Tomography and Sampling Theory

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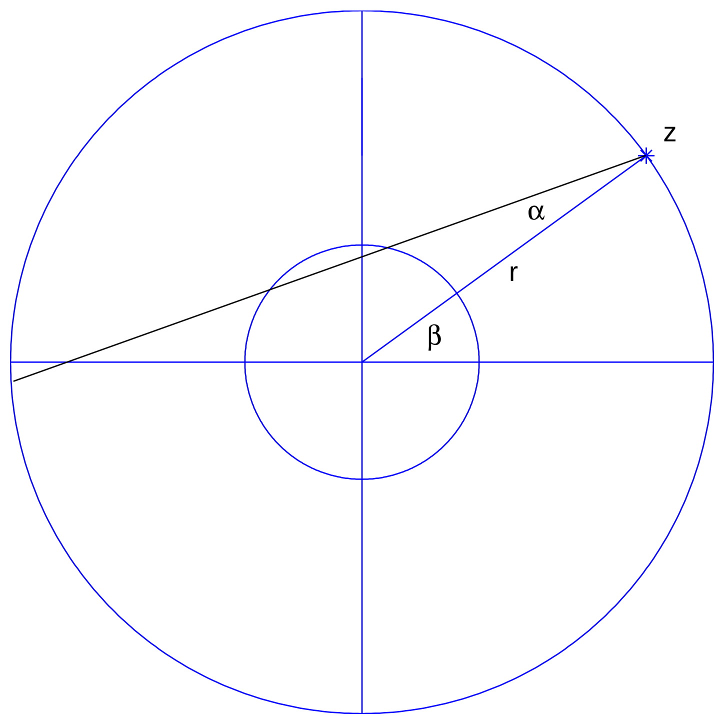
The Divergent Beam X-Ray Transform

$$D_z f(\omega) = \int_0^\infty f(z + t\omega)\, dt, \quad z \in^n, \ \omega \in S^{n-1}.$$

Parametrization in 2D:

$D_z f(\omega) = Df(\beta, \alpha), \ z = (r\cos\beta, r\sin\beta)$



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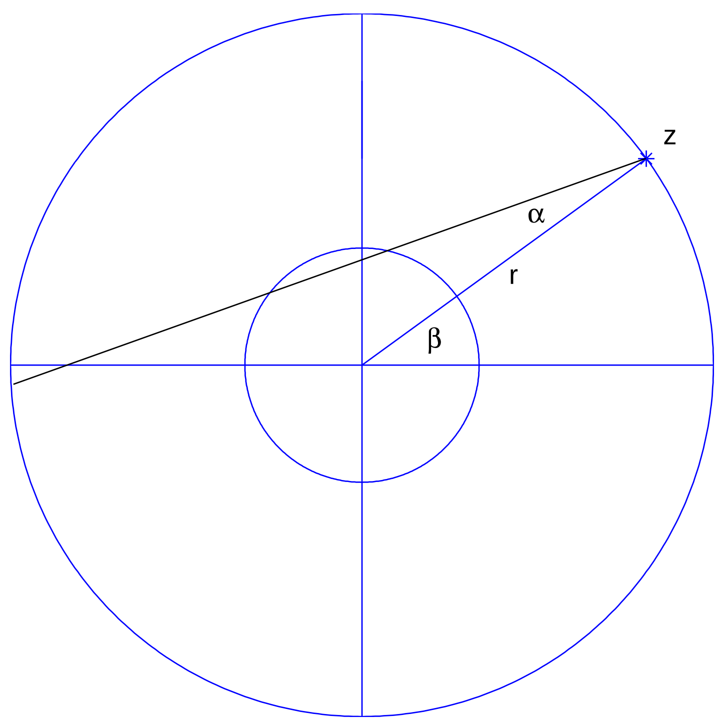
Symmetry:

$$Df(\beta, \alpha) = Df(\beta + 2\alpha + \pi, -\alpha).$$



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The goal of tomography

The goal is to reconstruct $f(x)$ from finitely many measurements of line integrals $Df(\beta, \alpha)$.

Fundamental Question

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The answer depends on the frequency content of the data function Df, that is both on the size and the shape of the support of the Fourier transform of Df.

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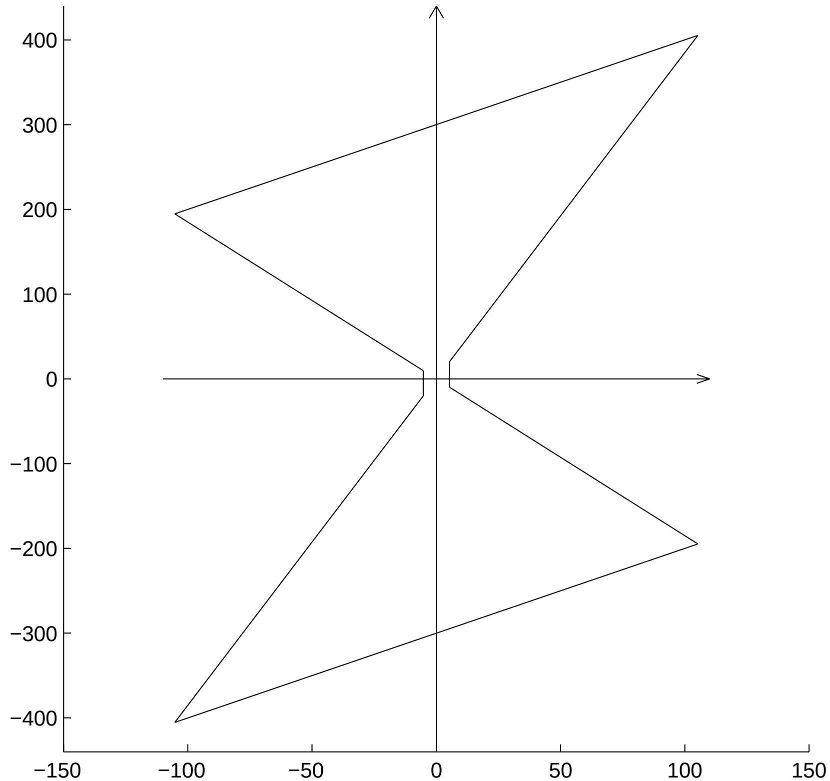
The answer depends on the frequency content of the data function Df, that is both on the size and the shape of the support of the Fourier transform of Df.

Change of variables:
$g(s,t) = Df(\beta,\alpha)$, $(s,t) \in [0,1)^2 = \mathbb{T}^2$.

The essential support of $\hat{g}$



A-priori information used: f has support in unit disk and 'essential bandwidth' $b = 100$.

Sampling lattices

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Characterization of sampling lattices

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\begin{aligned}
\mathbf{L} \;=\; \{(s_j, t_{jl}) :\; & s_j = j\,\Delta s,\ \Delta s = 1/P, \\
& t_{jl} = l\,\Delta t + \delta_j,\ \Delta t = 1/Q,\ \ \delta_j = jN/(PQ) \\
& j = 0, \ldots, P-1,\ l = 0, \ldots, Q-1\}.
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Let $g(x)$ be such that $\hat{g}(\xi) = 0$ for $\xi \notin K = [-b, b)$.

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- g(x) is sampled on a subgroup $\mathbf{L} = h\mathbb{Z}$ of $\mathbb{R}$.
- $2b$ sinc(bx) is the inverse Fourier transform of the indicator function $\chi_K(\xi)$ of K.
  $\chi_K(\xi) = 1$ for $\xi \in K$ and zero otherwise.

Motivation of the Sampling Theorem

Assume $\hat{g}$ is very small outside a set K.

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$$\mathbf{L}^\perp = \{\eta \in \widehat{G} : e^{2\pi i \langle y, \eta \rangle} = 1 \text{ for all } y \in \mathbf{L}\}$$

Hence $\mathsf{DFT}(g) \simeq \hat{g}(\xi)$ if $\xi \in K$ and $\xi + \eta \notin K$, that is, if the translates $K + \eta$, $\eta \in \mathbf{L}^\perp$ are disjoint.

Interpolation

$$
\begin{aligned}
Sg(x) &= \mathsf{IFT}\left[\,\chi_K\,\mathsf{DFT}(g)\,\right] \\
&= c\sum_{y\in\mathbf{L}}\tilde{\chi}_K(x-y)\,g(y)
\end{aligned}
$$

$\chi_K =$ indicator fct. of K

$\tilde{\chi}_K = \mathsf{IFT}\,[\chi_K]$

Sampling Theorem on $\mathbb{T}^2$

Theorem 1 *Let* $\mathbf{L} = \mathbf{L}(N, P, Q)$ *a sampling lattice such that* $K + \eta$, $\eta \in \mathbf{L}^{\perp}$ *are disjoint. For* $z \in \mathbb{T}^2$ *define*

$$Sg(z) = \frac{1}{PQ} \sum_{y \in \mathbf{L}} \tilde{\chi}_K(z - y) g(y).$$

Then

$$|g(z) - Sg(z)| \;\; \leq \;\; 2 \int_{\mathbb{Z}^2 \setminus K} |\hat{g}(\zeta)| \, d\zeta.$$

Achieving Efficiency

1. Find lattices $\mathbf{L}$ as sparse as possible so that the translates $K + \eta$, $\eta \in \mathbf{L}^\perp$ are disjoint.

2. Exploit the symmetry relation

$$Df(\beta, \alpha) = Df(\beta + 2\alpha + \pi, -\alpha).$$

   This requires sampling theorems for sampling sets which are not lattices.

Choice of lattices

Find lattices $\mathbf{L}$ as sparse as possible so that the translates $K + \eta$, $\eta \in \mathbf{L}^{\perp}$ are disjoint.

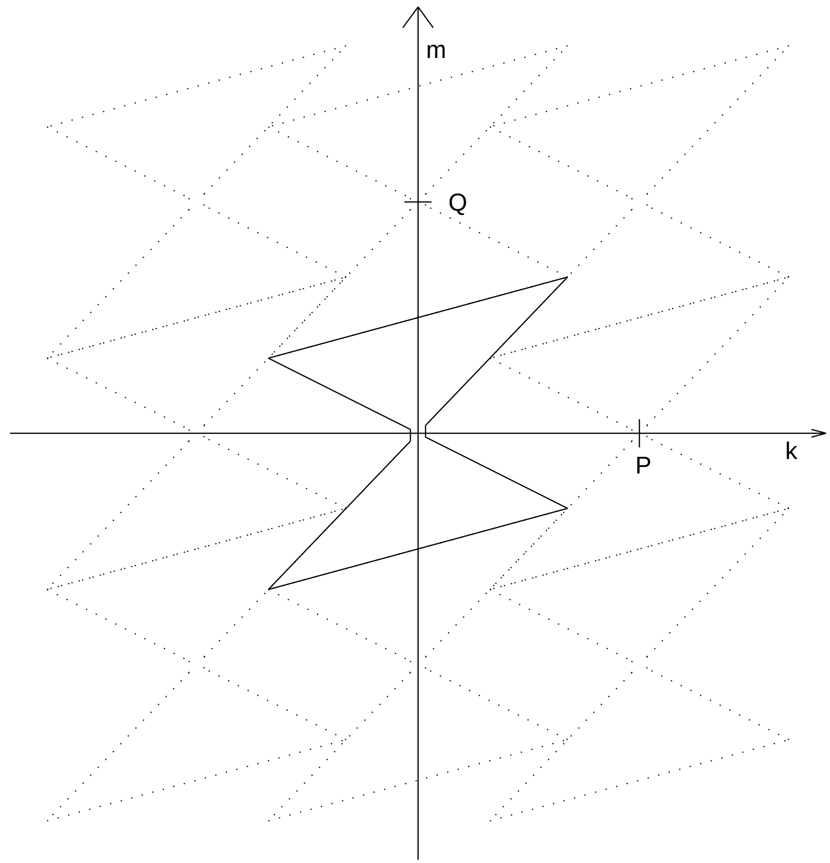
Example: Standard lattice (N=0).

$$\mathbf{L}_S(P,Q) = \{(j/P,\ l/Q)\,, \quad j = 0,\ldots,P-1$$

$$l = 0,\ldots,Q-1\}$$

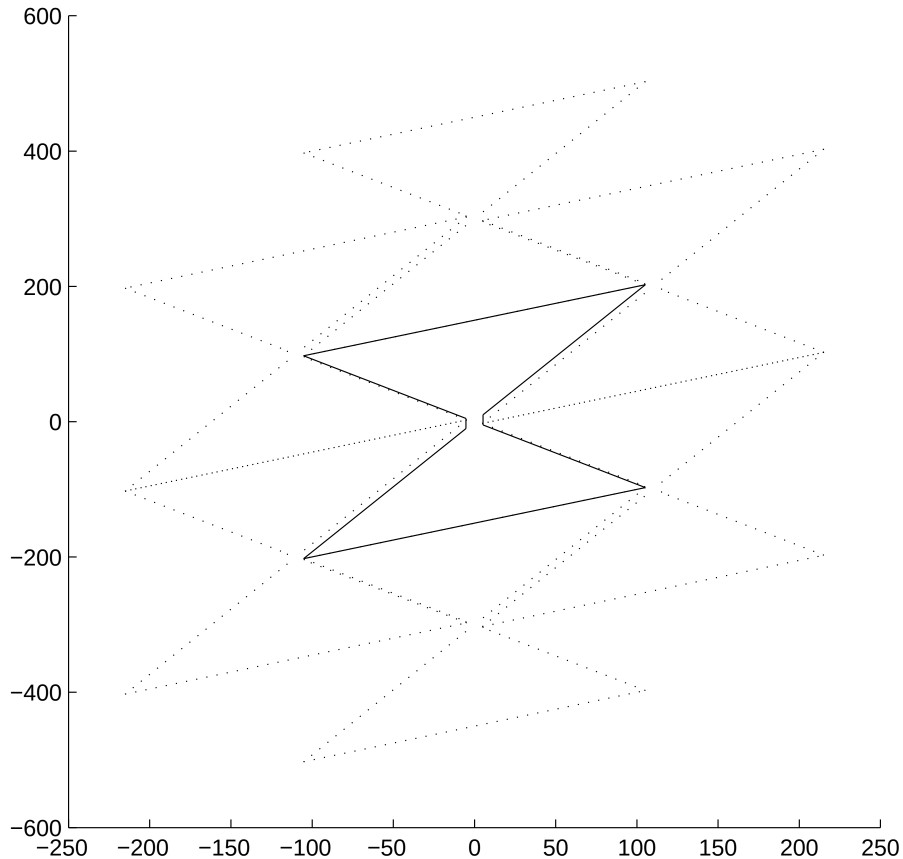
$$\mathbf{L}_S^{\perp} = \{(Pk,\ Qm)\,, \quad k,m \in \mathbb{Z}\}$$

Translates $K + \eta$, $\eta \in \mathbf{L}_S^{\perp}$



$$N = 0, \quad P = 156, \quad Q = 600, \quad |\mathbf{L}_S| = PQ = 93,600$$

More efficient lattice



$$N = 110, \quad P = 330, \quad Q = 200 \quad |\mathbf{L}| = PQ = 66,000$$

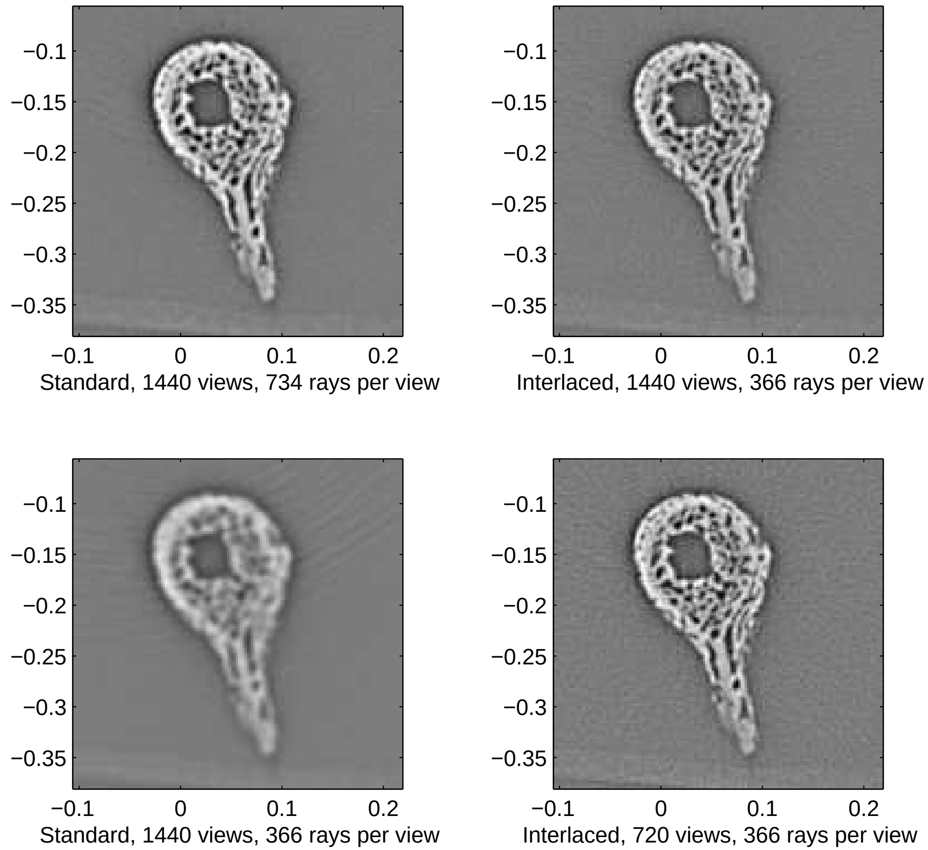
Reconstruction

Two basic strategies for reconstructing the function f from samples of Df.

1. **Direct.** Reconstruct directly from the sampled data. (This is the only possibility for local tomography.) Need for error analysis of reconstruction algorithm.

2. **Interpolated.** First use sampling theorem to interpolate data onto a denser grid. Then reconstruct from the interpolated data.

Direct local reconstruction of Λf

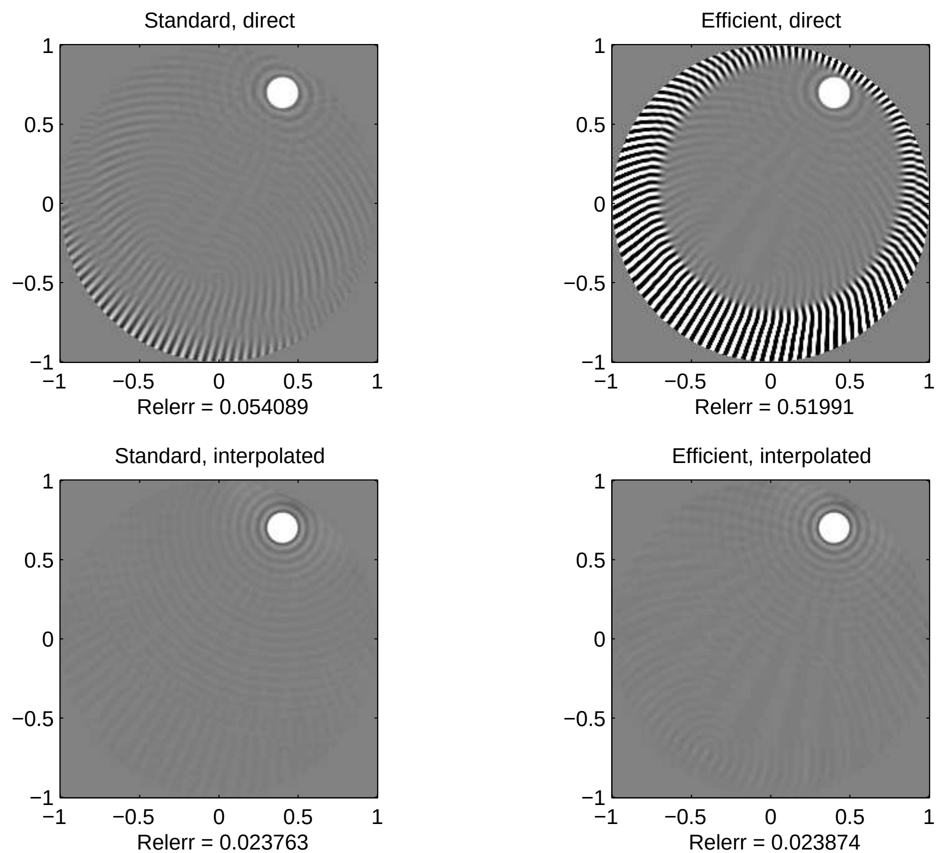


Left: Standard. Right: Efficient.

Parallel-beam. (F. & Ritman, 2000)

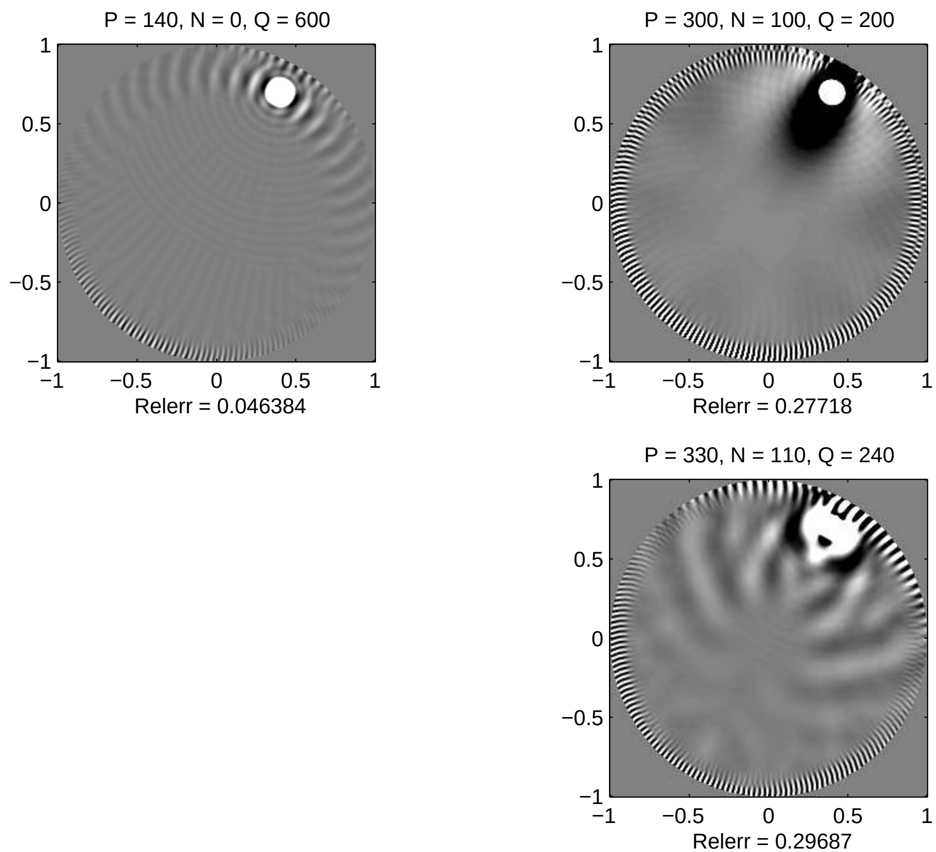
Direct vs. interpolated for fan-beam

$$f(x) = \left(1 - 100\,|x - x_o|^2\right)_+^3, \quad x_0 = (0.4,\, 0.7)$$



Theoretical explanation: Izen (2005)

Artifacts from undersampling

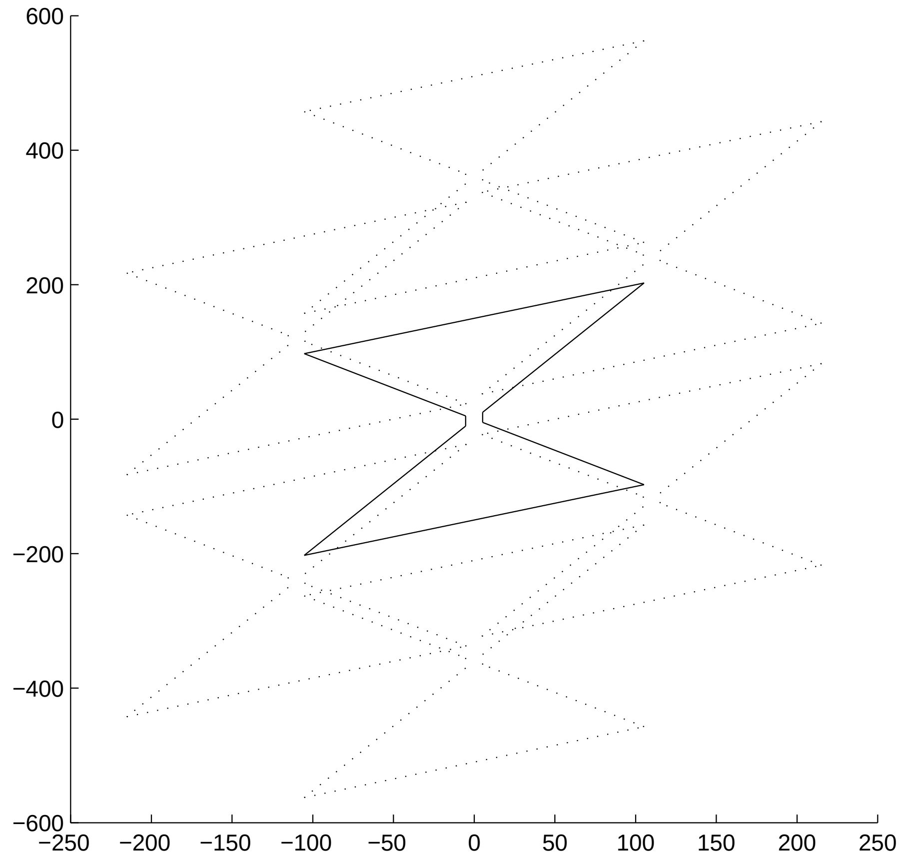


Left: Standard   Right: Efficient.

Top: P too small

Bottom: Q too large (!)

Undersampling by sampling more data



$$N = 110, P = 330, Q = 240 > 200.$$

Extensions of the Sampling Theorem

Goal: Extend to sampling sets which are not subgroups but still retain group structure.

- Periodic sampling: $S = \bigcup_{n=1}^{m} (x_n + \mathbf{L})$
  Sampling set is invariant with respect to shifts by elements of $\mathbf{L}$. (Well understood; see, e.g., Kohlenberg (1954), F. (1994), Izen (2005) and many other authors.)

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- Of course, these are not the only extensions! (See, e.g., Aldroubi & Unser, Feichtinger & Gröchenig, Zayed, . . .

Applications of periodic sampling in CT

- Additional efficient 2D sampling schemes
- "‘Preferred pitch"’ in 3D helical CT
- Higher resolution in 2D fan-beam CT

Applications of non-periodic sampling

- Higher resolution in 2D fan-beam CT

Exploiting Symmetry

$$Df(\beta, \alpha) = Df(\beta + 2\alpha + \pi, -\alpha).$$

Standard lattice with constant detector shift:

$$\mathbf{L}_S = \{(\beta_j, \alpha_l) : \beta_j = \frac{2\pi j}{P}, \;\; \alpha_l = \frac{\pi(l+\delta)}{Q},$$

$$j = 0, \ldots, P-1, \; l = -Q/2, \ldots, Q/2-1, \; \delta \geq 0\}$$

'Reflected lattice'

$$\mathbf{L}_R = \{(\beta_j + 2\alpha_l + \pi, -\alpha_l) : (\beta_j, \alpha_l) \in \mathbf{L}_S\}.$$

L_S and L_R are cosets of two different subgroups.

Key observation by Izen et al. (2005)

$\mathbf{L}_S \cup \mathbf{L}_R$ is a union of $Q/\gcd(P, Q/2)$ shifted copies of the smaller lattice

$$L_P \;=\; \{(2\pi j/P,\ \pi l/\gcd(P, Q/2)),$$

$$j = 0, \ldots, P-1,\ |l| \le \gcd(P, Q/2)\}$$

So the periodic sampling theorem can be applied!

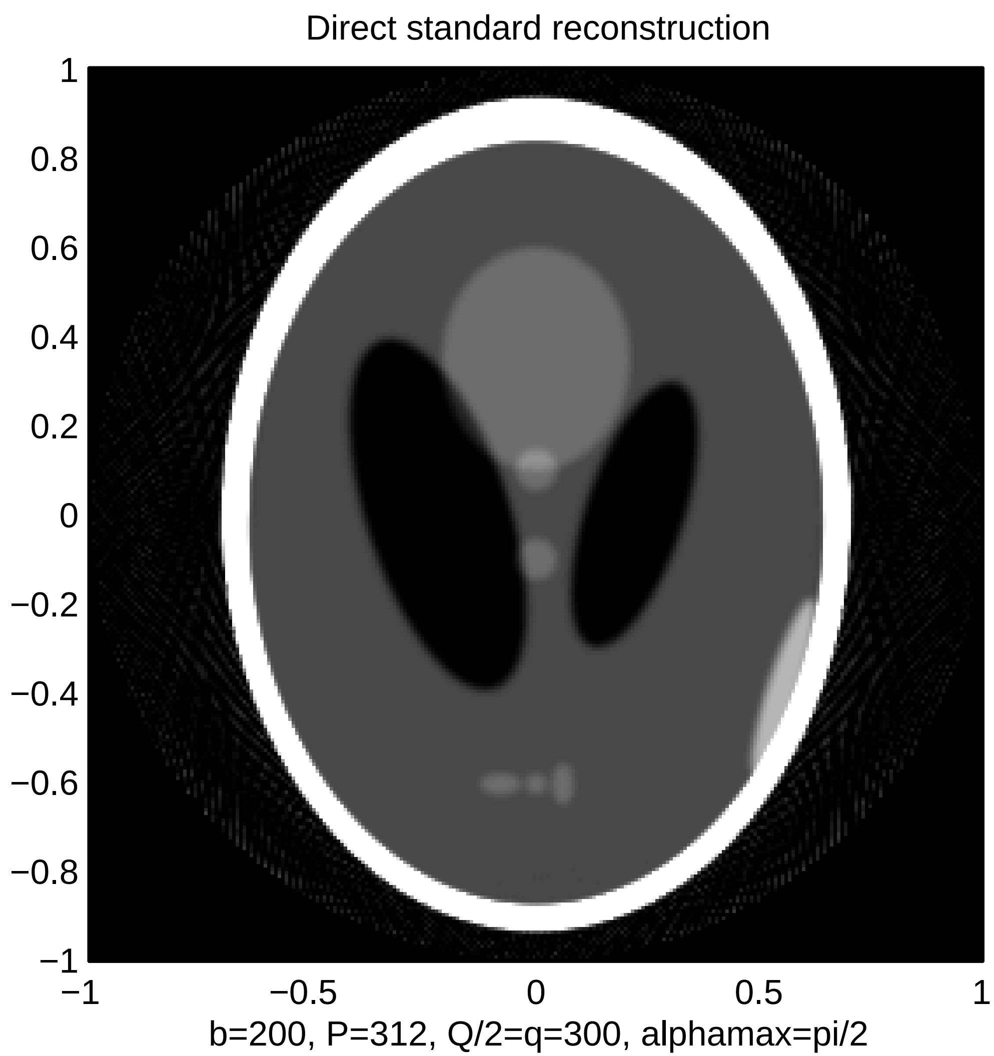
(Izen, Rohler & Sastry (2005) used an alternative reconstruction method.)

Thus effective bandwidth b can be doubled by only having to double P but not Q.

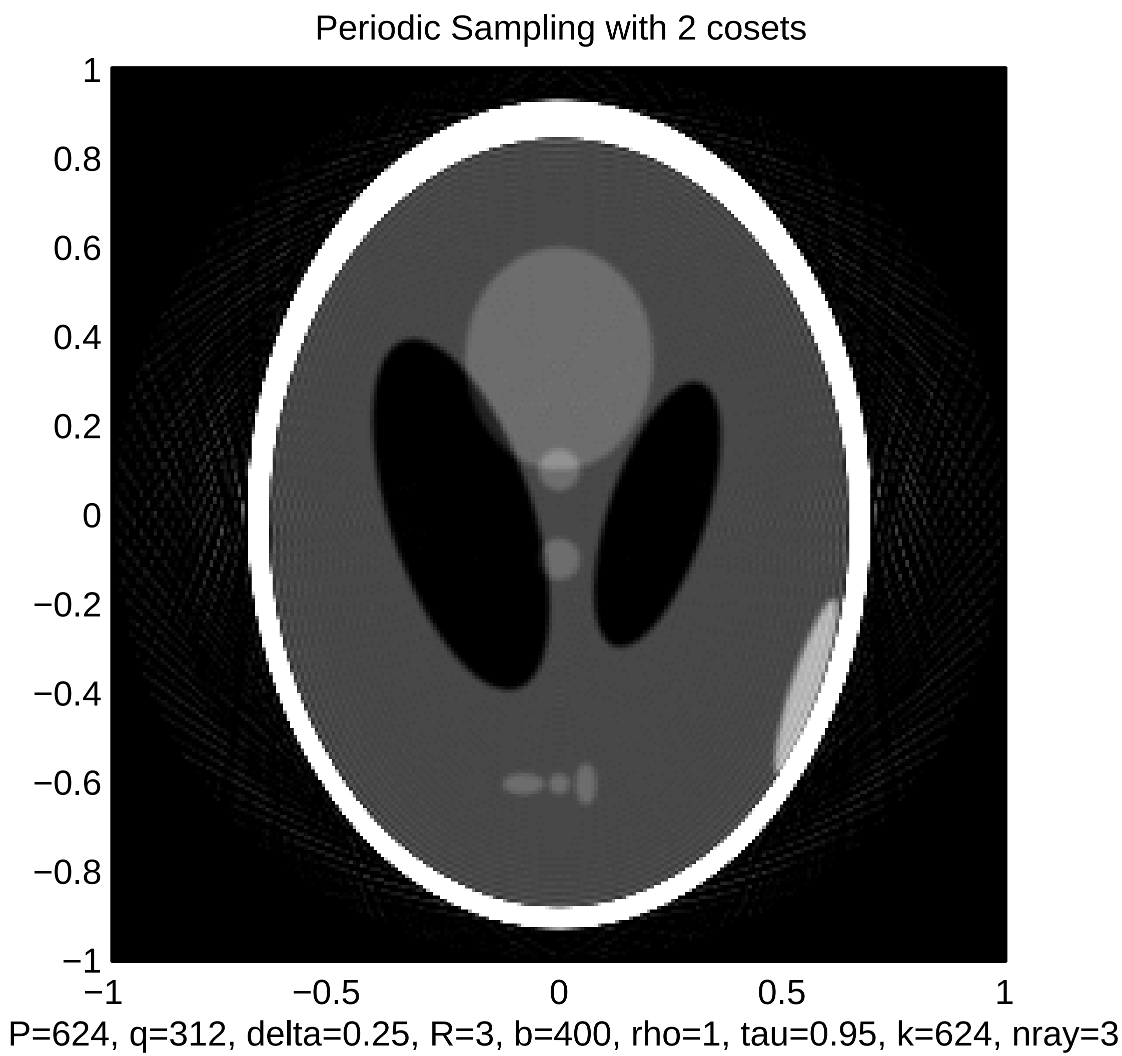
Use of periodic sampling

Mitchell (2005) used the periodic sampling theorem to incorporate the reflected data.

Standard (Direct) Reconstruction



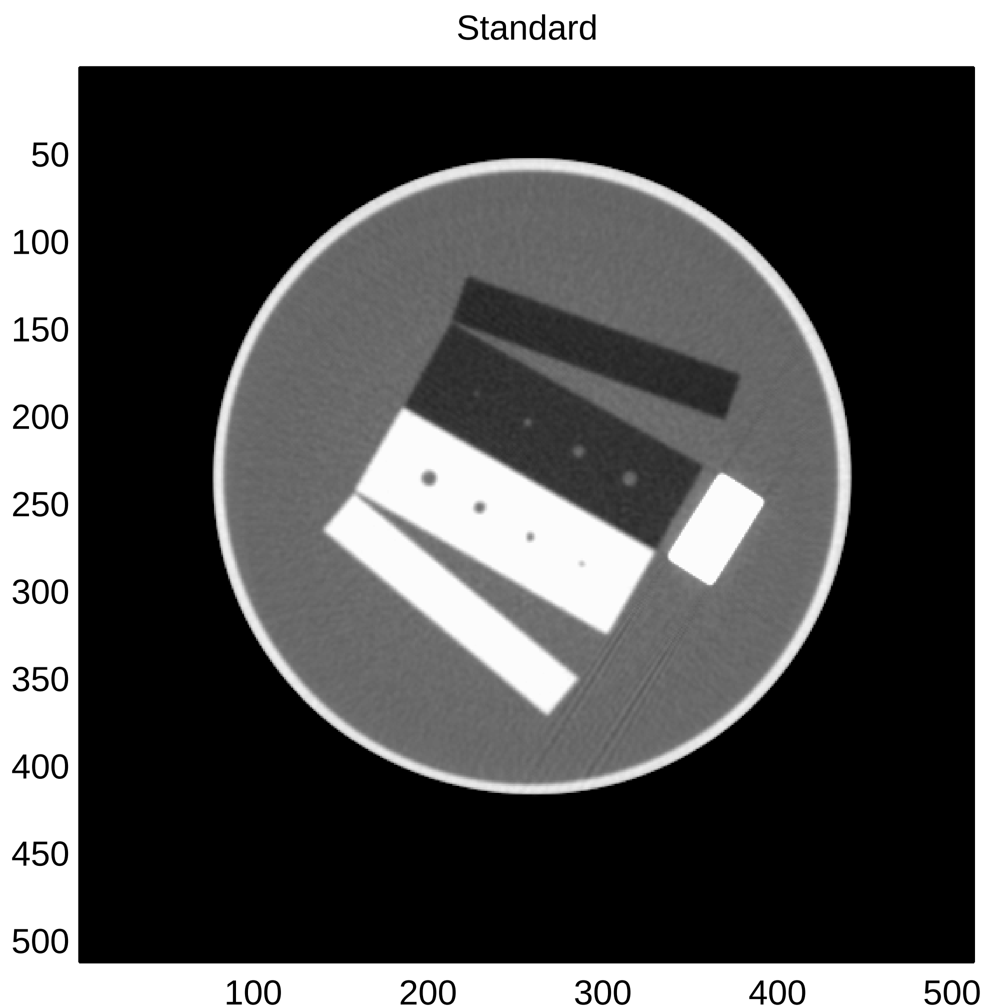
High-Resolution with periodic sampling



Use of non-periodic sampling

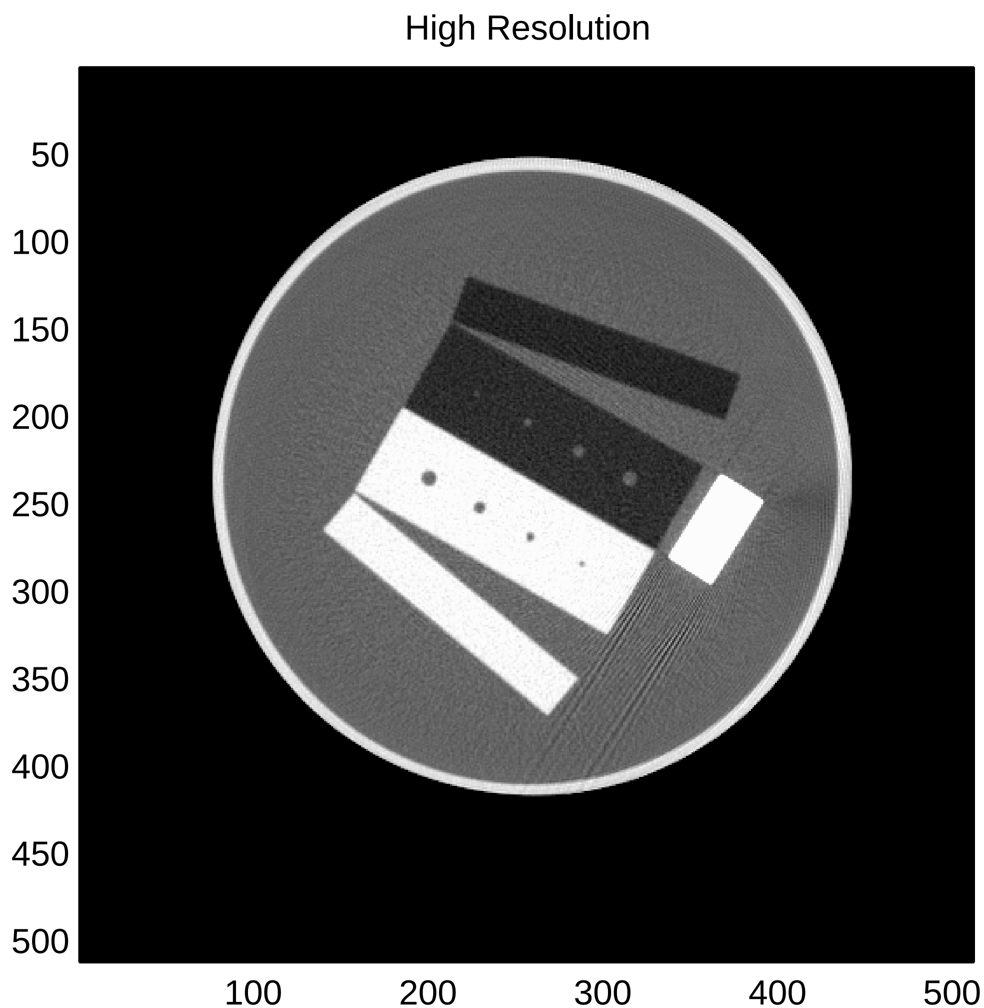
Gratton (2005) found an ingenious way to show that a non-periodic sampling theorem can be applied to the sampling set $\mathbf{L}_S \cup \mathbf{L}_R$ to exploit the symmetry.

Standard Rec. of Calibration Object



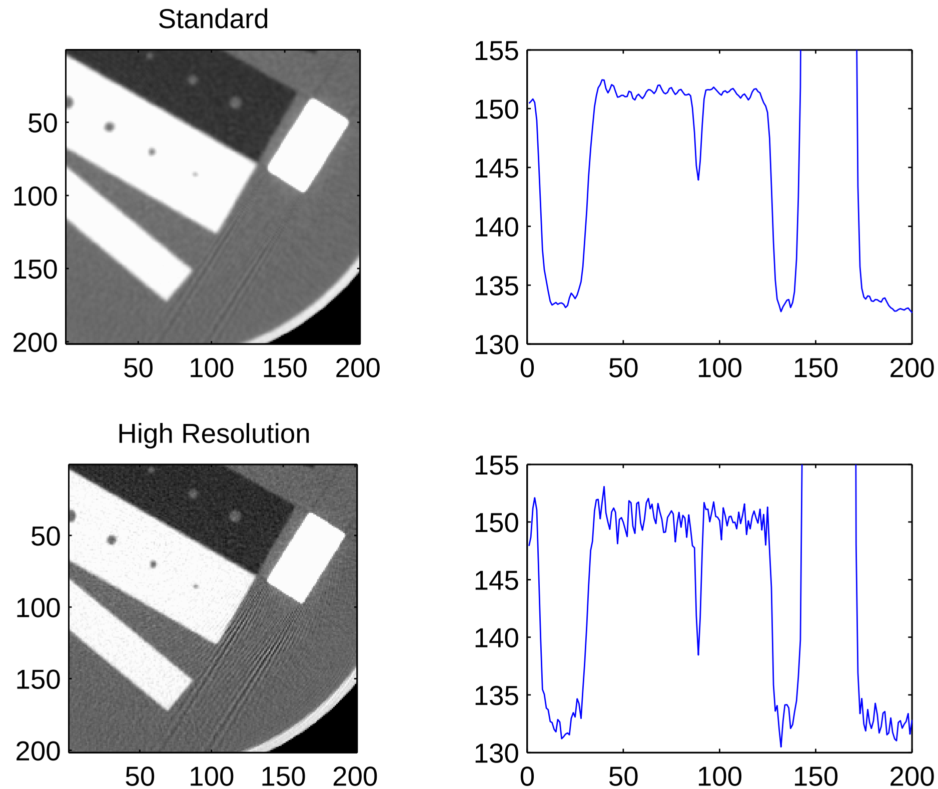
Real data.

High-Resolution Reconstruction



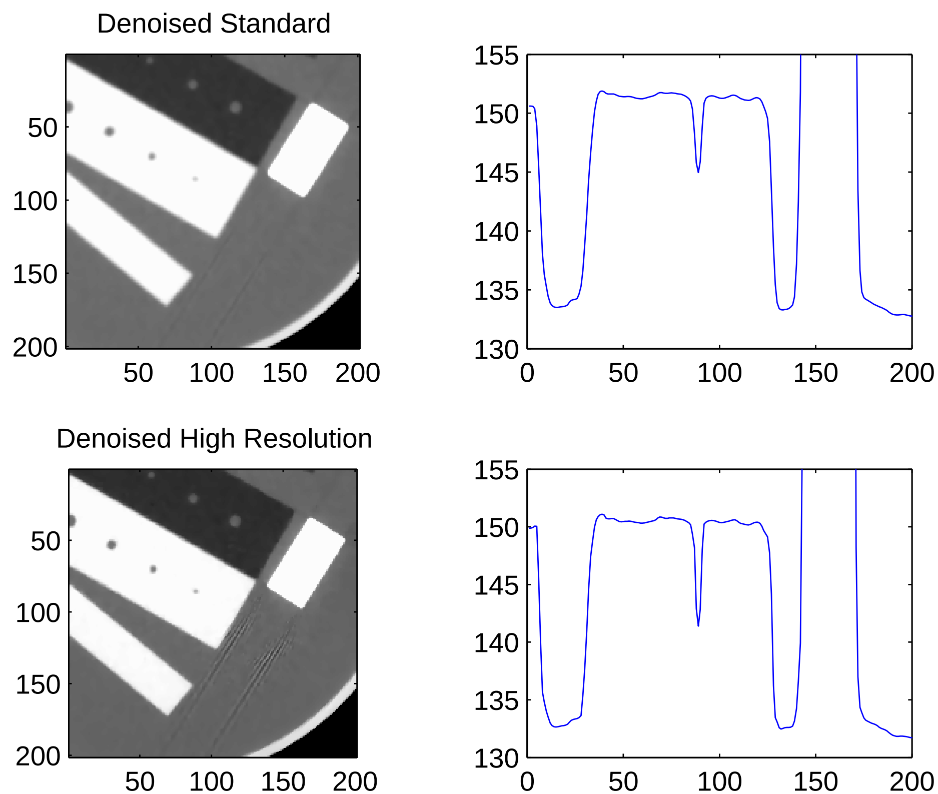
Using non-periodic sampling. Gratton (2005)

Standard vs. High-Resolution

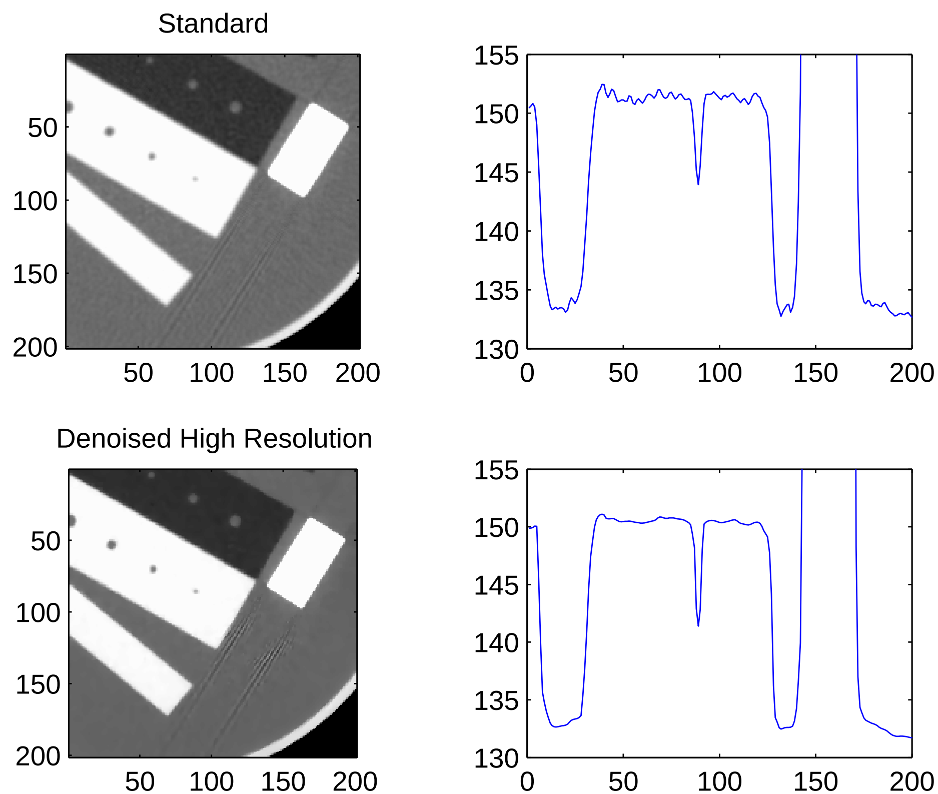


Challenge: Increased noise.
Possible remedy: Edge preserving denoising with TV-based algorithm (R. Hass, 2005).

Denoised Standard vs. Denoised High-R.



Standard vs. Denoised High-Resolution



Three dimensions

- Rich in potential sampling geometries. (Family of lines has 4 parameters. Need only a 3 parameter subfamily).
- Investigation of sampling issues has begun (see, e.g, Desbat and Grangeat (2004), Gratton (2005)) but very much remains to be done.

Conclusions

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- Both size and shape of the bandregion (spectrum) of g matter.
- Awareness and proper use of a-priori information is important. Here we only assumed that f has support in the unit disk and 'essential bandwidth' b.

Conclusions, continued

- Reconstruction can be done directly from the measured data or after interpolating data on a denser grid with the sampling theorem. The latter is often better.

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- Theory strictly applies only to sufficiently smooth functions, but turns out to give valuable guidance for other functions as well.
- Extensions of the classical sampling theorem allow to exploit the symmetry relation and also to construct additional efficient sampling schemes.
- Noise is a challenge for efficient sampling.
  Post-processing with edge preserving denoisers is promising.