Obstacle reconstruction from the scattering amplitude

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Weltbekannter Grundriss: Markgraf Karl Wilhelm von Baden-Durlach gründete die Fächerstadt 1715 entlang strahlenförmiger Alleen, die von seinem Schloss in Richtung Süden ausgingen. Die Stadtväter von Washington nahmen sich den Karlsruher Grundriss zum Vorbild.

Summary

- The direct and the inverse scattering problem
- Historical review
- Minimization algorithm
- Factorization method for sound-soft and sound-hard obstacles
- $F_{\#}$-algorithm
- Mixed BVP

The direct problem

$D \subset \mathbb{R}^3$ open; $\Gamma = \partial D \in C^2$;

$U = \mathbb{R}^3 \backslash \overline{D}$ connected.

$\theta \in S^2 = \left\{ x \in \mathbb{R}^3 : |x| = 1 \right\}$ direction.

Incident plane wave

$$u^i(x, \theta) = \exp\left(ik\theta \cdot x\right), \quad x \in U,$$

or point souce

$$u^i(x, \theta) = \mathcal{E}_z\left(x\right) = \frac{1}{4\pi} \frac{\exp(ik\left|x - z\right|)}{\left|x - z\right|}, \quad x \in U \setminus \left\{z\right\},$$

Scattered field

$$u^s = u^s(\cdot, \theta) \in C^2(U) \cap C\left(\overline{U}\right)$$

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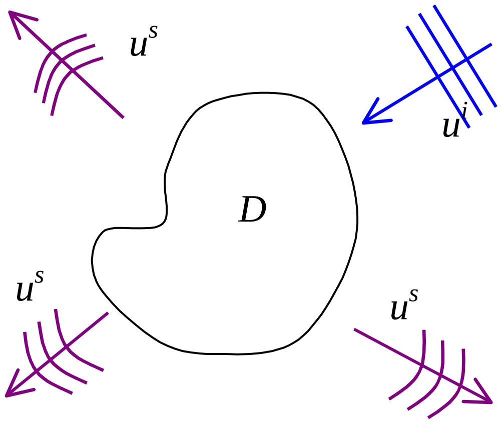


Equations and boundary conditions

$$\Delta u^s + k^2 u^s = 0 \quad \text{in } U, \qquad \text{(Helmholtz)}$$

$$\frac{\partial u^s}{\partial |x|} - iku^s = \mathcal{O}\left(\frac{1}{|x|^2}\right), \quad |x| \to \infty. \qquad \text{(Sommerfeld)}$$

The total field $u = u^i + u^s$ satisfies

$$u = 0 \text{ on } \Gamma, \qquad \text{(Dirichlet)}$$

or

$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma. \qquad \text{(Neumann)}$$

or

$$\frac{\partial u}{\partial n} + \lambda u = 0 \text{ on } \Gamma,\ \operatorname{Im}\lambda \geq 0. \qquad \text{(Robin)}$$

The exterior boundary value problem (Dirichlet case)

Theorem 1. *Let $k > 0$ be a fixed frequency.*
For any function $f \in H^{1/2}(\Gamma)$ the exterior BVP (Helmholtz), (Sommerfeld) has a unique solution $u \in H^1_{loc}(\mathbb{R}^3 \backslash \overline{D})$ which has the trace f on Γ :

$$u = f \quad \textit{on } \Gamma. \qquad \text{(Dirichlet)}$$

In our case:

$$u^s = -u^i(\cdot, \theta) = -\exp\left(ik\theta \cdot x\right) \text{ on } \Gamma.$$

The exterior BVP (Neumann or Robin case)

Theorem 2. *For any function* $g \in H^{-1/2}(\Gamma)$ *the exterior BVP (Helmholtz), (Sommerfeld) has a unique solution* $u \in H^1_{loc}(\mathbb{R}^3 \backslash \overline{D})$ *which satisfies*

$$\frac{\partial u}{\partial \nu} + \lambda u = g \;\; \textit{on } \Gamma. \qquad \text{(Robin)}$$

In our case:

$$\frac{\partial u^s}{\partial \nu} + \lambda u^s = -\frac{\partial u^i(\cdot, \theta)}{\partial \nu} - \lambda u^i(\cdot, \theta) \text{ on } \Gamma.$$

Far field pattern and scattering amplitude

Theorem 3. *For any radiating wave u holds*

$$u\left(x\right)=\frac{\exp\left(ik\left|x\right|\right)}{4\pi\left|x\right|}u_{\infty}\left(\hat{x}\right)+\mathcal{O}\left(\left|x\right|^{-2}\right),\ \hat{x}=\frac{x}{\left|x\right|},$$

for $x\rightarrow\infty$. The analytic function u_{∞} is called the **far field pattern** *of the wave.*

Scattering amplitude u_{∞} :

$$u^{s}(x,\theta)=\frac{1}{4\pi}\frac{\exp(ik\left|x\right|)}{\left|x\right|}u_{\infty}(\hat{x},\theta)+\mathcal{O}(\left|x\right|^{-2}).$$

Inverse scattering problem

Given: the scattering amplitude (far field pattern)

$$u_\infty(\eta, \theta) \quad \text{for } \eta, \theta \in W \subset S^2,$$

Find D.

Far field operator (FFO) $F : L_2(W) \to L_2(W), W \subset S^2$:

$$Fh(\eta) := \int_W u_\infty(\eta, \theta) h(\theta) d\sigma(\theta), \quad \eta, \theta \in W,$$

Newton-type and Landweber iterations (local)

- The forward problem is interpreted as the nonlinear differentiable operator

$$L : \Gamma \rightarrow F$$

- One tries to solve $L(\Gamma) = F$ by Newton-type methods.

- A starting point = an initial guess of the shape and position of the scatterer

- An evident drawback: one must solve the forward scattering problem at each step of the algorithm.

Analytical continuation

Proposed by Kirsch and Kress (80th)

- analytical continuation of the far or near field: $u_\infty \rightsquigarrow u^s$ (a linear ill-posed problem)

- $\Gamma =$ the zero-level set of $u = u^s + u^i$ (nonlinear)

The factorization method

The main idea: consider a data-to-pattern operator G:

$$G : u|_{\Gamma} \longmapsto u_{\infty}(\cdot)\,, \qquad \text{(Dirichlet case)}$$

with $G_D : H^{1/2}(\Gamma) \to L_2\left(S^2\right)$,

resp.

$$G : \left(\frac{\partial u}{\partial \nu} + \lambda u\right)\Big|_{\Gamma} \longmapsto u_{\infty}(\cdot)\,, \qquad \text{(Robin case)}$$

with $G_R : H^{-1/2}(\Gamma) \to L_2\left(S^2\right)$.

Domain reconstruction via the operator G

Criterion 4. *Denote* $\Phi_z(\theta) = \exp(-ikz\cdot\theta)$. *Then it holds for* $z \in \mathbb{R}^3$*:*

$$z \in D \iff \Phi_z \in \mathcal{R}(G).$$

Proof: Φ_z is the far field pattern of the point source

$$\mathcal{E}_z(x) = \frac{1}{4\pi}\frac{\exp(ik|x-z|)}{|x-z|}$$

i.e.

$$\frac{1}{4\pi}\frac{\exp(ik|x-z|)}{|x-z|} = \frac{1}{4\pi}\frac{\exp(ik|x|)}{|x|}\underbrace{e^{-ikz\cdot\theta}}_{u_\infty(\theta)} + \mathcal{O}\left(\frac{1}{|x|^2}\right).$$

The Herglotz operator H

$H : L_2\left(S^2\right) \to H^{1/2}(\Gamma)$ and $H^* : H^{-1/2}(\Gamma) \to L_2\left(S^2\right)$

$$H\psi(x) = \int_{S^2} e^{ik\theta\cdot x}\psi(\theta)\, d\theta, \quad x \in \Gamma.$$

$$H^* g(\theta) = \int_{\Gamma} e^{-ik\theta\cdot y} g(y)\, d\sigma(y), \quad \theta \in S^2,$$

- H and H^* does not depend on boundary condition;
- (generic case) G and H^* have the same ranges:

$$\mathcal{R}(H^*) = \mathcal{R}(G).$$

Auxiliary lemma from the functional analysis

Lemma 5. *Let X, Y be two reflexive Banach spaces, and let $L : X \to Y$ be an injective operator with the dense range.*

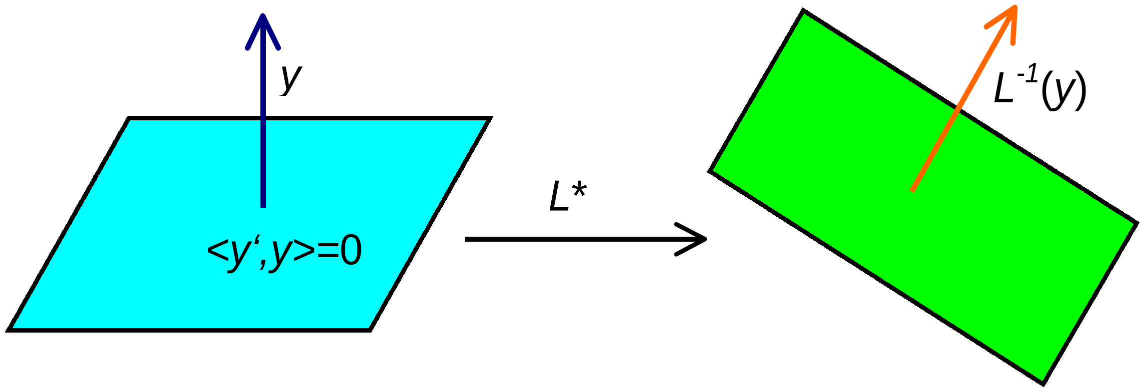
Then it holds for each $y \in Y$:

$$y \in \mathcal{R}(L) \iff \dim\left[X' \setminus L^*\left(y^{\perp}\right)\right] = 1,$$

$$y \notin \mathcal{R}(L) \iff L^*\left(y^{\perp}\right) \text{ is dense in } X.$$

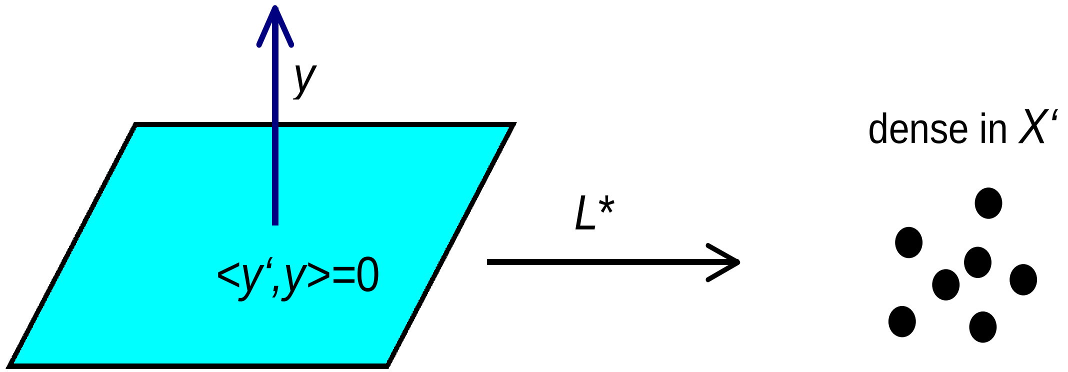
$y \in \mathcal{R}(L)$

$L^* \left(y^\top\right) \subset \left(L^{-1} y\right)^\top$ (a hyperplane):



$y \notin \mathcal{R}(L)$

L^* disperses a hyperplane $y^\top$:



Equivalent formulation

Consider a translated orthogonal complement $y^{\perp}$

$$T_y = \{\langle y', y\rangle = 1\} \subset Y'$$

Then

$$y \notin \mathcal{R}(L) \quad \Longleftrightarrow \operatorname{dist}(L^*(T_y), 0) = \inf\{\|L^* t\|_{X'} : \langle t, y\rangle = 1\} = 0, \quad (1)$$

$$y \in \mathcal{R}(L) \Longleftrightarrow \quad \operatorname{dist}(L^*(T_y), 0) \geq \frac{1}{\|L^{-1} y\|} > 0.$$

Domain characterization via H

$$X = H^{-1/2}(\Gamma)\,,\ Y = L_2\left(S^2\right),\ L = H^*,\ y = \Phi_z(\cdot) = \exp(-ikz\cdot\theta)$$

Criterion 6. *Denote* $\Phi_z(\theta) = \exp(-ikz\cdot\theta)$. *Then it holds for* $z \in \mathbb{R}^3$*:*

$$z \in D \iff \Phi_z \in \mathcal{R}(H^*)$$

$$\Updownarrow$$

$$\inf\left\{\|Hg\|_{H^{-1/2}(\Gamma)} : \langle g, \Phi_z\rangle = 1\right\} > 0$$

Factorization $F = H^*TH$ of the far field operator

$T : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ **coercive**:

$$\mathsf{c}\ |\langle T\varphi, \varphi\rangle| \le \|\varphi\|^2 \le\ \mathsf{C}\ |\langle T\varphi, \varphi\rangle|$$

The factorization implies:

$$\langle Fg, g\rangle = < T\underbrace{Hg}_{\varphi}, \underbrace{Hg}_{\varphi} >$$

$$\boxed{\mathsf{c}\ |\langle Fg, g\rangle| \le \|Hg\|^2 \le \mathsf{C}\ |\langle Fg, g\rangle|}.$$

Minimization algorithm

Theorem 7. F*-characterization:*

$$\boxed{z \in D \Longleftrightarrow \inf\{|\langle Fg, g\rangle| : \langle g, \Phi_z\rangle = 1\} > 0.}$$

Proof:

$$\begin{aligned} z \in D \quad &\Longleftrightarrow \quad \Phi_z \in \mathcal{R}(H^*) \\ &\Longleftrightarrow \inf\{\|Hg\| : \langle g, \Phi_z\rangle = 1\} > 0 \\ &\Longleftrightarrow \inf\{|\langle Fg, g\rangle| : \langle g, \Phi_z\rangle = 1\} > 0. \end{aligned}$$

On the way to fast reconstruction algorithm

Lemma 8. *Let* X_1, X_2, Y *be reflexive Banach spaces, and* $L_j : X_j \to Y$, $j = 1, 2$, *be injective operator with dense ranges. And let the following factorization equations hold*

$$\boxed{L_1 T_1 L_1^* = L_2 T_2 L_2^*} = A$$

with some coercive operators $T_j : X_j' \to X_j$. *Then it holds:*

$$y_0 \in \mathcal{R}(L_j) \Longleftrightarrow \inf \{ |\langle Ay, y \rangle| : \langle y, y_0 \rangle = 1 \} > 0.$$

In particular, it holds:

$$\boxed{\mathcal{R}(L_1) = \mathcal{R}(L_2)}.$$

The case of normal FFO

Let $\{\phi_j, \lambda_j\}$, $j = 1, 2, ...$be the eigensystem of F. Then

$$F\left(\sum c_j \phi_j\right) = \sum \lambda_j c_j \phi_j.$$

$$\underbrace{|F|^{1/2}}_{L_1} \underbrace{(\operatorname{sgn} F)}_{T_1} \underbrace{|F|^{1/2}}_{L_1^*} = \underbrace{H^*}_{L_2} \underbrace{T}_{T_2} \underbrace{H}_{L_2^*}.$$

The middle operator $\operatorname{sgn} F$ is coercive!!!

Square root characterization for sound-soft and sound-hard obstacles

Theorem 9. *In the case of normal FFO (Dirichlet or Neumann boundary condition) it holds* $\mathcal{R}(H^*) = \mathcal{R}\left(|F|^{1/2}\right)$.
Let $\{\phi_j, \lambda_j\}$, $j = 1, 2, ...$*be the eigensystem of the FFO.*

Then a point $z \in \mathbb{R}^3$ *belongs to* D ***iff*** *the series*

$$\sum_{n=1}^{\infty} \frac{|\langle \Phi_z, \phi_j \rangle|^2}{|\lambda_j|}$$

with $\Phi_z(\theta) = \exp(-ikz \cdot \theta)$ *converges.*

Square root algorithm for the impedance case / limited aperture

We take

$$\boxed{F_{\#} = |\mathrm{Re}\, F| + \mathrm{Im}\, F}.$$

$F_{\#}$ is positive and self-adjoint and satisfies

$$F_{\#} = H^* T_{\#} H$$

with appropriate $T_{\#}$. It implies $\boxed{\mathcal{R}(H^*) = \mathcal{R}\left(F_{\#}^{1/2}\right)}$.

It provides for D the visualization via the square root algorithm with F substituted by $F_{\#}$.

Mixed boundary value problem

$D = D_1 \cup D_2$ with $\partial D_1 \cap \partial D_2 = \varnothing$.

Boundary condition (Γ_1 and Γ_2 have different "gender"):

$$v = 0 \text{ on } \Gamma_1,$$
$$\frac{\partial v}{\partial n} + \lambda v = 0 \text{ on } \Gamma_2.$$

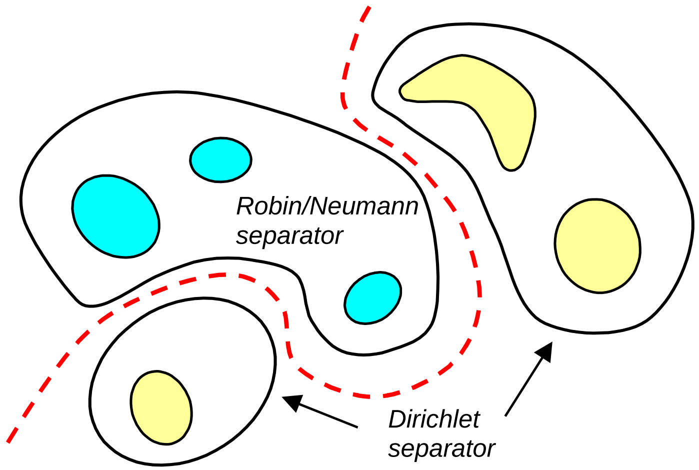
The factorization $F = H^* T H$ holds with a Fredholm operator T.

The problem:

T is **not** coercive!
The criterion
$$z \in D \iff \inf\{|\langle Fg, g\rangle| : \langle g, \Phi_z\rangle = 1\} > 0$$
does not hold!

Partial solution

A-priori known superdomains $D'_{1,2} \subset \mathbb{R}^3$ with $\overline{D_j} \subset D'_j, \quad j = 1, 2$:



Lagrange-multiplicators

We denote by $H_j : L_2(S^2) \to H^{1/2}(\partial D'_j),\ j = 1, 2$, the Herglotz operators

$$H_j \varphi\,(x) = \int_{S^2} e^{ik\theta\cdot x} \varphi\,(\theta)\, d\sigma\,(\theta)\,,\ x \in \partial D'_j.$$

and introduce the modified data operators

$$F_1 = F + \rho H_1^* H_1, \ \ F_2 = F - \rho H_2^* H_2,$$

with some positive constant ρ (arbitrary but fixed).

Theorem 10. *For any point* $z \in D'_j$, $j = 1, 2$ *it holds:*

$$z \in D_j \iff \Phi_z \in \mathcal{R}\left(L_{\#}^{1/2}\right),$$
$$L_{\#} = |\mathrm{Re}\, L| + \mathrm{Im}\, L$$

with

$$L = F_j = F \pm \rho H_j^* H_j \text{ for } j = 1, 2.$$

A point $z \in \mathbb{R}^3$ *belongs to* D *iff the series*

$$\sum_{n=1}^{\infty} \frac{|\langle \Phi_z, \phi_j \rangle|^2}{|\lambda_j|}$$

with $\Phi_z(\theta) = \exp(-ikz \cdot \theta)$ *converges. Here,* $\{\phi_j, \lambda_j\}$, $j = 1, 2, ...$ *is a singular system of* L.