On Reconstruction in Thermoacoustic Tomography

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Introduction

Thermoacoustic Tomography (TAT or TCT) is one of the promising new methods of medical imaging (e.g.,[16],[28]-[31]). TAT procedure: a short microwave (MW) or radiofrequency (RF) electromagnetic pulse is sent through a soft tissue. At each internal location x certain energy $H(x)$ is absorbed. The cancerous cells absorb several times more MW (or RF) energy than the normal ones. The resulting heating causes thermoelastic expansion of cancerous masses, which in turn creates a pressure wave. The acoustic wave is detected by ultrasound transducers outside the object. Assuming the sound speed c constant (a reasonable assumption for mammography), the acoustic signal detected at a moment t comes from the locations at the distance $r = ct$ from the transducer. Thus, one effectively measures the integrals of $H(x)$ over all spheres centered at the transducers' locations. To recover $H(x)$ and hence detect the tumors, one needs to invert the generalized Radon transform of H that integrates a function over all such spheres.

Definition 1 *The spherical Radon transform of f is defined as*

$$Rf(p,r) = \int_{|y-p|=r} f(y)d\sigma(y),$$

where $d\sigma(y)$ is the surface area on the sphere $|y-p| = r$.

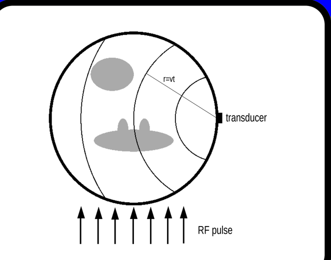
Dimension considerations show that it should be sufficient to run the transducers along a curve in the $2D$ case or a surface in $3D$. We thus denote

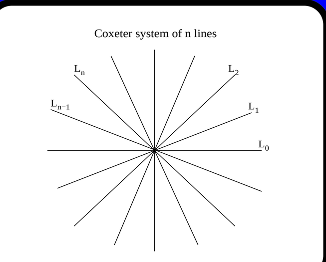
$$R_S f(p,r) = Rf(p,r)|_{p\in S}$$

for a surface (curve) S. The most implemented geometries of these surfaces (curves) have been spheres (circles), planes, and cylinders [28]-[30].

Such transforms have been previously studied in relation to problems of approximation theory, integral geometry, PDEs, sonar and radar imaging, and other applications ([1]–[7], [8]–[11], [12]–[13], [14]–[19], [20]–[26], [28]–[31]).

The central problems are: uniqueness of reconstruction, reconstruction formulas and algorithms, stability of the reconstruction, range conditions, incomplete data problems, and numerical implementation.





The Coxeter system of n lines $L_0, \dots, L_{n-1}$:

$$L_k = \{te^{i\pi k/n} | t \in \mathbb{R}\}$$

II. Uniqueness of Reconstruction

The following problem has been around for quite a while ([17, 18, 2, 9]).

Problem 1 *Describe all sets of injectivity for the circular Radon transform R on $C_c(\mathbb{R}^n)$.*

Here set S is called a set of injectivity for R, if the mapping R_S is injective on $C_c(\mathbb{R}^n)$.

A major breakthrough was made in the fundamental work by M. Agranovsky and E. T. Quinto [2] where they gave a complete solution for the 2D case.

Theorem 1 ***(M. L. Agranovsky, E. T. Quinto [2])*** *A set $S \in \mathbb{R}^2$ is a set of injectivity for the circular Radon transform R on $C_c(\mathbb{R}^2)$ if and only if S is not contained in any set of the form $\omega(\Sigma_N)\bigcup F$, where ω is a rigid motion in the plane, Σ_N is a Coxeter system of N lines and F is a finite set.*

The techniques involved microlocal analysis and known geometric structure of level sets of harmonic polynomials in $2D$. These methods, however, restrict applicability of the proof. The microlocal tool works at an edge of the support and is not applicable for non-compactly-supported functions. The geometry of level sets of harmonic polynomials, on the other hand, does not work in dimensions 3 and higher or on more general Riemannian manifolds. Thus, one looks for alternative approaches.

An alternative reformulation [2, 15] of the problem stems from the known relations between spherical integrals and the wave equation (e.g., [8, 14]). Consider the initial value problem:

$$u_{tt} - \triangle u = 0, \quad x \in \mathbb{R}^n, t \in \mathbb{R} \tag{1}$$

$$u|_{t=0} = 0, \quad u_t|_{t=0} = f. \tag{2}$$

Then

$$u(x,t) = \frac{1}{(n-2)!}\frac{\partial^{n-2}}{\partial t^{n-2}}\int_0^t r(t^2-r^2)^{(n-3)/2}(Rf)(x,r)dr, \quad t \ge 0.$$

The original problem of inverting R_S is equivalent to the problem of recovering $u_t(x,0)$ from the value of $u(x,t)$ on subsets of $S\times(-\infty,\infty)$.

Lemma 1 *[2, 15] A set S is a non-injectivity set for $C_c(\mathbb{R}^n)$ if and only if there exists a non-zero compactly supported continuous function f such that the solution $u(x,t)$ of the problem (1),(2) vanishes for any $x \in S$ and any t.*

In [3], following the ideas of [11], we used simple PDE tools (finite speed of propagation and domain of dependence for the wave equation), to prove new results concerning geometry of non-injectivity sets, as well as to re-prove some known results with simpler means. For instance, in $2D$ we reprove Theorem 1 in case of "not very bad" supports of functions (Condition A of [3] essentially requires the boundary of the support to have curvature bounded from below).

The sufficient conditions for S to be an injectivity set in [3] hold in any dimension. E.g.,

Theorem 2 ***(G.A. and P.K. [3])*** *Let $S \subset \mathbb{R}^n$ and $f(\neq 0) \in C_c(\mathbb{R}^n)$ be such that $R_S f = 0$. If the external boundary of the support of f is connected and satisfies Condition A, then surface S is ruled.[a]*

The conditions of the theorem are satisfied for instance when the support of f contains the boundary of its convex hull, or when the support's external boundary is connected and of the class C^2.

The approach of [3] bears a potential for considering non-compactly-supported functions and for generalizations to other Riemannian manifolds, in particular to the hyperbolic plane.

[a] A **ruled surface** is the union of a family of lines.

III. Range Description

As it is common for transforms of Radon type, the range of spherical Radon transform has infinite co-dimension in standard function spaces. Range descriptions are known to be very important for computed tomography for completing incomplete data, detecting and correcting measurement errors and hardware imperfections, recovering unknown attenuation, etc. In [4], we provided a complete range description in case of the spherical acquisition geometry in $2D$. The conditions include the recently found set of moment type conditions [25], which happened to be incomplete, as well as the rest of conditions that have less standard form.

Theorem 3 ***(G.A. and P.K. [4])*** *A function $g(p,r)$ on $S^1 \times \mathbb{R}$ can be representable as $R_S f(p,r)$ for some $f \in C_0^\infty(D)$, if and only if the following conditions are satisfied:*

1. $g \in C_0^\infty(S^1 \times (0,2))$.

2. For any n, the $2k$-th moment $\int_0^\infty r^{2k} g_n(r)dr$ of the n-th polar Fourier coefficient of g vanishes for integers $0 \le k < |n|$.

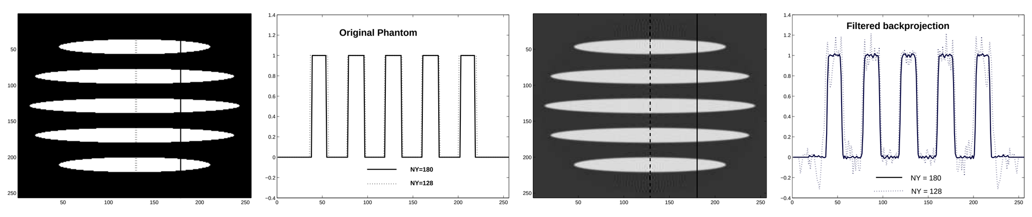
3. For any $n \in \mathbb{Z}$, function $\mathcal{H}_0\{g_n(r)/r\}(\sigma) = \int_0^\infty J_0(\sigma r)g_n(r)dr$ vanishes at any zero $\sigma \neq 0$ of Bessel function J_n.

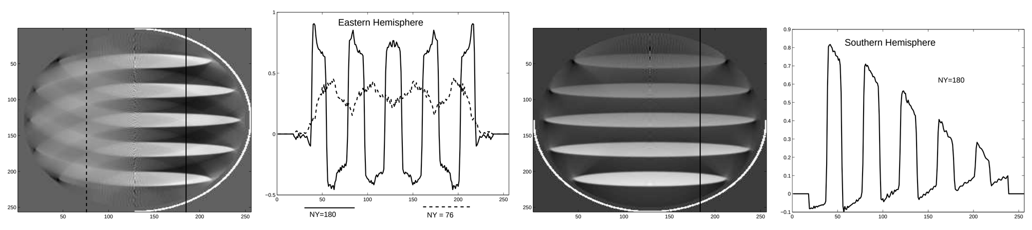
The proof of the theorem is based on the properties of Bessel functions and the Hankel transform, as well as on a known inversion formula [22] for the spherical Radon transform in $2D$. A similar approach should work for the spherical transform in higher dimensions.

IV. Implementation of Reconstruction Formulas

Explicit reconstruction formulas are known for the cases of planar, cylindrical, and spherical acquisition surfaces [7, 10, 11, 12, 13, 20, 21, 22, 23, 24]. Although there is no uniqueness of reconstruction for S being a plane [8, 14], functions supported on one side of the plane can be recovered. Filtered backprojection (FBP) type formulas are known when S is a hyperplane (in any dimension), as well as when S is a sphere in odd dimensions only. In a joint work of the first author with Sarah Patch [5, 6] it was shown that the recently found formula for spherical acquisition in $3D$ [11] allows successful numerical implementation. It is interesting to mention that the FBP formula reconstructs only functions supported inside the transducer sphere S and fails to do so outside.

In spite of absence of exact FBP formulas in $2D$, approximate ones that preserve all the singularities of the image can be easily written and then improved by successive iterative corrections [31].





Reconstruction formulas for spherical geometry in 3D
(D. Finch, Rakesh, and S.Patch, 2004)

Filtered Back Projection (FBP)

$$f(x) = \frac{-1}{8\pi^2}\left(\int_{|p|=1}\frac{1}{|x-p|}Rf''(p,|x-p|)dp\right)$$

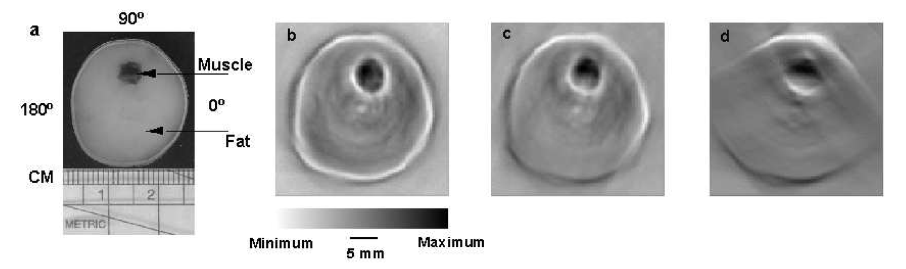
ρ-filtered Back Projection (RoBP)

$$f(x) = \frac{-1}{8\pi^2}\triangle_x\left(\int_{|p|=1}\frac{1}{|x-p|}Rf(p,|x-p|)dp\right)$$

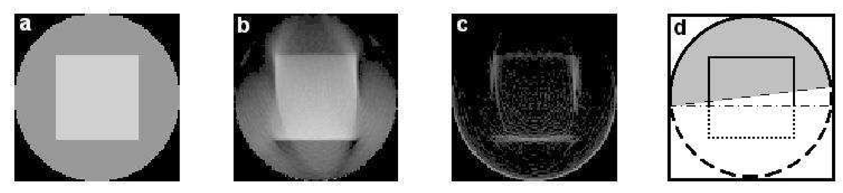
V. Incomplete Data - What is Visible?

If only a part of the sphere is used for data collection, the uniqueness theorem guarantees reconstructability. However, some parts of the image are not recoverable, due to arising strong instabilities. Namely, some parts of the wavefront set $WF(f)$ of the image will be lost [19, 27, 31].

In [31] we considered the limited-view problem for TCT in $2D$ and $3D$. It was shown analytically and numerically, that a point $(x,\xi) \in WF(f)$ can be stably detected from the Radon data, if and only if Rf includes data obtained from a sphere passing through x and co-normal to ξ. In other words, one can see only those parts of image singularities, that can be tangentially touched by spheres of integration centered at available transducer locations. The rest of the edges will be blurred. The numerical verification using exact inversion formulas was provided for $3D$ in in [6].



(a) Photograph of the experimental sample. (b)-(d) TAT reconstructions of the sample using detection arcs of 360 degrees, 202 degrees (from $-18°$ to $184°$), 92 degrees (from $50°$ to $142°$), and 360 degrees, respectively.



(a) A square phantom inside a circular detection curve. (b) FBP reconstruction. (c) Local tomography reconstruction. The boundary is emphasized. (d) The diagram showing the detection curve (solid part of the outer circle), the "visible" (solid) and "invisible" (dashed) boundaries of the objects, and the "detection region" (shaded).

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