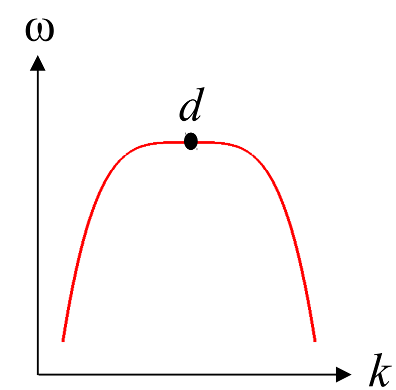
Regular band edge: $\omega \approx \omega_g - \frac{\omega_0''}{2}\left(k-k_g\right)^2$:

$$\max\left(W\right) \mu\ W_I \left(\frac{N}{s}\right)^2$$



Degenerate band edge: $\omega \approx \omega_d - \frac{\omega_0''''}{24}\left(k-k_d\right)^4$:

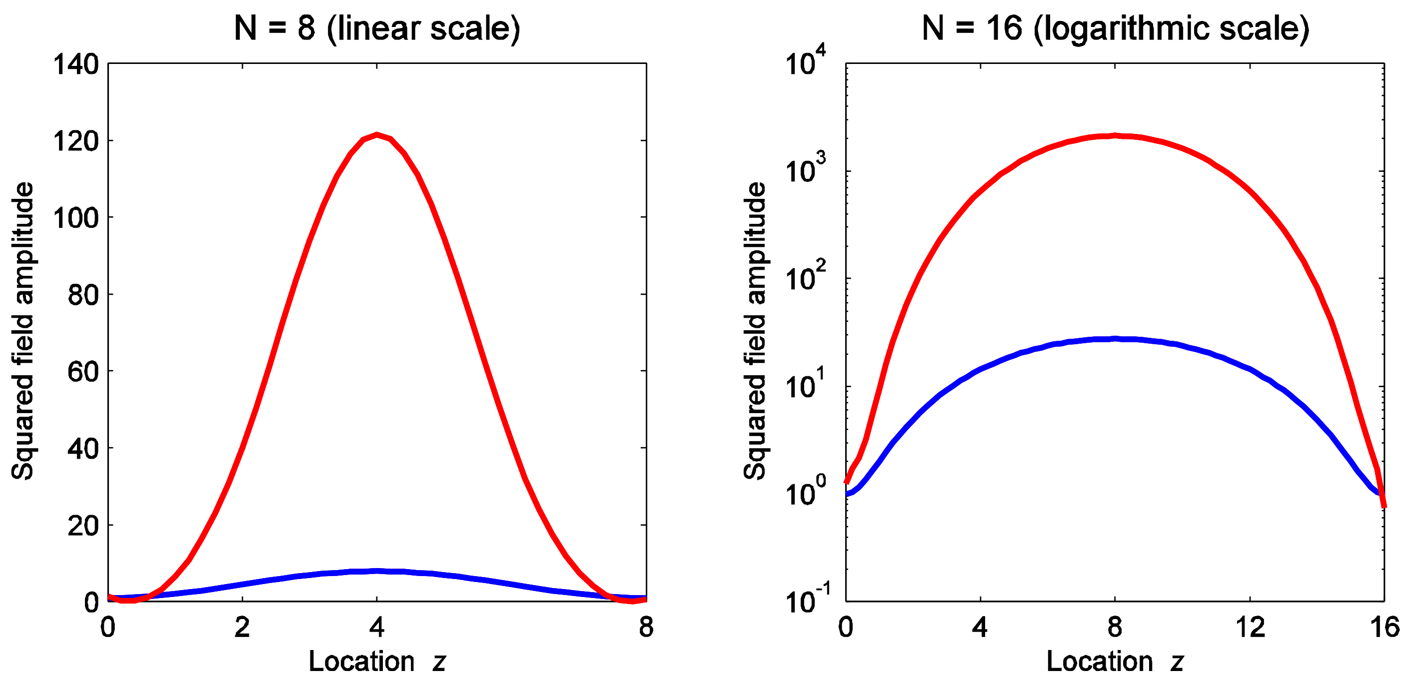
$$\max\left(W\right) \mu\ W_I \left(\frac{N}{s}\right)^4$$



A stack of 8 layers with a degenerate photonic BE performs as well as a stack of 64 layers with a regular photonic BE !

Regular Band Edge vs. Degenerate Band Edge.

Example: photonic slabs with $N = 8$ and $N = 16$



Electromagnetic energy density distribution inside photonic cavity at frequency of transmission band edge resonance

RBE: Regular photonic band edge (Energy density ~ N^2)

DBE: Degenerate photonic band edge (Energy density ~ N^4)

Additional advantages of the giant transmission resonance at oblique incidence [9]:

- The oblique effect can occur in much simpler periodic array involving just two alternating layers A and B. No misaligned or off-plane anisotropy is required.

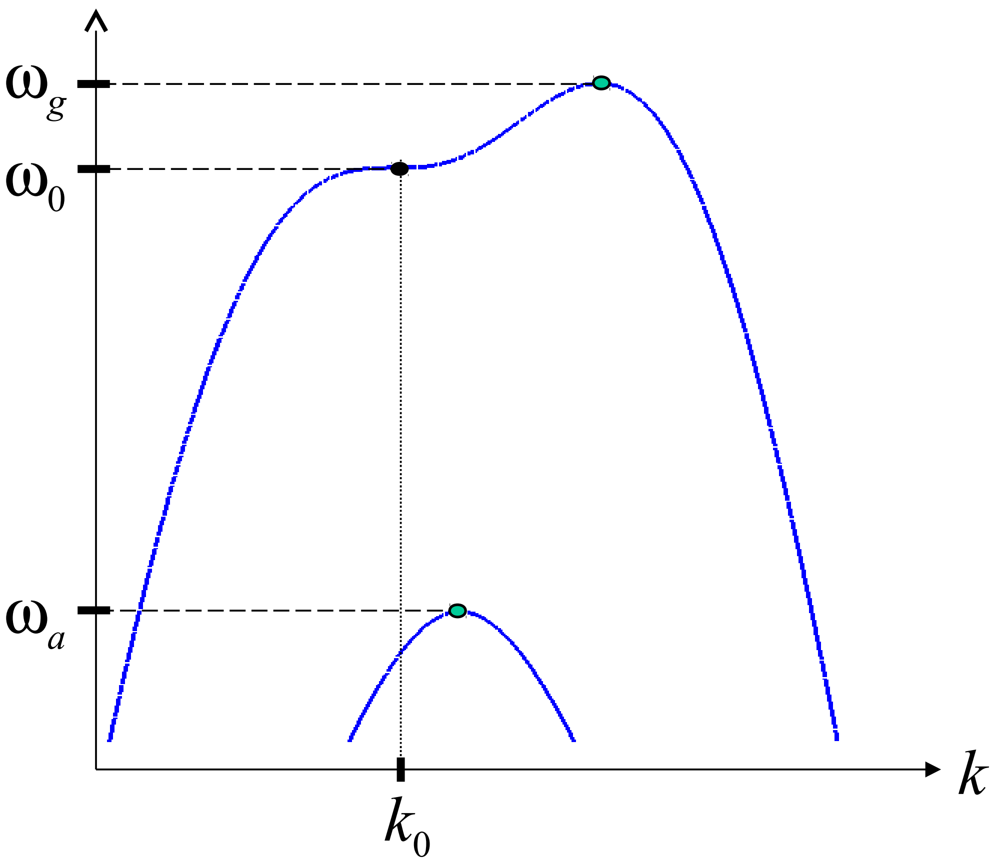
- The oblique resonance frequency can be changed continuously within a wide frequency range by adjusting the direction of incidence. There is no need to change any physical parameters of the periodic structure.

- Extreme directional selectivity can be utilized in transmitting and receiving antennas.

2. Frozen mode regime derived from the Maxwell equations

Eigenmodes composition at different frequencies

Dispersion relation $\omega(k)$



Regular frequencies:

$\omega < \omega_a$: 4 ex.

$\omega > \omega_g$: 4 ev. (gap)

$\omega_a < \omega < \omega_g$: 2 ex. + 2 ev.

Stationary points:

$\omega = \omega_a$: 3 ex. + 1 Floq.

$\omega = \omega_g$: 2 ev. + 1 ex. + 1 Floq.

$\omega = \omega_0$: 2 ex. + 2 Floq.

$\omega = \omega_d$: 4 Floq. (not shown)

Time-harmonic Maxwell equations in layered media

$$\nabla \times \vec{E}(\vec{r}) = \frac{i\omega}{c}\hat{\mu}(z,\omega)\vec{H}(\vec{r}); \quad \nabla \times \vec{H}(\vec{r}) = -\frac{i\omega}{c}\hat{\varepsilon}(z,\omega)\vec{E}(\vec{r})$$

In case of transverse electromagnetic waves propagating in the z direction, the time-harmonic Maxwell equations reduce to

$$\partial_z \Psi(z) = i\frac{\omega}{c}M(z,\omega)\Psi(z),$$

where

$$\Psi(z) = \begin{bmatrix} E_x(z) \\ E_y(z) \\ H_x(z) \\ H_y(z) \end{bmatrix}, \quad M = JA, \quad A^\dagger = A, \quad J = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = J^\dagger = J^{-1}$$

Propagating and evanescent eigenemodes in periodic layered media

At any given frequency ω, the reduced Maxwell equation

$$\partial_z \Psi(z) = i\frac{\omega}{c} M(z,\omega)\,\Psi(z), \;\; M(z+L) = M(z)$$

has four solutions which normally can be chosen in Bloch form:

$$\Psi_{k_i}(z+L) = e^{ik_i L}\Psi_{k_i}(z), \qquad i = 1,2,3,4$$

Every Bloch eigenmode is either extended or evanescent:

$\Psi_k(z)$ is extended if $\operatorname{Im} k = 0$,

$\Psi_k(z)$ is evanescent if $\operatorname{Im} k \neq 0$.

The dispersion relation:

$$\omega \leftrightarrow \{k_1, k_2, k_3, k_4\} \equiv \{k_1^*, k_2^*, k_3^*, k_4^*\}$$

Transfer matrix formalism

The reduced time-harmonic Maxwell equations in layered media

$$\partial_z \Psi(z) = i\frac{\omega}{c} M(z,\omega)\Psi(z);\ \ \Psi(z) = \begin{bmatrix} E_x(z) \\ E_y(z) \\ H_x(z) \\ H_y(z) \end{bmatrix},\ M^{\dagger} = JMJ$$

The respective Cauchy problem

$$\partial_z \Psi(z) = i\frac{\omega}{c} M(z,\omega)\Psi(z),\quad \Psi(z_0) = \Phi$$

has a unique solution

$$\boxed{\Psi(z) = T(z,z_0)\Psi(z_0)}$$

where $T(z,\ z_0)$ is the transfer matrix

$$T(z,z_0) = T(z,z')\,T(z',z_0),\qquad T(z,z_0) = T^{-1}(z_0,z)$$

The respective Cauchy problem for the transfer matrix is

$$\partial_z T(z, z_0) = i\frac{\omega}{c} M(z, \omega)\, T(z, z_0), \;\; T(z, z) = I$$

which implies that $T(z,\ z_0)$ is a $J-$ unitarity matrix

$$T^{\dagger} = J T^{-1} J$$

The transfer matrix T_S of an arbitrary stratified medium is

$$T_S = \prod_m T_m$$

Explicit expressions for the transfer matrices of individual homogeneous layers are known (they are very cumbersome)

$$T_m = T_m\left(\omega, \hat{\varepsilon}_m, \hat{\mu}_m, k_x, k_y\right)$$

The transfer matrix of a unit cell

$$T_L = \prod_m T_m$$

Bloch eigenmodes are the T_L eigenvectors

$$T_L \Psi_k(z) = \Psi_k(z+L) = \varsigma \Psi_k(z), \qquad \varsigma = e^{ikL}$$

The characteristic polynomial

$$P(\varsigma) = \det[T_L - \varsigma I] = \varsigma^4 + P_3\varsigma^3 + P_2\varsigma^2 + P_1\varsigma + 1 = 0$$

determines the dispersion relation:

$$\omega \leftrightarrow \{k_1, k_2, k_3, k_4\} \equiv \{k_1^*, k_2^*, k_3^*, k_4^*\}$$

Symmetric dispersion relation (if $T_L^{-1} = U^{-1} T_L U$):

for any ω, $\{-k_1, -k_2, -k_3, -k_4\} = \{k_1, k_2, k_3, k_4\}$

Jordan normal form of the transfer matrix T_L of the periodic layered array correspomding to each of the stationary points

$$\frac{\partial\omega}{\partial k} \neq 0: \quad \bar{T}_L = \begin{bmatrix} \varsigma_1 & 0 & 0 & 0 \\ 0 & \varsigma_2 & 0 & 0 \\ 0 & 0 & \varsigma_3 & 0 \\ 0 & 0 & 0 & \varsigma_4 \end{bmatrix}. \qquad \text{RBE:} \quad \bar{T}_L = \begin{bmatrix} \varsigma_1 & 0 & 0 & 0 \\ 0 & \varsigma_2 & 0 & 0 \\ 0 & 0 & \varsigma_a & 1 \\ 0 & 0 & 0 & \varsigma_a \end{bmatrix}$$

$$\text{SIP:} \quad \bar{T}_L = \begin{bmatrix} \varsigma_1 & 0 & 0 & 0 \\ 0 & \varsigma_0 & 1 & 0 \\ 0 & 0 & \varsigma_0 & 1 \\ 0 & 0 & 0 & \varsigma_0 \end{bmatrix}. \qquad \text{DBE:} \quad \bar{T}_L = \begin{bmatrix} \varsigma_0 & 1 & 0 & 0 \\ 0 & \varsigma_0 & 1 & 0 \\ 0 & 0 & \varsigma_0 & 1 \\ 0 & 0 & 0 & \varsigma_0 \end{bmatrix}$$

The eigenmodes at $\omega = \omega_0$

Consider a Bloch solution

$$\Psi_k(z) = e^{ikz}\varphi_k(z), \quad \varphi_k(z) = \varphi_k(z+L)$$

of the reduced Maxwell equation

$$\partial_z \Psi_k(z) = i\frac{\omega}{c} M(z,\omega)\Psi_k(z), \quad A(z+L) = A(z).$$

At the frozen mode frequency $\omega_0 = \omega(k_0)$ defined by

$$\partial_k \omega(k)_{k=k_0} = 0, \quad \partial_{kk}^2 \omega(k)_{k=k_0} = 0,$$

there are two extended Bloch solutions $\Psi_{k_1}(z)$ and $\Psi_{k_0}(z)$.

The other two solutions are related to the frozen mode $\Psi_{k_0}(z)$

$$\Psi_{01}(z) = \left(\frac{\partial}{\partial k}\Psi_k(z)\right)_{k=k_0}, \qquad \Psi_{02}(z) = \left(\frac{\partial^2}{\partial k^2}\Psi_k(z)\right)_{k=k_0}$$

or, explicitely:

$$\boxed{\begin{aligned}\Psi_{01}(z) &= \bar{\Psi}_{k_0}(z) + ik_0 z\Psi_{k_0}(z), \quad (\sim z)\\ \Psi_{02}(z) &= \bar{\bar{\Psi}}_{k_0}(z) + 2iz\bar{\Psi}_{k_0}(z) - z^2\Psi_{k_0}(z), \quad (\sim z^2)\end{aligned}}$$

where

$$\bar{\Psi}_{k_0}(z) = \left(e^{ikz}\frac{\partial}{\partial k}\varphi_k(z)\right)\Big|_{k=k_0}, \quad \bar{\bar{\Psi}}_{k_0}(z) = \left(e^{ikz}\frac{\partial^2}{\partial k^2}\varphi_k(z)\right)\Big|_{k=k_0}$$

are auxiliary Bloch functions (not eigenmodes !!!)

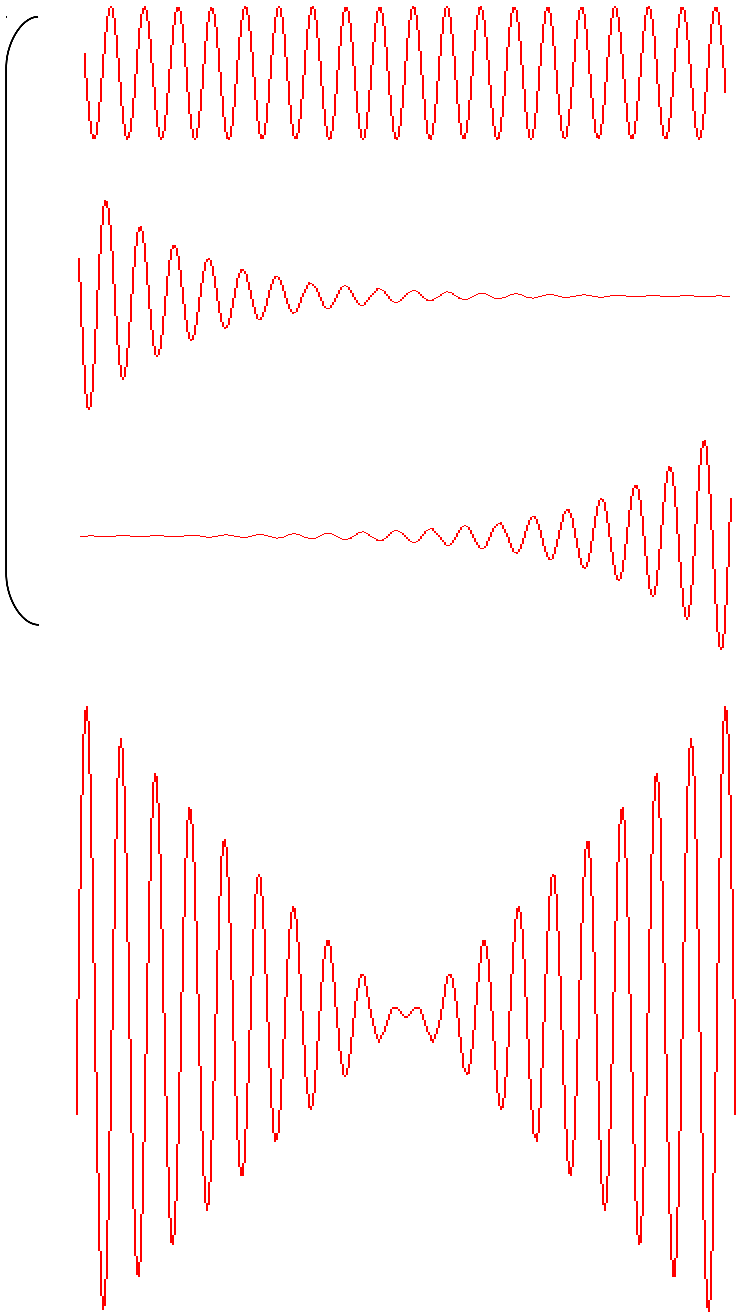
At $\omega = \omega_0$ there are only two (not four !!!) Bloch solutions:

1. $\Psi_{k_0}(z)$ – extended (frozen) mode with $k = k_0$ and $u = 0$
2. $\Psi_{k_1}(z)$ – extended mode with $k = k_1$ and $u < 0$

The other two solutions are the non-Bloch Floquet eigenmodes

$$\Psi_{01}(z) \propto z \quad \text{and} \quad \Psi_{02}(z) \sim z^2$$

Bloch eigenmodes



Extended mode: Im $k = 0$

Evanescent mode: Im $k > 0$

Evanescent mode: Im $k < 0$

Non-Bloch eigenmode

Floquet mode: $\Psi_{01}(z) \sim z$