An Introduction to the Mathematics of Tomography

Todd Quinto

Tufts University

Todd.Quinto@tufts.edu
http://www.tufts.edu/∼equinto

References:

CD E. Candès and D. Donoho, Proc. SPIE 4119 (2000).

FFRS A. Faridani, E.L. Ritman, and K.T. Smith, *SIAM J. Appl. Math.* **52**(1992), +Finch II: **57**(1997).

KLM P. Kuchment, P. Lancaster, M. Mogilevskaya, *Inverse Problems* **11**(1995).

LR A. Louis and A. Rieder, *Numer. Math.* **56**(1989).

Na Frank Natterer, *The Mathematics of Computerized Tomography*, (SIAM 2001).

NaW Frank Natterer, Frank Wuebbling, *Mathematical Methods in Image Reconstruction*, SIAM, 2001.

NAC *Mathematics and Physics of emerging biomedical Imaging* National Academy Press, 1996

OQ G. Ólafsson and E.T. Quinto, editors, *Lecture Notes in Appl. Math.*, Vol 63, 2006.

Q1 E.T. Quinto, *Inverse Problems* **Medical: 4**(1988); **Industrial: 14**(1998).

Q2 E.T. Quinto, *SIAM J. Math. Anal.* **24**(1993).

Q3 E.T. Quinto, *J. Computational and Applied Math.*, 2006.

RK A. Ramm and A. Katsevich *Pseudolocal Tomography*, CRC Press, 1996.

Outline:

- X-ray tomography and the Radon transform
- Limited Data Tomography and Lambda CT
- Singularity detection in tomography

X-ray Computed Tomography (CT)

$f(x) =$ density of the cross-section at $x \in \mathbb{R}^2$ (slide 1).

X-ray Computed Tomography (CT)

$f(x) =$ density of the cross-section at $x \in \mathbb{R}^2$ (slide 1).

$L =$ the line of X-rays, $I(x) =$ the intensity of X-rays at $x \in L$.

X-ray Computed Tomography (CT)

$f(x) =$ density of the cross-section at $x \in \mathbb{R}^2$ (slide 1).

$L =$ the line of X-rays, $I(x) =$ the intensity of X-rays at $x \in L$.

Physics: $I(x)$ is decreased proportional to $f(x)$:

$$dI/dx = -f(x)I(x) \qquad dI/I = -f(x)dx.$$

X-ray Computed Tomography (CT)

$f(x) =$ density of the cross-section at $x \in \mathbb{R}^2$ (slide 1).

$L =$ the line of X-rays, $I(x) =$ the intensity of X-rays at $x \in L$.

Physics: $I(x)$ is decreased proportional to $f(x)$:

$$dI/dx = -f(x)I(x) \qquad dI/I = -f(x)dx.$$

Measured data: $\dfrac{I_{in}}{I_{out}} = e^{Rf(L)}$

X-ray Computed Tomography (CT)

$f(x) =$ density of the cross-section at $x \in \mathbb{R}^2$ (slide 1).

$L =$ the line of X-rays, $I(x) =$ the intensity of X-rays at $x \in L$.

Physics: $I(x)$ is decreased proportional to $f(x)$:

$$dI/dx = -f(x)I(x) \qquad dI/I = -f(x)dx.$$

$$\text{Measured data: } \frac{I_{in}}{I_{out}} = e^{Rf(L)}$$

where the total attenuation along L:

$$Rf(L) := \int_{x \in L} f(x)ds$$

— the total “material” along L — The **Radon Transform** of f on L (pure math: [Radon 1917]).

The Goal of CT:

Recover an approximation to $f(x)$ from X-ray CT data over a finite number of lines.

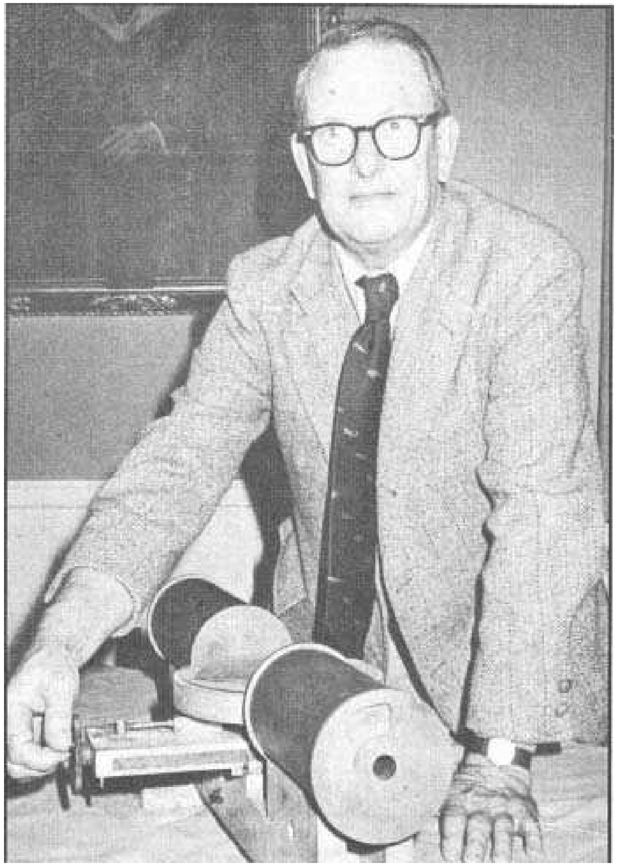
The Goal of CT:

Recover an approximation to $f(x)$ from X-ray CT data over a finite number of lines.

With uniformly distributed data (lines throughout the object with fairly evenly spaced angles), good, stable reconstruction methods exist, such as Filtered Backprojection [Na, NaW] (Faridani).

More CT Scanners:

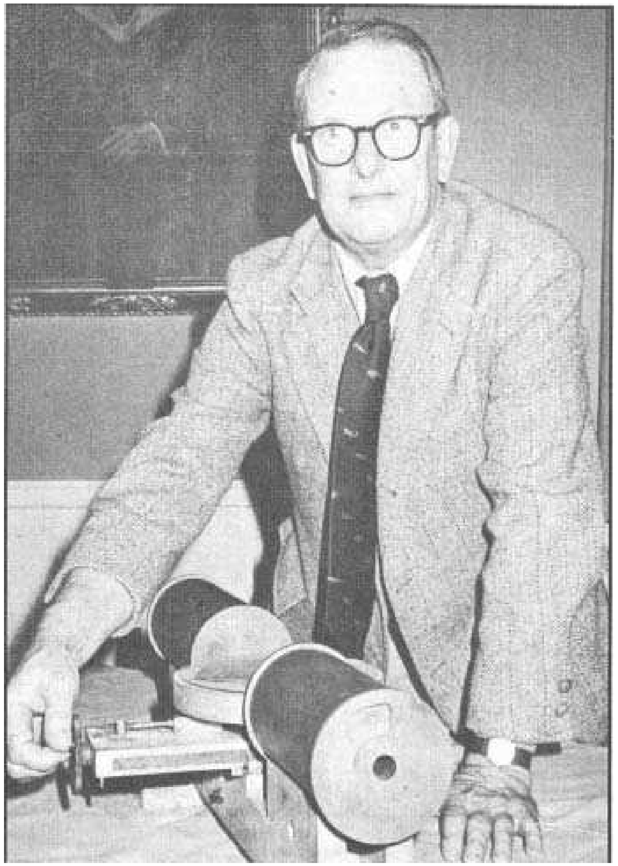
Cormack's Scanner:



Cost: $300

More CT Scanners:

Cormack's Scanner:



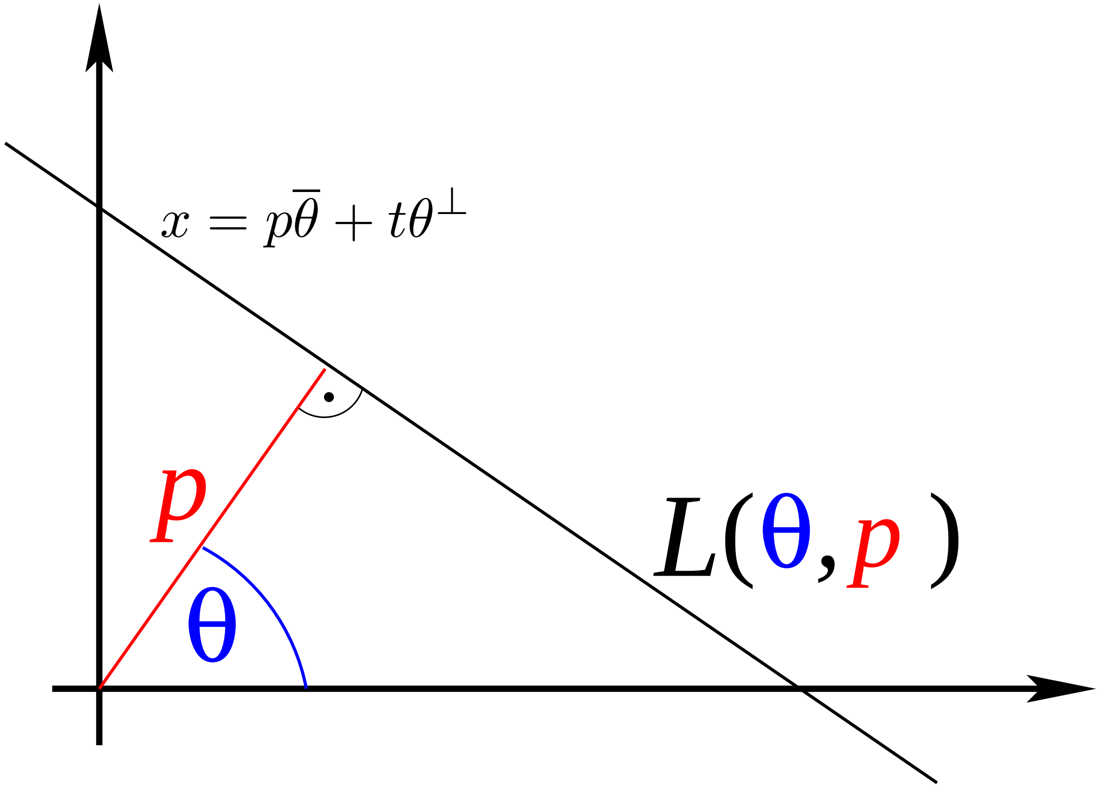
Cost: $300

Modern GE Scanner (©GE):

Cost: $1,500,000

Parallel beam parameterization of line

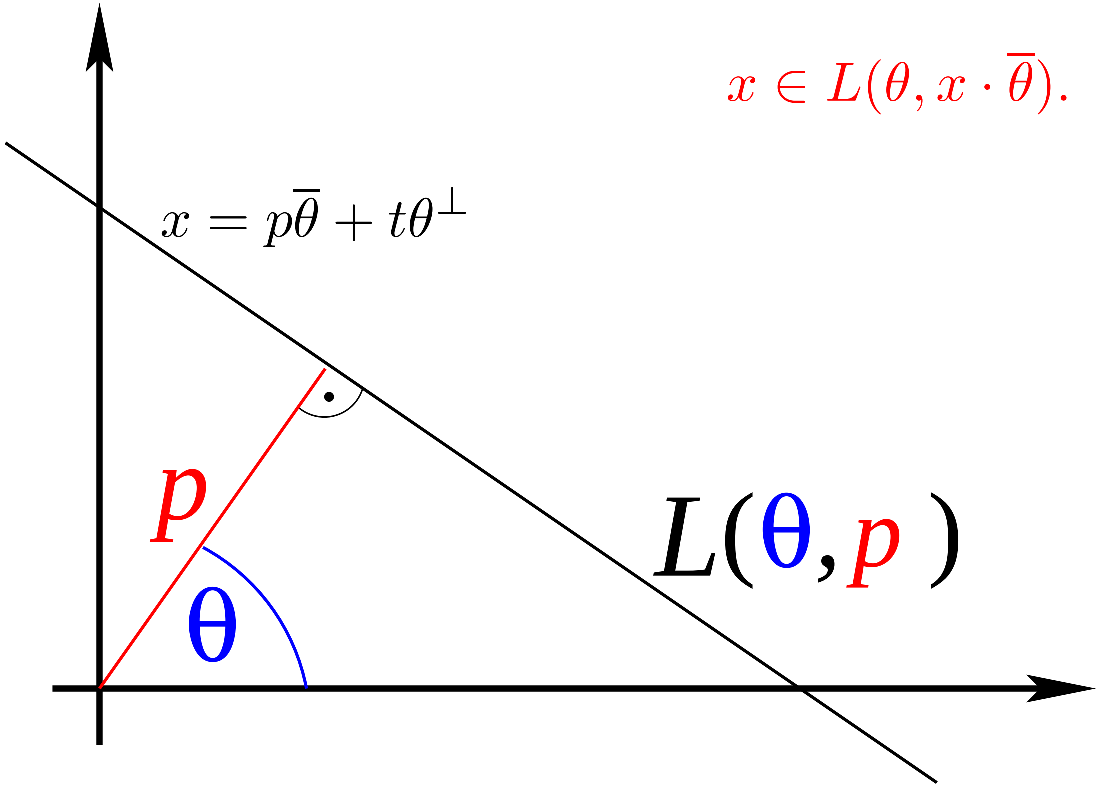
$$L(\theta, p) = \{x \in \mathbb{R}^2 \,|\, x \cdot \overline{\theta} = p\} \qquad \overline{\theta} = (\cos\theta, \sin\theta) \quad \theta^\perp = (-\sin\theta, \cos\theta).$$



Identity for points and lines:

$$L(\theta, p) = \{x \in \mathbb{R}^2 \big| x \cdot \overline{\theta} = p\} \qquad \overline{\theta} = (\cos\theta, \sin\theta) \quad \theta^\perp = (-\sin\theta, \cos\theta).$$

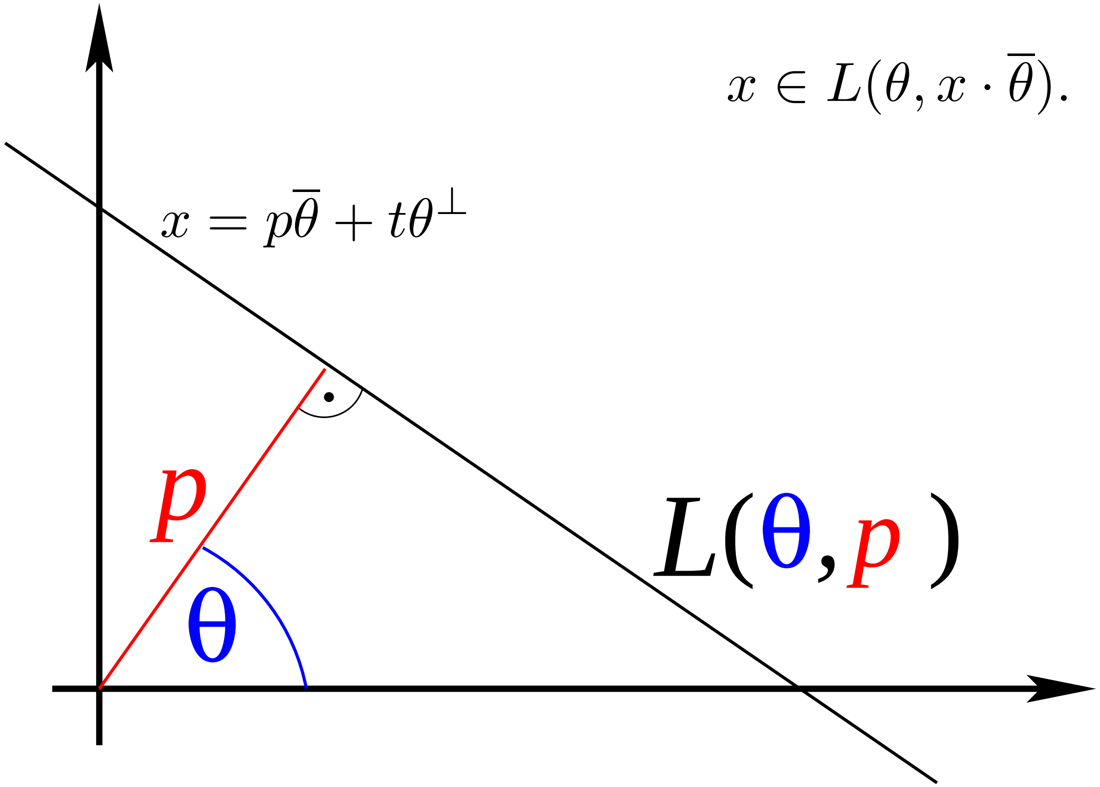
$$x \in L(\theta, x \cdot \overline{\theta}).$$



Parameterization of Radon Transform

$$L(\theta,p)=\{x\in\mathbb{R}^2\,|\,x\cdot\overline{\theta}=p\}\qquad \overline{\theta}=(\cos\theta,\sin\theta)\quad \theta^{\perp}=(-\sin\theta,\cos\theta).$$

$$x\in L(\theta,x\cdot\overline{\theta}).$$



$$Rf(\theta,p)=\int_{x\in L(\theta,p)} f(x)ds=\int_{t=-\infty}^{\infty} f(p\overline{\theta}+t\theta^{\perp})dt.$$

Transforms:

2-D Fourier Transform: $f \in L^1(\mathbb{R}^2)$

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = \frac{1}{2\pi}\int_{x\in\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\,dx\,.$$

$$\mathcal{F}^{-1}f(x) = \mathcal{F}f(-x)$$

Transforms:

2-D Fourier Transform: $f \in L^1(\mathbb{R}^2)$

$$\widehat{f}(\xi) = \mathcal{F}f(\xi) = \frac{1}{2\pi}\int_{x\in\mathbb{R}^2} e^{-ix\cdot\xi} f(x)\,dx\,.$$

$$\mathcal{F}^{-1}f(x) = \mathcal{F}f(-x)$$

1-D Fourier Transform: $g \in L^1([0,2\pi]\times\mathbb{R})$

$$\mathcal{F}_1 g(\theta,\tau) = \frac{1}{\sqrt{2\pi}}\int_{p=-\infty}^{\infty} e^{-i\tau p} g(\theta,p)\,dp\,.$$

Projection Slice Theorem:

Projection Slice Theorem:

$$\sqrt{2\pi}\,\widehat{f}(\tau\overline{\theta}) = \mathcal{F}_1 Rf(\theta,\tau)\,. \tag{1}$$

Projection Slice Theorem:

Projection Slice Theorem:

$$\sqrt{2\pi}\,\widehat{f}(\tau\overline{\theta}) = \mathcal{F}_1 Rf(\theta,\tau)\,. \qquad (1)$$

Therefore, R is injective on domain $L^1(\mathbb{R}^2)$.

Proof: Fubini's Theorem!

$\overline{\theta} = (\cos\theta, \sin\theta), \quad \theta^{\perp} = (-\sin\theta, \cos\theta)$. Slide 3

First calculate $\widehat{f}(\tau\overline{\theta})$, $\tau \in \mathbb{R}$, by integrating over lines *perpendicular* to $\overline{\theta}$, then in the $\overline{\theta}$ direction, $x = p\overline{\theta} + t\theta^{\perp}$:

$$2\pi\widehat{f}(\tau\overline{\theta}) = \int_{x\in\mathbb{R}^2} e^{-i(\tau\overline{\theta})\cdot x} f(x)dx$$

$$= \int_{p=-\infty}^{\infty}\int_{t=-\infty}^{\infty} e^{-i\tau p} f(p\overline{\theta} + t\theta^{\perp})\, dt\, dp$$

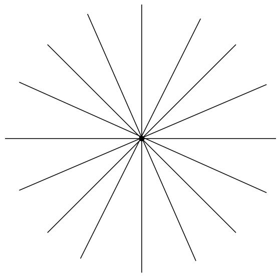
since $\tau\overline{\theta}\cdot(p\overline{\theta} + t\theta^{\perp}) = \tau p$. So,

$$2\pi\widehat{f}(\tau\overline{\theta}) = \int_{p=-\infty}^{\infty} e^{-i\tau p} Rf(\theta, p)\, dp = \sqrt{2\pi}\mathcal{F}_1 Rf(\theta, \tau).$$

Inversion Theorem

Dual Radon transform: $g \in C([0,2\pi] \times \mathbb{R})$

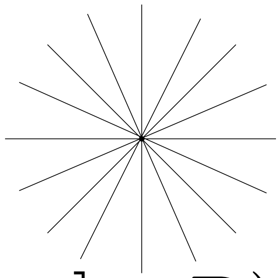
$$R^*g(x) = \int_0^{2\pi} g(\theta, x \cdot \overline{\theta}) d\theta$$



Inversion Theorem

Dual Radon transform: $g \in C([0,2\pi] \times \mathbb{R})$

$$R^* g(x) = \int_0^{2\pi} g(\theta, x \cdot \overline{\theta}) d\theta$$



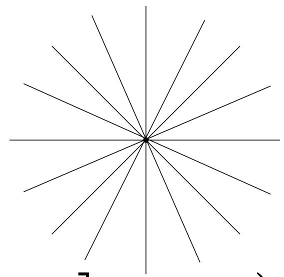
Lambda Operator: $g \in C_c^\infty([0,2\pi] \times \mathbb{R})$

$$\Lambda_p g(\theta, p) = \sqrt{-d^2/dp^2}\, g = \mathcal{F}_1^{-1}\big(|\tau| \mathcal{F}_1 g(\theta, p)\big)$$

Inversion Theorem

Dual Radon transform: $g \in C([0, 2\pi] \times \mathbb{R})$

$$R^* g(x) = \int_0^{2\pi} g(\theta, x \cdot \overline{\theta}) d\theta$$



Lambda Operator: $g \in C_c^\infty([0, 2\pi] \times \mathbb{R})$

$$\Lambda_p g(\theta, p) = \sqrt{-d^2/dp^2}\, g = \mathcal{F}_1^{-1}\big(|\tau| \mathcal{F}_1 g(\theta, p)\big)$$

Filtered Backprojection: $f \in C_c^\infty(\mathbb{R}^2)$,

$$f(x) = \frac{1}{4\pi} R^*(\Lambda_p R f)(x).$$

Filtered Back Projection (FBP):

Filtered Back Projection Inversion formula:

$$f = \frac{1}{4\pi} R^* \Lambda_p R f$$

Filtered Back Projection (FBP):

Filtered Back Projection Inversion formula:

$$f = \frac{1}{4\pi} R^* \Lambda_p R f$$

- FBP reconstructs f.

Filtered Back Projection (FBP):

Filtered Back Projection Inversion formula:

$$f = \frac{1}{4\pi} R^* \Lambda_p R f$$

- FBP reconstructs f.
- Reconstruction is not local.

Filtered Back Projection (FBP):

Filtered Back Projection Inversion formula:

$$f = \frac{1}{4\pi} R^* \Lambda_p R f$$

- FBP reconstructs f.
- Reconstruction is not local.

  Implement FBP by choosing an approximation to Λ_p:

$$f(x) \equiv R^*(\phi *_p Rf)(x)$$

Lambda CT [K. Smith [FFRS], E. Vainberg]:

Lambda reconstruction:

$$\Lambda_x f = \frac{1}{4\pi} R^* \left(-d^2/dp^2 \right) R f$$

Lambda CT [K. Smith [FFRS], E. Vainberg]:

Lambda reconstruction:

$$\Lambda_x f = \frac{1}{4\pi} R^* \left(-d^2/dp^2 \right) Rf$$

- The algorithm replaces Λ_p by $-d^2/dp^2$ and it reconstructs $\Lambda_x f$ **not** f.

Lambda CT [K. Smith [FFRS], E. Vainberg]:

Lambda reconstruction:

$$\Lambda_x f = \frac{1}{4\pi} R^* \left(-d^2/dp^2\right) R f$$

- The algorithm replaces Λ_p by $-d^2/dp^2$ and it reconstructs $\Lambda_x f$ **not** f.
- Reconstruction **is** local.

Lambda CT [K. Smith [FFRS], E. Vainberg]:

Lambda reconstruction:

$$\Lambda_x f = \frac{1}{4\pi} R^* \left(-d^2/dp^2 \right) R f$$

- The algorithm replaces Λ_p by $-d^2/dp^2$ and it reconstructs $\Lambda_x f$ **not** f.
- Reconstruction **is** local.
- Reconstructions look great

Lambda CT [K. Smith [FFRS], E. Vainberg]:

Lambda reconstruction:

$$\Lambda_x f = \frac{1}{4\pi} R^* \left(-d^2/dp^2 \right) Rf = \frac{1}{2\pi} \int_{\xi \in \mathbb{R}^2} e^{ix\cdot\xi} \, |\xi| \, \widehat{f}(\xi) d\xi$$

- The algorithm replaces Λ_p by $-d^2/dp^2$ and it reconstructs $\Lambda_x f$ **not** f.
- Reconstruction **is** local.
- Reconstructions look great since Λ_x preserves and emphasizes singularities.

Lambda CT [K. Smith [FFRS], E. Vainberg]:

Lambda reconstruction:

$$\Lambda_x f = \frac{1}{4\pi} R^* \left(-d^2/dp^2\right) Rf = \frac{1}{2\pi} \int_{\xi \in \mathbb{R}^2} e^{ix\cdot\xi} \, |\xi| \, \widehat{f}(\xi) d\xi$$

- The algorithm replaces Λ_p by $-d^2/dp^2$ and it reconstructs $\Lambda_x f$ **not** f.
- Reconstruction **is** local.
- Reconstructions look great since Λ_x preserves and emphasizes singularities.
- Lambda CT is made for ROI CT and easy to adapt to other limited data problems......

Limited Data Tomography:

When data are not uniformly distributed.

Limited Data Tomography:

When data are not uniformly distributed.

- **Limited Angle CT:** data in a limited angular range.

Limited Data Tomography:

When data are not uniformly distributed.

- **Limited Angle CT:** data in a limited angular range. Inversion methods: Louis, Davison and Grünbaum, Madych and Nelson, Katsevich, [LR]. Slide 4

Limited Data Tomography:

When data are not uniformly distributed.

- **Limited Angle CT:** data in a limited angular range. Inversion methods: Louis, Davison and Grünbaum, Madych and Nelson, Katsevich, [LR]. Slide 4
  - **Warning:** Inversion is highly ill posed [F, L, DG].

Limited Data Tomography:

When data are not uniformly distributed.

- **Limited Angle CT:** data in a limited angular range. Inversion methods: Louis, Davison and Grünbaum, Madych and Nelson, Katsevich, [LR]. Slide 4
  - **Warning:** Inversion is highly ill posed [F, L, DG].
- **Region of interest (ROI) CT:** Data passing through a (small) region [CD, FFRS, KLM, LR, RK, R-FLBW, V]. $\longrightarrow$

Limited Data Tomography:

When data are not uniformly distributed.

- **Limited Angle CT:** data in a limited angular range. Inversion methods: Louis, Davison and Grünbaum, Madych and Nelson, Katsevich, [LR]. Slide 4
  - **Warning:** Inversion is highly ill posed [F, L, DG].
- **Region of interest (ROI) CT:** Data passing through a (small) region [CD, FFRS, KLM, LR, RK, R-FLBW, V].$\longrightarrow$ Use: medical CT and nondestructive evaluation of small parts of objects.

Limited Data Tomography:

When data are not uniformly distributed.

- **Limited Angle CT:** data in a limited angular range. Inversion methods: Louis, Davison and Grünbaum, Madych and Nelson, Katsevich, [LR]. Slide 4
  - **Warning:** Inversion is highly ill posed [F, L, DG].
- **Region of interest (ROI) CT:** Data passing through a (small) region [CD, FFRS, KLM, LR, RK, R-FLBW, V]. $\longrightarrow$
  Use: medical CT and nondestructive evaluation of small parts of objects.
  - Inversion is not possible.

Limited Data Tomography:

When data are not uniformly distributed.

- **Limited Angle CT:** data in a limited angular range. Inversion methods: Louis, Davison and Grünbaum, Madych and Nelson, Katsevich, [LR]. Slide 4
  - **Warning:** Inversion is highly ill posed [F, L, DG].
- **Region of interest (ROI) CT:** Data passing through a (small) region [CD, FFRS, KLM, LR, RK, R-FLBW, V].$\longrightarrow$ Use: medical CT and nondestructive evaluation of small parts of objects.
  - Inversion is not possible.
  - All singularities are visible!

Limited angle region of interest (ROI) CT:

Limited angle data in a small region of interest. Slide 4

Limited angle region of interest (ROI) CT:

Limited angle data in a small region of interest. Slide 4

Use: Electron microscopy (dentistry, NDE of circuit boards)!

Limited angle region of interest (ROI) CT:

Limited angle data in a small region of interest. Slide 4

Use: Electron microscopy (dentistry, NDE of circuit boards)!

GOAL: See individual objects the size of 8 Å (now around 50-80 Å) *in situ* or *in vitro* for molecules that are not structurally rigid (as opposed to reconstructions of rigid objects such as crystals).

Limited angle region of interest (ROI) CT:

Limited angle data in a small region of interest. Slide 4

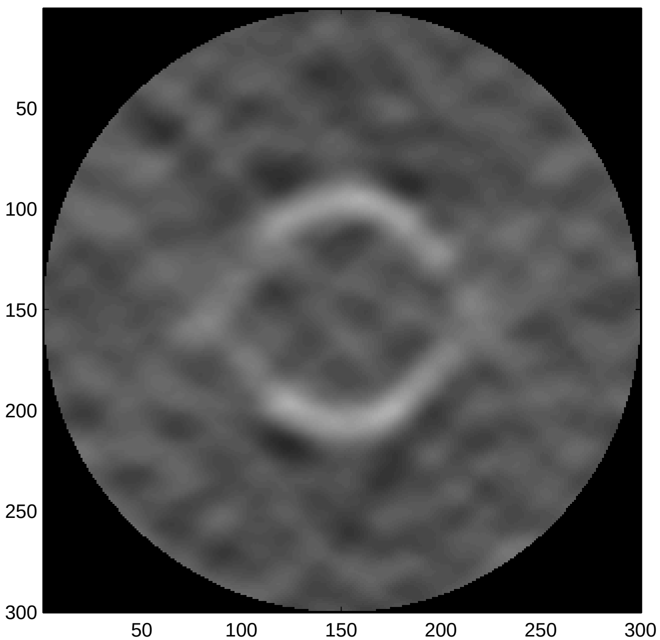
Use: Electron microscopy (dentistry, NDE of circuit boards)!

GOAL: See individual objects the size of 8 Å (now around 50-80 Å) *in situ* or *in vitro* for molecules that are not structurally rigid (as opposed to reconstructions of rigid objects such as crystals).

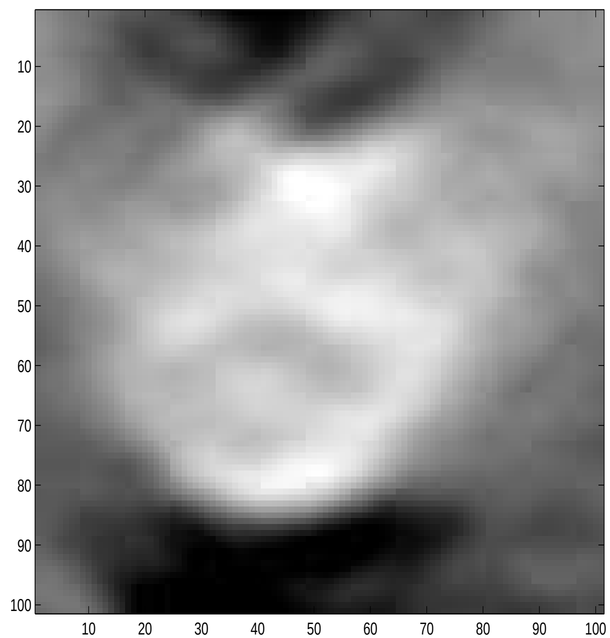
I developed a limited angle Lambda CT algorithm (see also [KLM]), and I'm working with Ulf Skoglund and Ozan Öktem of the Karolinska Institute/Sidec Technologies, Stockholm, applying it to electron tomography. Slide 3

My limited angle ROI Reconstruction

Simulated electron microscope data $\theta \in [30^\circ, 150^\circ]$



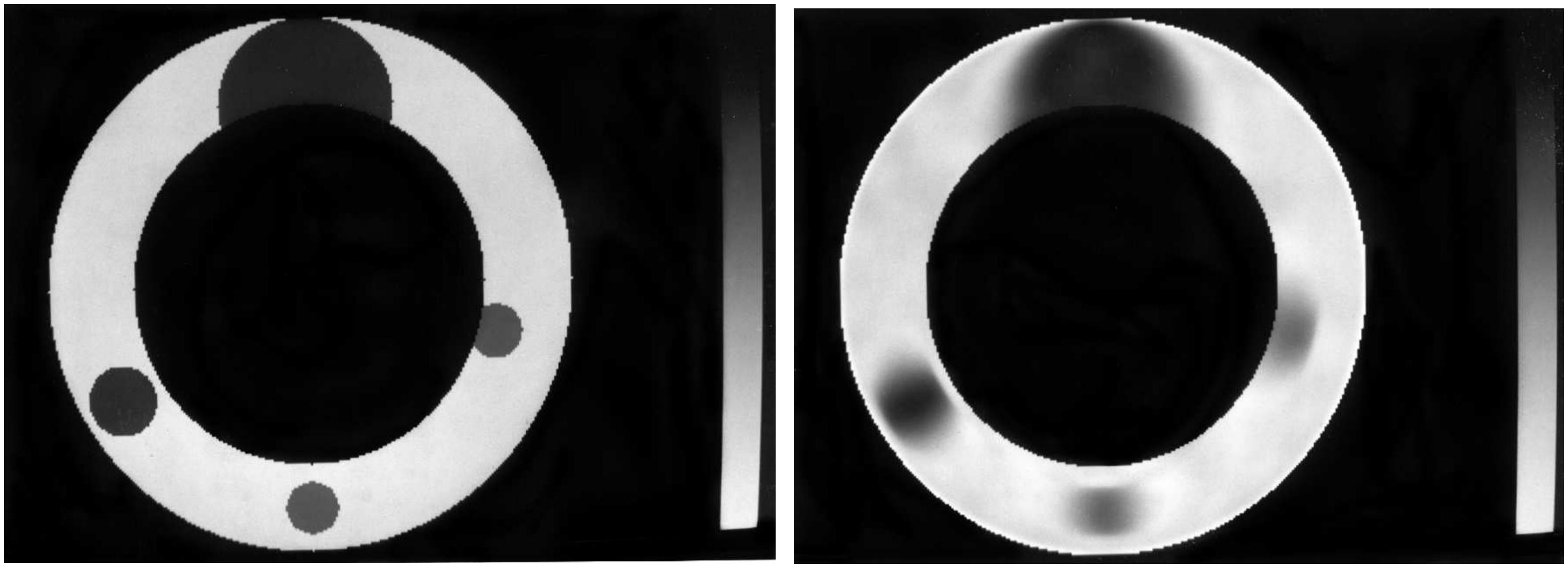
Reconstruction of real virus, 25 angles



From Sidec/Karolinska Institute data.

Exterior reconstruction [Q1]:

Exterior data: lines that are exterior to (do not meet) the big black central circle.



Phantom (left) and ERA reconstruction from simulated data [Q1].

Big Question:

Why are some singularities (object boundaries) more clearly visible from limited data than others.

Big Question:

Why are some singularities (object boundaries) more clearly visible from limited data than others.

Tools for our Answer:

Big Question:

Why are some singularities (object boundaries) more clearly visible from limited data than others.

Tools for our Answer:

- Characterize singularities of objects.

Big Question:

Why are some singularities (object boundaries) more clearly visible from limited data than others.

Tools for our Answer:

- Characterize singularities of objects.
- Learn what the Radon transform does to singularities.

What are singularities?

What are singularities?

Practically: density jumps, boundaries between regions.

What are singularities?

Practically: density jumps, boundaries between regions.

Mathematically: where the density of the object is not smooth...

What are singularities?

Practically: density jumps, boundaries between regions.

Mathematically: where the density of the object is not smooth...

The Fourier Transform:

$$
\begin{aligned}
(\mathcal{F}f)(\xi) &= \widehat{f}(\xi) = \frac{1}{2\pi}\int_{x\in\mathbb{R}^2} e^{-ix\cdot\xi} f(x)dx, \\
f(x) &= \frac{1}{2\pi}\int_{\xi\in\mathbb{R}^2} e^{ix\cdot\xi}(\mathcal{F}f)(\xi)d\xi
\end{aligned}
\tag{2}
$$

What are singularities?

Practically: density jumps, boundaries between regions.

Mathematically: where the density of the object is not smooth...

The Fourier Transform:

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \frac{1}{2\pi}\int_{x\in\mathbb{R}^2} e^{-ix\cdot\xi} f(x)dx, \qquad f(x) = \frac{1}{2\pi}\int_{\xi\in\mathbb{R}^2} e^{ix\cdot\xi}(\mathcal{F}f)(\xi)d\xi \tag{2}$$

Key idea: *Rapid decrease (or $L^2(\mathbb{R}^2,(1+|\xi|)^2)^s)$ integrability) of $\mathcal{F}f$ $\sim$ smoothness (or L^2 derivatives) of f.....*

Singularities done rigorously:

$$\mathcal{F}f(\xi) = \frac{1}{2\pi}\int_{x\in\mathbb{R}^2} e^{-ix\cdot\xi} f(x)dx,$$

Singularities done rigorously:

$$\mathcal{F}f(\xi) = \frac{1}{2\pi}\int_{x\in\mathbb{R}^2} e^{-ix\cdot\xi} f(x)dx,$$

- **Localize:** multiply f by a *cut-off function* $\varphi \in C_c^\infty(\mathbb{R}^2)$.

$$\mathcal{F}(\varphi f)(\xi) = \frac{1}{2\pi}\int_{x\in\mathbb{R}^2} e^{-ix\cdot\xi}\varphi(x) f(x)dx,$$

Singularities done rigorously:

$$\mathcal{F}f(\xi) = \frac{1}{2\pi}\int_{x\in\mathbb{R}^2} e^{-ix\cdot\xi} f(x)dx,$$

- **Localize:** multiply f by a *cut-off function* $\varphi \in C_c^\infty(\mathbb{R}^2)$.

$$\mathcal{F}(\varphi f)(\xi) = \frac{1}{2\pi}\int_{x\in\mathbb{R}^2} e^{-ix\cdot\xi}\varphi(x) f(x)dx,$$

  - **Singular Support:** f is not smooth at x_0 iff for every smooth cut-off φ near x_0 $(\varphi(x_0) \neq 0)$, $\mathcal{F}(\varphi f)$ is **not** rapidly decreasing.

Singularities done rigorously:

$$\mathcal{F}f(\xi) = \frac{1}{2\pi}\int_{x\in\mathbb{R}^2} e^{-ix\cdot\xi} f(x)dx,$$

- **Localize:** multiply f by a *cut-off function* $\varphi \in C_c^\infty(\mathbb{R}^2)$.

$$\mathcal{F}(\varphi f)(\xi) = \frac{1}{2\pi}\int_{x\in\mathbb{R}^2} e^{-ix\cdot\xi}\varphi(x) f(x)dx,$$

  - **Singular Support:** f is not smooth at x_0 iff for every smooth cut-off φ near x_0 ($\varphi(x_0) \neq 0$), $\mathcal{F}(\varphi f)$ is **not** rapidly decreasing.
- **Microlocalize:** Find *directions* where $\mathcal{F}(\varphi f)$ is not rapidly decreasing:

Singularities done rigorously:

$$\mathcal{F}f(\xi) = \frac{1}{2\pi}\int_{x\in\mathbb{R}^2} e^{-ix\cdot\xi} f(x)dx,$$

- **Localize:** multiply f by a *cut-off function* $\varphi \in C_c^\infty(\mathbb{R}^2)$.

$$\mathcal{F}(\varphi f)(\xi) = \frac{1}{2\pi}\int_{x\in\mathbb{R}^2} e^{-ix\cdot\xi}\varphi(x) f(x)dx,$$

  - **Singular Support:** f is not smooth at x_0 iff for every smooth cut-off φ near x_0 $(\varphi(x_0)\neq 0)$, $\mathcal{F}(\varphi f)$ is **not** rapidly decreasing.
- **Microlocalize:** Find *directions* where $\mathcal{F}(\varphi\, f)$ is not rapidly decreasing:
  - **Wavefront Set [Hö, P]....**

Wavefront Set:

Definition 1: Let $x_0 \in \mathbb{R}^2$ and $\xi_0 \in \mathbb{R}^2 \setminus 0$. The function f is *smooth at x_0 in direction ξ_0* **iff** $\exists$ a cut-off function φ near x_0 such that

$$\mathcal{F}(\varphi f)(\xi) = \frac{1}{2\pi} \int_{x \in \mathbb{R}^2} e^{-ix\cdot\xi} \varphi(x) f(x)\, dx \qquad (3)$$

is rapidly decreasing in some open cone $\mathcal{C}$ from the origin containing ξ_0.

Wavefront Set:

Definition 1: Let $x_0 \in \mathbb{R}^2$ and $\xi_0 \in \mathbb{R}^2 \setminus 0$. The function f is *smooth at x_0 in direction ξ_0* **iff** $\exists$ a cut-off function φ near x_0 such that

$$\mathcal{F}(\varphi f)(\xi) = \frac{1}{2\pi} \int_{x \in \mathbb{R}^2} e^{-ix \cdot \xi} \varphi(x) f(x) \, dx \qquad (3)$$

is rapidly decreasing in some open cone $\mathcal{C}$ from the origin containing ξ_0.
On the other hand, $(x_0, \xi_0) \in \mathrm{WF}(f)$ **iff** f is **not** smooth at x_0 in direction ξ_0. (Slide 5)

Example: $f = 1$ inside a circle, $f = 0$ outside.
What is $\mathrm{WF}(f)$? Slide 5

Example: $f = 1$ inside a circle, $f = 0$ outside.
What is $\mathrm{WF}(f)$? Slide 5

MORAL: If f has a jump singularity on a smooth curve C, then vectors (co)normal to C are in $\mathrm{WF}(f)$.

Singularities and R:

$f \in L^1(\mathbb{R}^2)$, Data: $Rf(\theta, p) = \int_{x \in L(\theta,p)} f(x)ds$ Slide 5

Singularities and R:

$f \in L^1(\mathbb{R}^2)$, Data: $Rf(\theta,p) = \displaystyle\int_{x\in L(\theta,p)} f(x)ds$ Slide 5

Theorem 1. [Q2] *There is a one-to-one correspondence between wavefront of f and wavefront of Rf.*

Let $L_0 = L(\theta_0, p_0)$. If for some $x_0 \in L_0$, $(x_0, \overline{\theta}_0) \in \mathrm{WF}(f)$, then Rf is not smooth near (θ_0, p_0). If f is smooth in directions $\pm\overline{\theta}_0$ at every point on L_0, then Rf is smooth near (θ_0, p_0). Undetected singularities (those not perpendicular to L_0) are smoothed by data $Rf(\theta,p)$ for (θ,p) near (θ_0,p_0).

Singularities and R:

$f \in L^1(\mathbb{R}^2)$, Data: $Rf(\theta,p) = \int_{x \in L(\theta,p)} f(x)ds$ Slide 5

Theorem 1. [Q2] *There is a one-to-one correspondence between wavefront of f and wavefront of Rf.*

Let $L_0 = L(\theta_0, p_0)$. If for some $x_0 \in L_0$, $(x_0, \overline{\theta}_0) \in \mathrm{WF}(f)$, then Rf is not smooth near (θ_0, p_0). If f is smooth in directions $\pm\overline{\theta}_0$ at every point on L_0, then Rf is smooth near (θ_0, p_0). Undetected singularities (those not perpendicular to L_0) are smoothed by data $Rf(\theta,p)$ for (θ,p) near (θ_0,p_0).

- **Reason:** Guillemin [1975]: R is an elliptic Fourier integral operator. Candès: curvelets and wavefront sets.

Singularities and R:

$f \in L^1(\mathbb{R}^2)$, Data: $Rf(\theta,p) = \int_{x\in L(\theta,p)} f(x)ds$ Slide 5

Theorem 1. [Q2] *There is a one-to-one correspondence between wavefront of f and wavefront of Rf.*

Let $L_0 = L(\theta_0,p_0)$. If for some $x_0 \in L_0$, $(x_0,\overline{\theta}_0) \in \mathrm{WF}(f)$, then Rf is not smooth near (θ_0,p_0). If f is smooth in directions $\pm\overline{\theta}_0$ at every point on L_0, then Rf is smooth near (θ_0,p_0). Undetected singularities (those not perpendicular to L_0) are smoothed by data $Rf(\theta,p)$ for (θ,p) near (θ_0,p_0).

- **Reason:** Guillemin [1975]: R is an elliptic Fourier integral operator. Candès: curvelets and wavefront sets.
- The Projection Slice Theorem.

Singularities and R:

$f \in L^1(\mathbb{R}^2)$, Data: $Rf(\theta,p) = \int_{x\in L(\theta,p)} f(x)ds$ Slide 5

Theorem 1. [Q2] *There is a one-to-one correspondence between wavefront of f and wavefront of Rf.*

Let $L_0 = L(\theta_0, p_0)$. If for some $x_0 \in L_0$, $(x_0, \overline{\theta}_0) \in \mathrm{WF}(f)$, then Rf is not smooth near (θ_0, p_0). If f is smooth in directions $\pm\overline{\theta}_0$ at every point on L_0, then Rf is smooth near (θ_0, p_0). Undetected singularities (those not perpendicular to L_0) are smoothed by data $Rf(\theta,p)$ for (θ,p) near (θ_0,p_0).

- **Reason:** Guillemin [1975]: R is an elliptic Fourier integral operator. Candès: curvelets and wavefront sets.
- The Projection Slice Theorem.
- This all is valid for $\mathrm{WF}^s(f)$ and $\mathrm{WF}^{s+1/2}(Rf)$ [Q2]! Slide 6

Morals:

I. Theorem 1$\longrightarrow$: the Radon transform, R, with limited data does a good job detecting a singularity of f when the line is perpendicular to the singularity (e.g., tangent to a boundary of part of the object).

Morals:

I. Theorem 1$\longrightarrow$: the Radon transform, R, with limited data does a good job detecting a singularity of f when the line is perpendicular to the singularity (e.g., tangent to a boundary of part of the object).

II. If a singularity is not perpendicular to any line in the data set, then, it will be harder to reconstruct stably.

Morals:

I. Theorem 1$\longrightarrow$: the Radon transform, R, with limited data does a good job detecting a singularity of f when the line is perpendicular to the singularity (e.g., tangent to a boundary of part of the object).

II. If a singularity is not perpendicular to any line in the data set, then, it will be harder to reconstruct stably.

Moral for limited data CT: *If L_0 is in a data set, then singularities of f perpendicular to L_0 will be visible, but other singularities....*

Morals:

I. Theorem 1$\rightarrow$: the Radon transform, R, with limited data does a good job detecting a singularity of f when the line is perpendicular to the singularity (e.g., tangent to a boundary of part of the object).

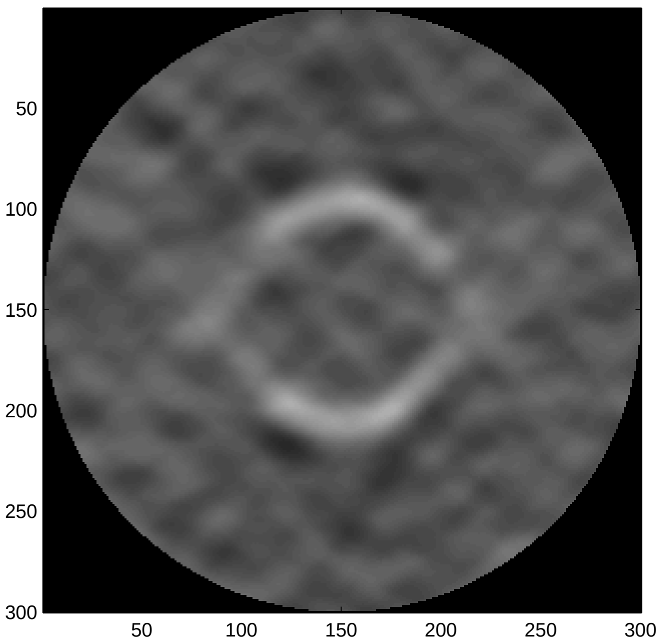
II. If a singularity is not perpendicular to any line in the data set, then, it will be harder to reconstruct stably.

Moral for limited data CT: *If* L_0 *is in a data set, then singularities of* f *perpendicular to* L_0 *will be visible, but other singularities....*

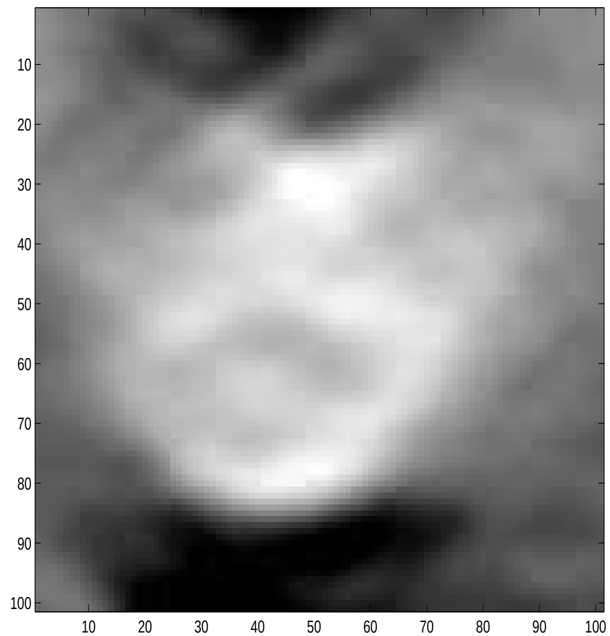
Examples: $\longrightarrow$

My limited angle ROI Reconstruction

Simulated electron microscope data $\theta \in [30^\circ, 150^\circ]$



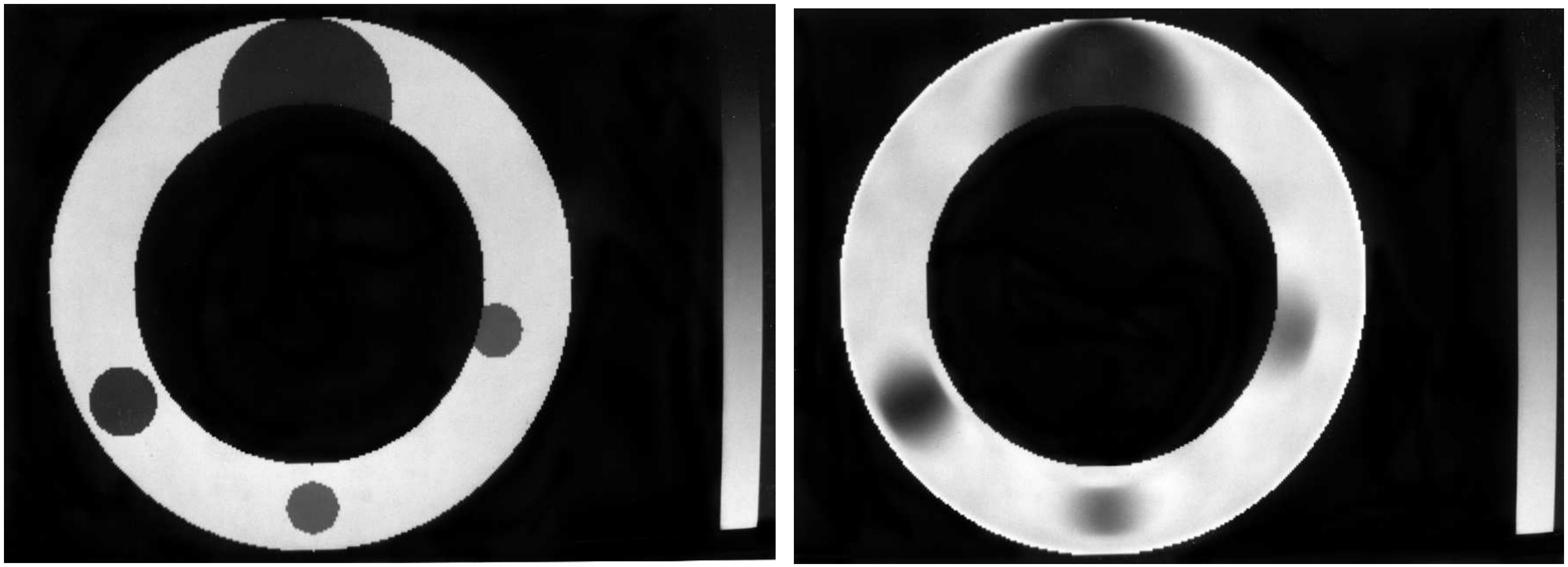
Reconstruction of real virus, 25 angles



From Sidec/Karolinska Institute data.

Exterior reconstruction [Q1]:

Exterior data: lines that are exterior to (do not meet) the big black central circle.



Phantom (left) and ERA reconstruction from simulated data [Q1].

The Moral:

- Boundaries tangent to lines in the data set <u>are</u> clearer.

The Moral:

- Boundaries tangent to lines in the data set <u>are</u> clearer.
- It is often easier to reconstruct object features (boundary shapes, etc.) than actual density values from limited data.

The Moral:

- Boundaries tangent to lines in the data set are clearer.
- It is often easier to reconstruct object features (boundary shapes, etc.) than actual density values from limited data.
- Singularity detection methods reconstruct only the “visible” singularities so the visible singularities look as good as can be expected [Q2] $\left(\ddot{\smile}\right)$.

The Moral:

- Boundaries tangent to lines in the data set <u>are</u> clearer.
- It is often easier to reconstruct object features (boundary shapes, etc.) than actual density values from limited data.
- Singularity detection methods reconstruct only the "visible" singularities so the visible singularities look as good as can be expected [Q2] $(\ddot\smile)$.
- **The Caveat:** However, "invisible" singularities can be blurred or distorted (depending on the algorithm).