Limiting Behavior of Networks

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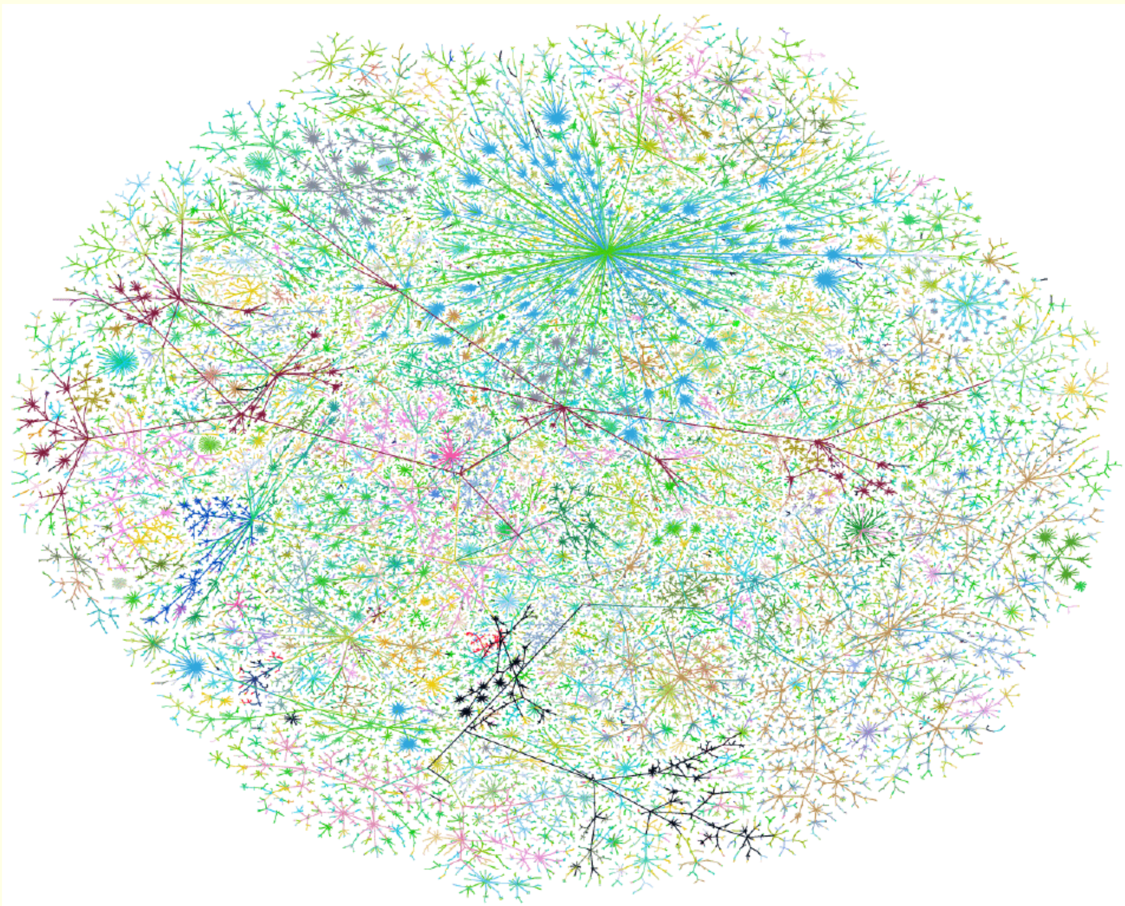
Internet Map (c 1999)



Maps

- http://plan9.bell-labs.com/who/ches/map/ (B. Cheswick)
- Traceroute-style probe to each registered entity.
- Builds a large rooted tree.
- Other mapping efforts:
  - Matrix.Net
  - The Cyber-Geography Research initiative http://www.cybergeography.org (M. Dodge)
- “Bowtie” structure of Web pages
  - A. Broder , R. Kumar, F. Maghoul, P. Raghavan, S. Rajagopalan, R. Stata, A. Tomkins, J. Wiener ($\sim$ 1999.)

Large Graphs

- Large graphs occur in nature
  - Communication networks are large
  - Many computational problems are modelled by large graphs
- Meanings of "large":
  - Size $\to \infty$;
  - Size $\cong \infty$
  - Size transfinite (A. H. Zemanian)
- Precedents:
  - Well-known analogies between graphs and manifolds: Laplacians, Cheeger constants.
  - M. Gromov. Combinatorial group theory.
  - L. Van-den-Dries and A. J. Wilkie ($\sim$ 1984.) Nonstandard proof of Gromov's theorem.

Goals of this Presentation

- Nonstandard mathematics provides useful and realistic abstraction for “large”.
- Show mathematical correspondence between large “nice” graphs and “nice” metric spaces.
  - *Geometrization*
- Motivate geometrization as a policy tool.
- Other technical applications (not discussed):
  - Expander graphs (F. Bien 1989, A. Lubotzky)

Outline

- Motivating example:
  - Network policies
- Vocabulary of Nonstandard Analysis
- Hyperfinite graphs.
- Taxicab geometry
- Polynomial growth
- Alfhors regularity and dimension.
- Conclusion.

Policy

- Enforcement and Management vs Policy
  - Policy: Books in reading area cannot leave reading area.
  - Enforcement: Install detectors at physical entrances.
- Policy for physical assets can be set by geometrical boundaries
- Network management via Routers and Filters may scale up to extremely large networks,
- Network policy needs "higher level" representation, such as
  - High level policy language
  - Geometrical representation
- Claim: It is easier to understand low-dimensional spatial relations than combinatorial ones.

High-Speed Course on Nonstandard Analysis

Intro

- "Problem" of Infinitesimals of historical interest,
- Satisfactory solution by A. Robinson ($\sim$ 1960)
- Based on "nonstandard models" of set theory.
  - Nonstandard means interpretation of $\mathbb{N}$ is nonstandard.
- Initial successes:
  - Invariant subspaces for polynomialy compact operators.
- E. Nelson introduced formalistic approach IST ($\sim$ 1977)
  - *The widely held belief that one cannot get something for nothing is a superstition*
- *Truth in advertising.* Field active, but marginal.

Application Areas

- Probability: Stochastic integrals, finance
- Lattice models in Physics
- Economics: Large numbers of consumers
- P.D.E's: Navier Stokes

Algebraic Structures

$$\mathbb{N} \subseteq {}^\star\mathbb{N}$$

$$\mathbb{Z} \subseteq {}^\star\mathbb{Z} \text{ (commutative ring)}$$

$$\mathbb{R} \subseteq {}^\star\mathbb{R} \text{ (ordered field)}$$

$$\mathbb{C} \subseteq {}^\star\mathbb{C} \text{ (algebraically complete field)}$$

Size

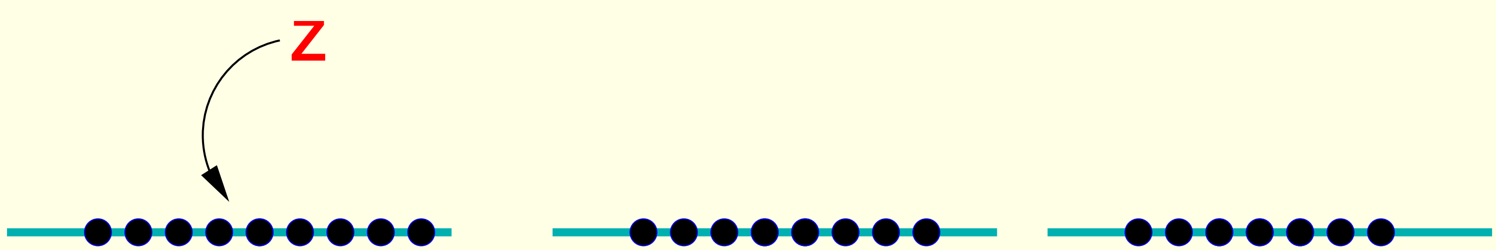
- Unlimited: $z \in {}^\star\mathbb{C}$ such that $|z|$ majorizes $\mathbb{N}$.
- Limited: $z \in {}^\star\mathbb{C}$ such that $|z| \leq n$ for some $n \in \mathbb{N}$.
- Infinitesimal: $z \in {}^\star\mathbb{C}$ such that $|z| \leq 1/n$ for all $n \in \mathbb{N}$.
- Standard part: $\text{st} : {}^\star\mathbb{C} \to \mathbb{S}_2$ (usual Riemann sphere)
- Fact: If z limited, $z - \text{st}(z)$ is infinitesimal.

Internal and Hyperfinite

- Internal sets:
  - $^\star\mathbb{N}$, $^\star\mathbb{Z}$, $^\star\mathbb{R}$ $^\star\mathbb{C}$
  - Closed balls $\{z \in {}^\star\mathbb{C} : |z| \leq r\}$,
  - Open balls $\{z \in {}^\star\mathbb{C} : |z| < r\}$,
  - Intervals $\{t \in {}^\star\mathbb{R} : a \leq t \leq b\}$.
- Internal functions:
  - $^\star\sin$, $^\star\cos$
  - $P(z) = \sum_{i=1}^{N} a_i z^i$,
    $\{a_i\}_{i=1}^{N}$ is an internal sequence.
- Hyperfinite set A:

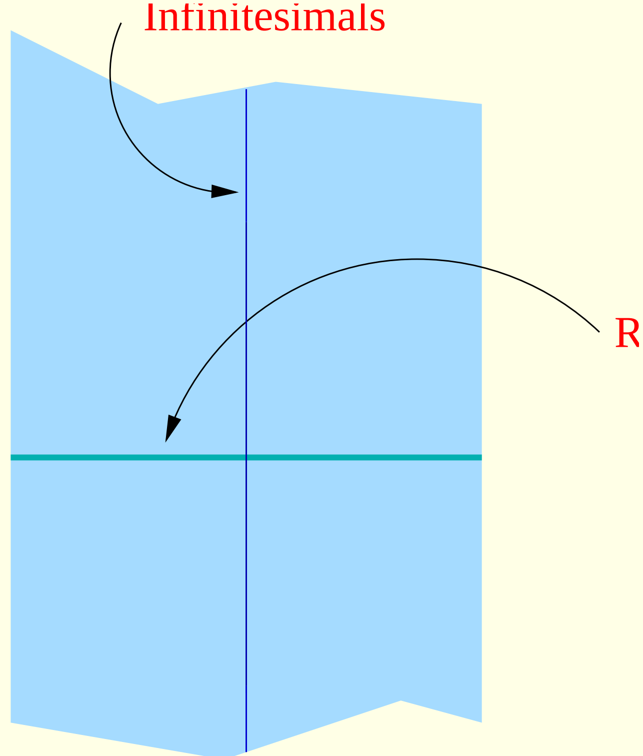
  $\exists N \in {}^\star\mathbb{N}$, $\exists$ internal bijection: $A \to \{1, \ldots, N\}$
- card $A = N$.

Nonstandard Integers



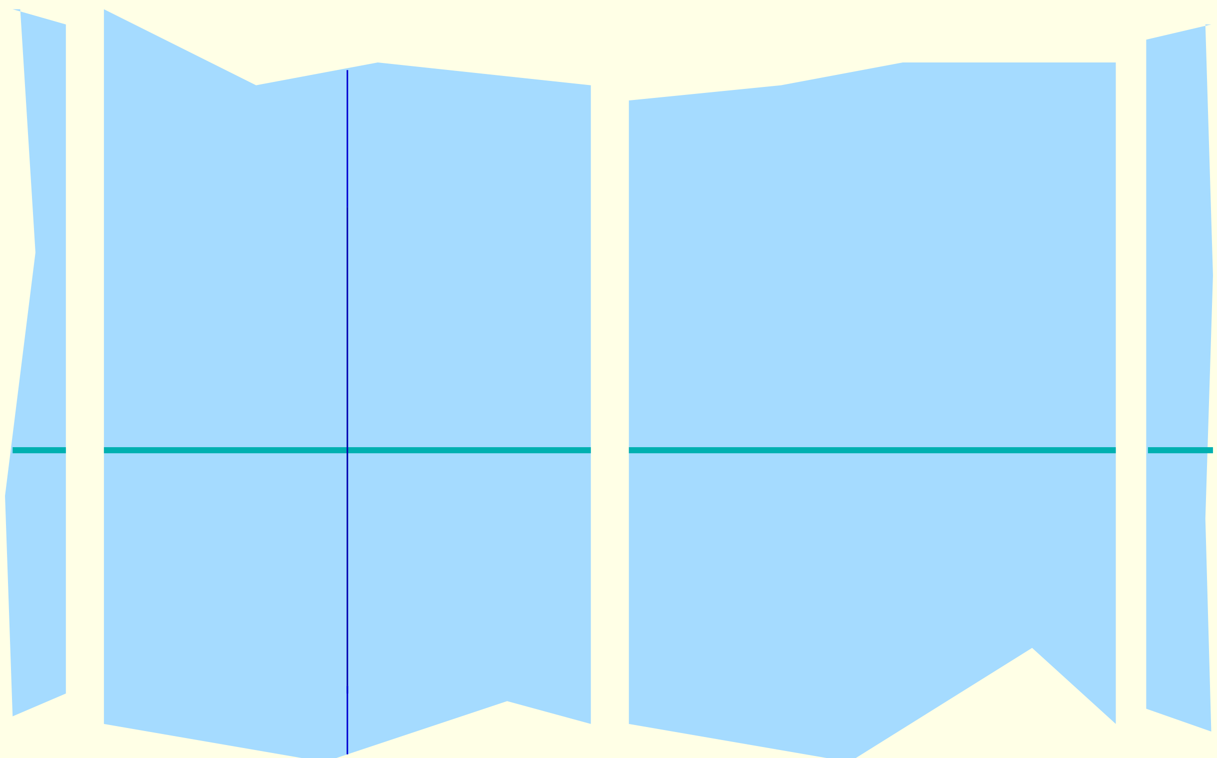
- ${}^\star\mathbb{Z}$ is disjoint union of *very* large number of copies of $\mathbb{Z}$.
- Principle of induction holds for *internal sets*

Bounded Hyperreals



- Standard part map is vertical projection.

Hyperreals



- Horizontal space is “long line”.

Graphs and Metric Spaces

Graphs

- Denote the adjacency between vertices x, y by $x \longleftrightarrow y$.

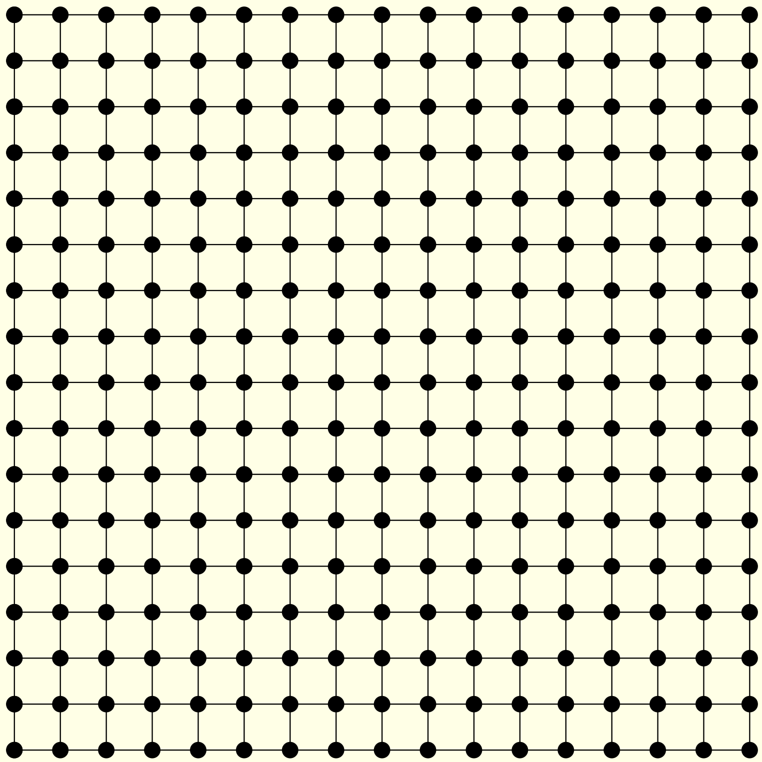
- Path: hyperfinite sequence of vertices $p = \langle a_0, a_1, a_2, \ldots, a_n \rangle$ of G such that $a_i \longleftrightarrow a_{i+1}$.

- Connected: Any two vertices are linked by a (hyperfinite) path.

- Graph distance: $\text{dist}(x, y) =$ length of shortest path from x to y.
  - Any internal set of ${}^\star\mathbb{Z}$ has least element!

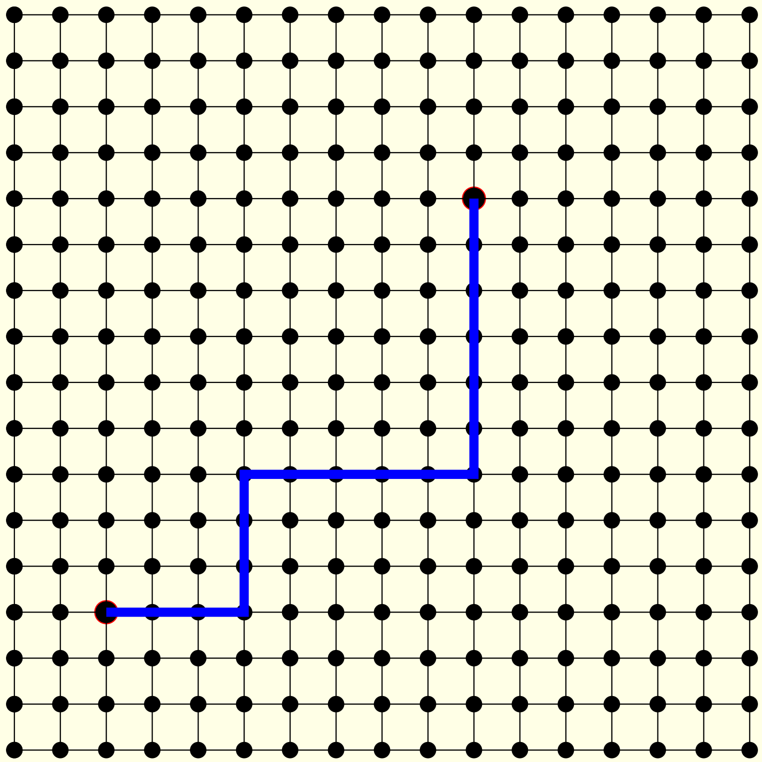
- <u>Fact</u>: $\text{dist}(x, y)$ has values in ${}^\star\mathbb{Z}^+$.

Example: Taxicab Metric

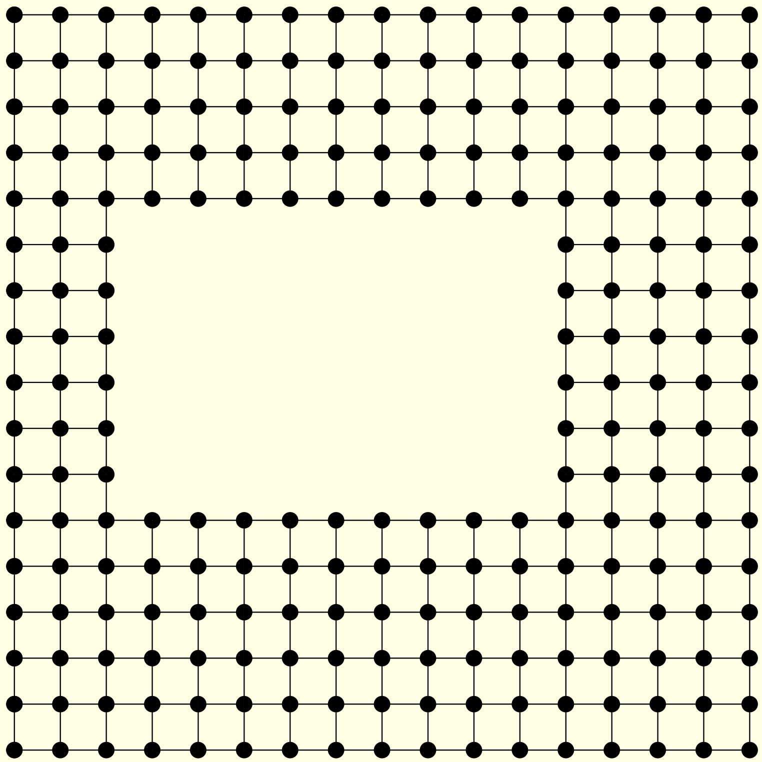


- Distance is length of shortest path along grid.

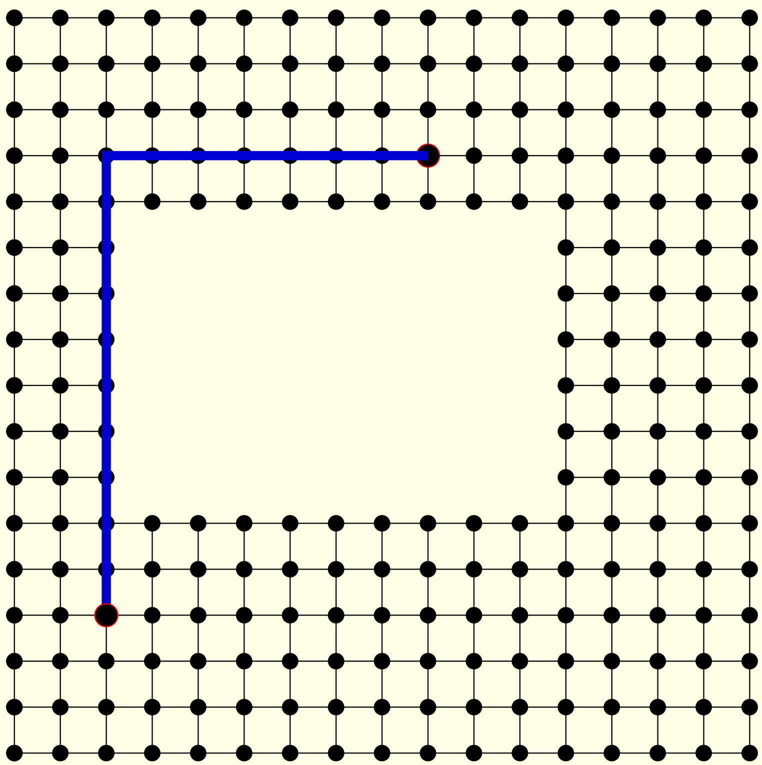
Shortest Path



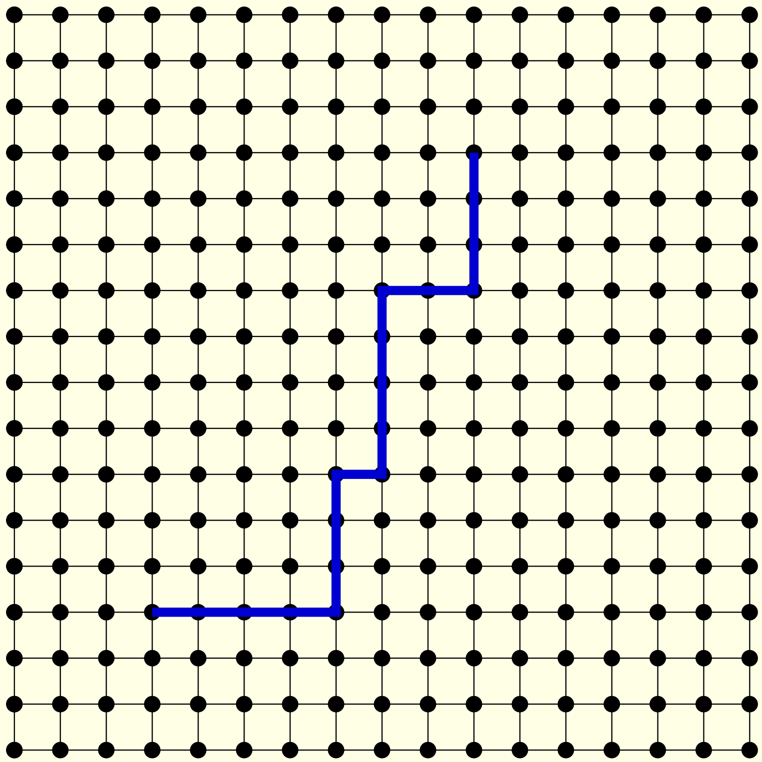
Example: Taxicab Metric



Shortest Path



Rescaling



- Change unit of distance to h.
- New metric $\mathrm{dist}_h(x,y) = h\ \mathrm{dist}(x,y)$.

Rescaling

- Qualitative behavior of (G, dist_h) changes when h infinitesimal.
- $\mathrm{card}\, G \in \mathbb{N} \implies \mathrm{diam}(G) \cong 0$.
  - Sum of two infinitesimals is infinitesimal!
- If $\mathrm{card}\, G \in {}^\star\mathbb{N} \setminus \mathbb{N}$, then graph may display nondiscrete behavior.

Infinitesimal Hull

$(\mathfrak{X}, d)$ an *internal* metric space.

- Infinite proximity: $x \cong y$ iff $d(x, y)$ is infinitesimal.
- Fact: "$\cong$" is equivalence relation.
- Infinitesimal Hull of $\mathfrak{X}$:

$$\diamond\mathfrak{X} := \mathfrak{X}/\cong$$

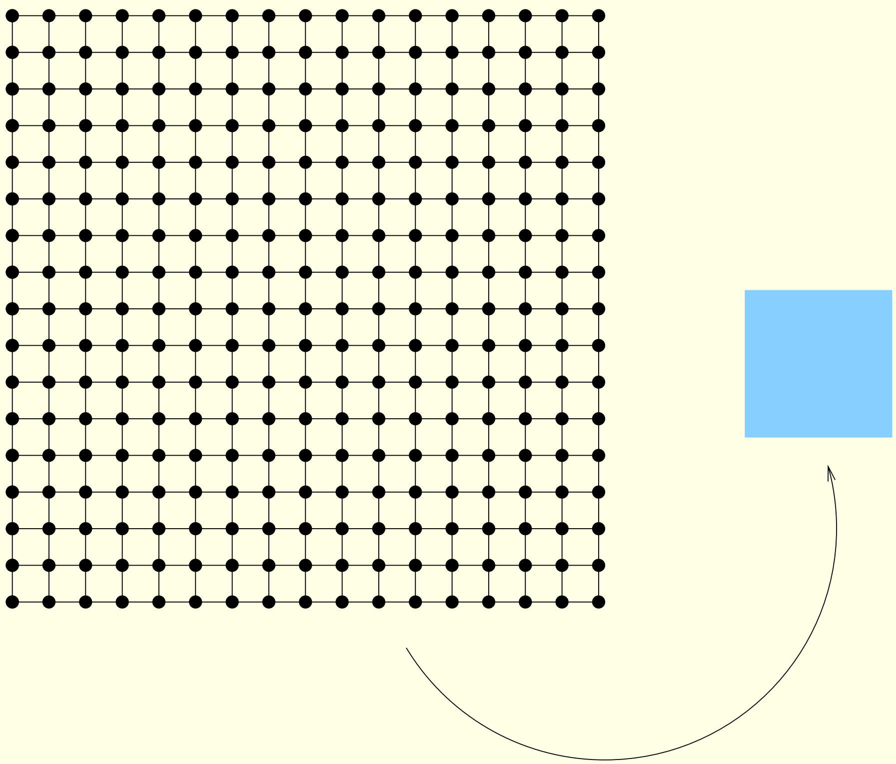
- Example: Assume

  N is unlimited,

  $\mathfrak{X} = \{0, 1/N, 2/N, \ldots, N-1/N, 1\}$.

  Then: $\diamond\mathfrak{X} \equiv [0, 1]$.

Example



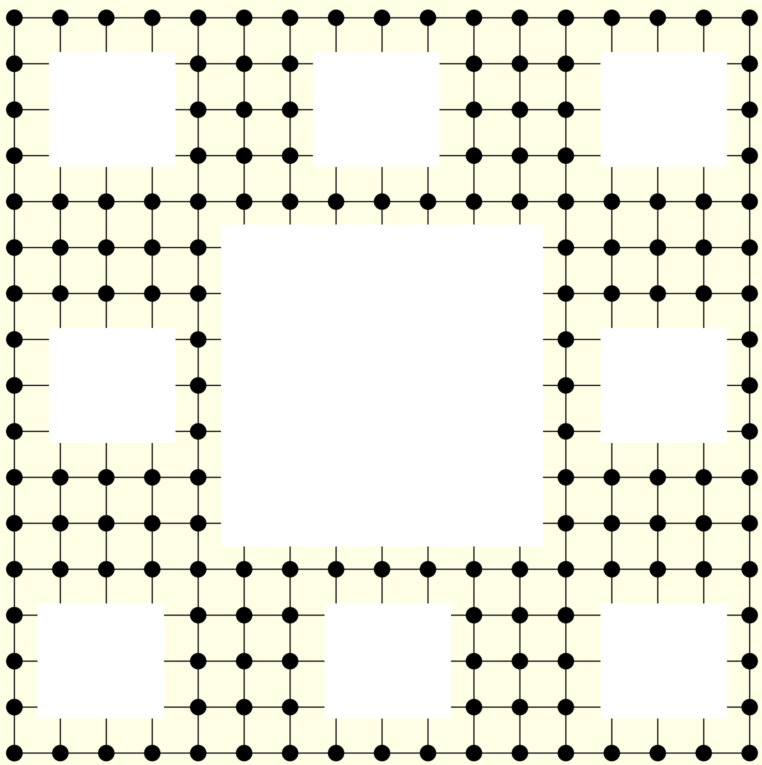
- $N \times N$ grid. N unlimited!
- Scaling parameter $h = \frac{1}{N}$.
- Blue square is $[0, 1] \times [0, 1]$ — *Taxicab metric*

Example

What happens if:

- N is limited?
- Scaling parameter h is infinitesimal but $h \times N$ is unlimited?
- Grid has middle third squares removed an unlimited number of times?

Iteration



Example

- N is limited?

  Answer: Hull is singleton

- Scaling parameter h is infinitesimal but $h \times N$ is unlimited?

  Answer: Disjoint union of *very* large number of copies of $\mathbb{R}^2$.

- Grid has middle third squares removed an unlimited number of times?

  Answer: Sierpinski Carpet.

Cayley Graphs

- Previous naive grid example can be generalized
- Cayley(G, V): $x \longleftrightarrow y$ iff $x^{-1}y \in V$.
- Rectangular grids are Cayley graphs of hyperfinite abelian groups.
- Nilpotent groups afford interesting examples.

Length Spaces

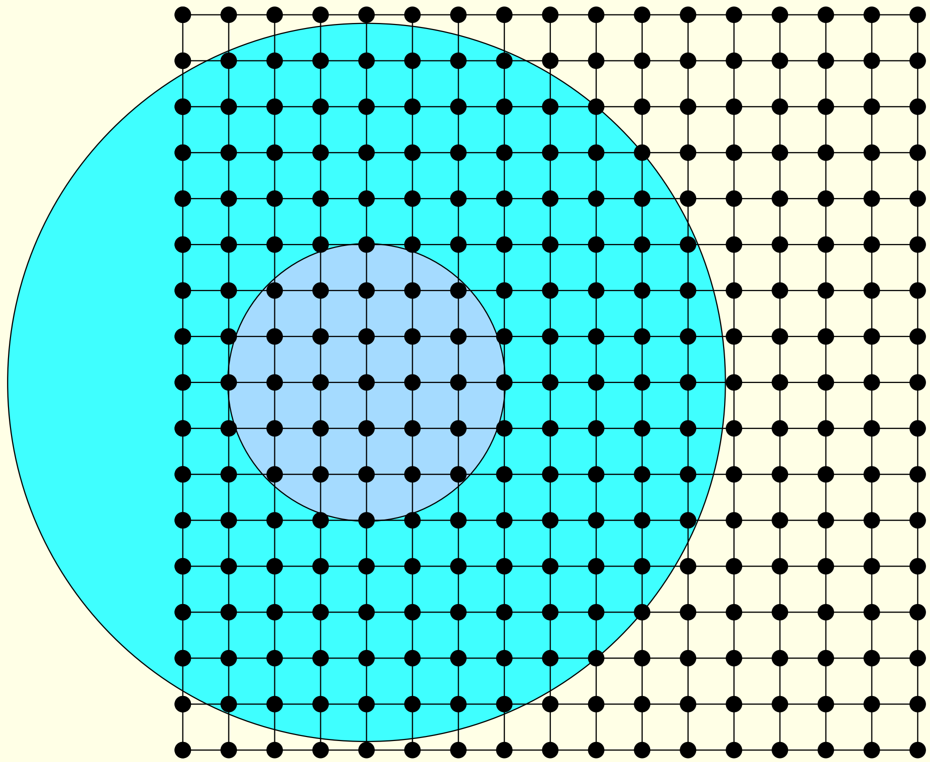
- Consider $\diamond(\mathfrak{X}, d_h)$ for rescaled graph spaces. e.g,, $d_h = h \times$ dist.

- Fact: If $h \cong 0$, $\diamond(\mathfrak{X}, d_h)$ is *length space*: For every $x, y \in \diamond(\mathfrak{X}, d_h)$,

$$\exists \text{ isometric map } f : [0, d(x,y)] \to X$$

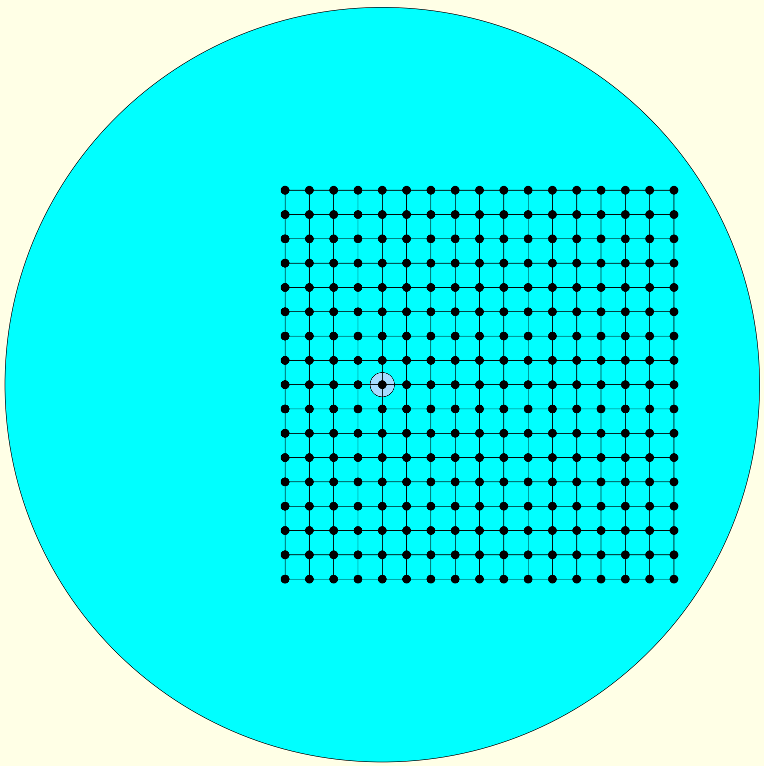
$$\text{with} \begin{cases} f(0) = x \\ f(d(x,y)) = y \end{cases}$$

Counting Grid Points



- Within bounds of R , # gridpoints $\sim R^2$.

Counting Grid Points



- R too large, $\frac{\#\text{ gridpoints}}{R^2} \to 0$.
- R too small $\frac{\#\text{ gridpoints}}{R^2} \to \infty$.

Polynomial Growth

$(\mathfrak{X}, d)$ is an internal metric space.

- Polynomial growth in interval $[m, M]$:

$$c \leq \frac{\text{card}\,\mathsf{B}(x,r)}{r^{\lambda}} \leq C \quad m \leq r \leq M$$

where $\begin{cases} c & \text{non-infinitesimal} \\ C & \text{limited} \end{cases}$

- Fact: If $(\mathfrak{X}, d)$ is of polynomial growth λ in $[m, M]$, then $(\mathfrak{X}, d_h)$ is of polynomial growth λ in $[h\,m, h\,M]$
- Renormalize d so that $M = 1$.

Measure

Assume $(\mathfrak{X}, d)$ is has polynomial growth in $[m, 1]$

- Consider the set function

$$\nu(A) = \frac{\text{card}\, A}{M^\lambda} \in {}^\star\mathbb{R}$$

  defined for internal $A \subseteq \mathfrak{X}$.

- <u>Facts</u>:
  - ν may have nonstandard values (e.g. $\in {}^\star\mathbb{R} \setminus \mathbb{R}$).
  - ν is finitely additive
  - There is an $\mathbb{R}$-valued *countably additive* measure μ such that $\mu(A) = \text{st}\, \nu(A)$. P. A. Loeb ($\sim$ 1975)
  - μ is defined on some σ-algebra of subsets of $\mathfrak{X}$

Measure on $\diamond\,\mathfrak{X}$

- For $m \leq r \leq 1$, $c\,r^\lambda \leq \mu(\mathrm{B}(x,r)) \leq C\,r^\lambda$
- μ pushes down to a *Borel* measure $\mathrm{P}(\mu)$ on the metric space $\diamond\,\mathfrak{X}$ with a similar estimate.
- Spaces with this property are *Ahlfors regular*.
- The measure $\mathrm{P}(\mu)$ is "Boundedly equivalent" to λ-dimensional Hausdorff measure.
- Converse is also true. Any Ahlfors regular space is covered by a graph space of polynomial growth,

Ahlfors Regularity, Dimension

Suppose (X, d) is a complete metric space and ν a Borel measure on X. If there are real numbers $R > 0$ and $\lambda > 0$ such that for $0 < r < R$

$$k\, r^{\lambda} \le \nu\big(\mathsf{B}(x, r)\big) \le K\, r^{\lambda}$$

then all closed balls in X of radius $< R$ are compact. Moreover, μ is boundedly equivalent to Hausdorff measure.

Proposition

Suppose (X, d) is an Ahlfors regular length space of dimension λ. Then there is a hyperfinite graph space $(\mathfrak{X}, d_{\mathfrak{X}})$, a Lispchitz equivalent metric on X and a hyperreal R_0 such that the following hold:

1. (X, d) is a bounded component of $\diamond(\mathfrak{X}, d_{\mathfrak{X}})$.
2. For some uniform hypermeasure ν on $\mathfrak{X}$, $\mathsf{P}_{\mathfrak{X}}(\nu)|_X = \mathcal{H}^\lambda$.
3. $(\mathfrak{X}, d_{\mathfrak{X}})$ is of polynomial growth λ on $[m, R_0]$ where $m \cong 0$ and $0 \ll R_0 \ll \infty$.

Mathematical Conclusion

- There is a “functor” from nice graph spaces to nice metric spaces.
- Functor is nearly surjective.
- Measure preserving automorphisms lift to bijections.

Conclusion

- Geometric view of graphs is a theoretical tool.
  - Use of Nonstandard Analysis can make heuristics rigorous.
- Suggested geometrization technique;
  - *Not necessarily the right one!*
- Geometric languages for network policy undeveloped.
- Policy statements currently
  - High-level and hard to map down to implementation,
  - Implementation oriented and hard to use in business strategy.