Symplectification procedure for the equivalence problem of vector distributions

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A rank ℓ distribution D on an n-dim manifold M (or shortly (ℓ,n)-distribution) is a subbundle of TM, $D=\{D(q)\}_{q\in M}$,

$D(q)\subset T_qM$, $\dim D(q)=\ell$.

Locally $\exists\ \ell$ smooth vector fields $\{X_i\}_{i=1}^{\ell}$ s.t. $D(q)=\mathrm{span}\ \{X_1(q),\dots,X_\ell(q)\}$.

The group of germs of diffeomorphisms acts on germs of (ℓ,n)-distributions:

$$D \xrightarrow{\text{Action of } F} F_*D$$

Question: When two germs of distributions are equivalent?

$$D^j = D^{j-1} + [D, D^{j-1}] = \mathrm{span}\left\{\begin{array}{l}\text{all Lie brackets}\\ \text{of the fields } X_i\\ \text{of the length} \le j\end{array}\right\}$$

$(\dim D, \dim D^2, \dots, \dim D^j, \dots)$ – small growth vector (s.g.v)

Generic germs of (ℓ,n)-distributions are equivalent one to each other only when

$\ell = 1$, $\quad$ $\ell = n-1$, $\quad$ $(\ell,n) = (2,4)$

$\ell = 1 \downarrow$ rectification of vector fields

$\ell = n-1 \downarrow$ Darboux normal form

$(\ell,n) = (2,4) \downarrow$ Engel normal form

In all other cases generic (ℓ,n)-distributions have functional invariants (moduli)

The cases treated before are

1) $(\ell,n) = (2,5)$ or $(3,5)$ (s.g.v $(2,3,5)$ and $(3,5)$)
E. Cartan, 1910 – G_2-valued Cartan connection

2) $(\ell,n) = (3,6)$ (s.g.v. $(3,6)$)
R. Bryant, 1979 – B_3-valued Cartan connection

3) $n = \frac{\ell(\ell+1)}{2}$, $\ell \geq 3$ (s.g.v. $(\ell, \frac{\ell(\ell+1)}{2})$)
Tanaka school – B_ℓ-valued Cartan connection.

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<u>Our main result</u>: Construction of the canonical frame (absolute parallelism) for $(2,n)$-distributions, $n \geq 6$, from some generic class.

$\Downarrow$

the canonical frame for variational problem $\int L(x, y, y', \ldots, y^{(n)})\,dx$, $n \geq 3$ considered up to divergent equivalence and constant multiplier.

<u>Our main tool</u>: <u>A symplectification procedure</u> which allows to reduce the equivalence problem to diff. geometry of certain curves of flags of isotropic and coisotropic subspaces in a linear symplectic space.

Symplectification procedure

Step 1 To distinguish a special submanifold of T^*M endowed with the characteristic 1-foliation (the foliation of abnormal extremals)

Let $T^*M = \{(p,q) : q \in M, p \in T_q^*M\}$ be the cotangent bundle; ω be the canonical symplectic form on it;

$(D^j)^\perp = \{(p,q) : p \cdot v = 0 \ \ \forall v \in D^j(q)\}$ be the annihilator of the jth power D^j of D.

$$\boxed{\widetilde{W}_D = \{\lambda \in D^\perp : \ker \omega|_{D^\perp}(\lambda) \neq 0\}}$$

- If rank of D is odd, then $\widetilde{W}_D = D^\perp$;
- If rank $D = 2$, then $\widetilde{W}_D = (D^2)^\perp$;
- If rank $D = 2k$, then $\widetilde{W}_D \cap \{\text{the fiber of } D^\perp\}$ is a zero level set of a polynomial of ~~order~~ degree k (the Pfaffian of certain $2k \times 2k$ antisymmetric matrix)

$\widetilde{W}_D$ is odd dimensional. Define $W_D \subset \widetilde{W}_D$:

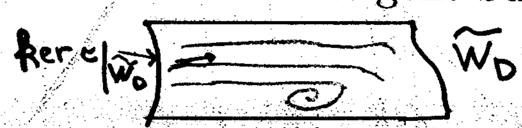
$$\boxed{W_D = \{\lambda \in \widetilde{W}_D : \ker \sigma|_{\widetilde{W}_D}(\lambda) \text{ is one-dimensional}\}}$$

W_D is open and dense in $\widetilde{W}_D$ for generic D.

Examples:

- If rank of D is 2, then $W_D = (D^2)^\perp/(D^3)^\perp$;
- If rank of D is 3, then $W_D = D^\perp/(D^2)^\perp$

The kernels of $\sigma|_{W_D}$ form *the characteristic line distribution* $\mathcal{C}$ on W_D. The integral curves of $\mathcal{C}$ are called *the* abnormal extremals *characteristic curves*, associated with D in the cotangent bundle.



$$\begin{tikzcd} \widetilde{W}_D \arrow[r, phantom, "\subset"] & T^*M \arrow[d, "\pi"] \\ W_D \arrow[u, phantom, "\cup"] & M \end{tikzcd}$$

Let $$\boxed{J(\lambda) = \{V \in T_\lambda W_D : \pi_* V \in D(\pi(\lambda))\}}$$

be the pull-back of D on W_D. Note that $\mathcal{C} \subset J$.

Step 2. To study the "dynamics" of $\mathcal{J}$ along C.

$$\mathcal{J}(\lambda) = \{ v \in T_\lambda W_D : \pi_* v \in D(\pi(\lambda)) \}$$

$$\mathcal{J}^{(i)} \overset{def}{=} \mathcal{J}^{(i-1)} + [C, \mathcal{J}^{(i-1)}], \quad \mathcal{J}^{(0)} = \mathcal{J}$$

Then $\dim \mathcal{J}^{(1)} - \dim \mathcal{J}^{(0)} \leq l-1$

$$\Downarrow$$

$$\dim \mathcal{J}^{(i+1)} - \dim \mathcal{J}^{(i)} \leq l-1$$

Let $\mathcal{J}_{(i)}(\lambda) \overset{def}{=} \left(\mathcal{J}^{(i)}(\lambda)\right)^{\angle} = \left\{ v \in T_\lambda W_D : \begin{array}{c} \sigma(v,w)=0 \\ \forall\, w \in \mathcal{J}^{(i)} \end{array} \right\}$

Then we get a flag in $T_\lambda W_D$:

$$\ldots \subset \mathcal{J}_{(2)} \subset \mathcal{J}_{(1)} \subset \mathcal{J}_{(0)} \subseteq \mathcal{J}^{(0)} \subset \mathcal{J}^{(1)} \subset \mathcal{J}^{(2)} \ldots$$

$$\dim \mathcal{J}_{(i+1)} - \dim \mathcal{J}_{(i)} \leq l-1$$

- $\operatorname{span}\{C, e\} \subset \mathcal{J}_{(i)}$ $\forall i$, where e is the Euler field (the generator of homotheties on the fibers of T^*M)

Let H be a vector field tangent to C.

$F^{(i)}(t) \stackrel{def}{=} (e^{-tH})_* J^{(i)}(e^{tH}\lambda) / \mathrm{span}\{C(\lambda), e(\lambda)\}$

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Then $\ldots \subset F_{(2)}(\cdot) \subset F_{(1)}(\cdot) \subset F_{(0)}(\cdot) \subseteq F^{(0)}(\cdot) \subset F^{(1)}(\cdot) \subset \ldots$

is the curve of flags in the linear symplectic space $\widehat{W}_\lambda = e(\lambda)^\angle / \mathrm{span}\{e(\lambda), C(\lambda)\}$ s.t.

1) $F^{(i+1)} = \frac{d}{dt} F^{(i)}$, $\dim F^{(i+1)} - \dim F^{(i)} \le l-1$

2) $F_{(i)} = \left(F^{(i)}\right)^\angle$, $\dim F_{(i+1)} - \dim F_{(i)} \le l-1$

3) $F^{(0)}$ is coisotropic (or Lagrangian) $\Rightarrow$
$F_{(0)}$ is isotropic (or Lagrangian) $\Rightarrow$
$F^{(i)}$ are coisotropic, $F_{(i)}$ are isotropic.

Any invariant of this curve (w.r.t. Symplectic Group) produces an invariant of the distribution D.

The case of rank 2 distribution

$\dim D^2 = 3$, $\dim D^3 > 3$, $\widetilde{W}_D = (D^2)^\perp$

$J_{(0)} = J^{(0)}$ $\qquad \dim J^{(1)} - \dim J^{(0)} = 1$

$$\ldots \subset J_{(2)} \subset J_{(1)} \subset J_{(0)} = J^{(0)} \subset J^{(1)} \subset J^{(2)} \subset \ldots$$

jump of dimensions ≤ 1

$$\boxed{\nu(\lambda) \overset{\text{def}}{=} \min\{i \in N : J^{(i)}(\lambda) = J^{(i+1)}(\lambda)\}, \ \lambda \in \widetilde{W}_D}$$

$$\Downarrow$$

$$1 \leq \nu(\lambda) \leq n-3$$

If $\nu(\lambda) = n-3$, the curve of flags

$$0 = F_{(n-3)}(\cdot) \subset \underbrace{F_{(n-4)}}_{\text{1-dim.}} \subset \ldots \subset F_{(1)} \subset \underbrace{F_{(0)} = F^{(0)}}_{\text{Lagranpian}} \subset F^{(1)} \subset \ldots \subset F^{(n-3)} = \widehat{W}_\lambda$$

is the curve of complete flags in $\widehat{W}_\lambda$

Moreover, the whole curve can be recovered (by differentiation) from $F_{(n-4)}(\cdot)$, which is the $\boxed{\text{curve in the projective space}}$ $\mathbb{P}\widehat{W}_\lambda$

$$\Downarrow$$

reduction to diff. geometry of curves in projective spaces.

$F_{(n-4)}(\cdot)$ is a curve in the projective space (Wilczynski, 1905) $\Rightarrow$ it possesses the canonical projective structure $\Rightarrow$ any characteristic curve of D is endowed with the canonical projective structure

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$\forall \lambda \in \widetilde{W}_D = (D^2)^\perp$: $\nu(\lambda) \overset{def}{=} \min\{i \in \mathbb{N} : J^{(i)}(\lambda) = J^{(i+1)}(\lambda)\}$

$\forall q \in M$: $\underbrace{m_D(q)}_{\text{the class of the distribution } D \text{ at } q} \overset{def}{=} \max\{\nu(\lambda) : \lambda \in W_D \cap \pi^{-1}(q)\}$

$m_D(q) \in \{1, 2, \dots, n-3\}$

Prop. Germs of $(2,n)$-distributions of the maximal class $n-3$ are generic.

Let P_λ be the set of all projective parameterizations $\varphi: \gamma \to \mathbb{R}$ of the char. curve γ, passing through λ s.t. $\varphi(\lambda)=0$.

$$\Sigma_D = \{(\lambda, \varphi) : \lambda \in (D^2)^\perp, \varphi \in P_\lambda\}$$

Σ_D is a principle bundle over $(D^2)^\perp$ with the structural group of all Möbius transformations, preserving 0.

In particular, $\dim \Sigma_D = 2n-1$

Theorem 1 For any $(2,n)$-distribution, $n>5$, of maximal class there exist 2 canonical frames on the corresponding $(2n-1)$-dim. manifold Σ_D, obtained one from another by a reflection. Any $(2,n)$-distribution, $n>5$, of maximal class with a $(2n-1)$-dim. group of symmetries is locally equiv. to the distribution, given by the following system of Pfaffian equations

$$\begin{cases} dy_i - y_{i+1}\,dx = 0 & 0 \le i \le n-4 \\ dz - y_{n-3}^2\,dx = 0 & \text{in } \mathbb{R}^n = (x, y_0, \dots, y_{n-3}, z) \end{cases} \qquad (*)$$

The algebra of infinitesimal symmetries of $(*)$ is isomorphic to a semidirect sum of $\mathfrak{gl}_2(\mathbb{R})$ and $(2n-5)$-dimensional Heisenberg algebra $\mathfrak{n}_{2n-5}$.

Remark The distribution $(*)$ is associated with the underdetermined ODE $\boxed{z'(x) = \left(y^{(n-3)}(x)\right)}$ w.r.t. $y(x)$ and $z(x)$.

Prop. Germs of $(2,n)$-distributions of the maximal class $n-3$ are generic.

For $n=5,6$ $m_D(q)$ is maximal $\Leftrightarrow$ the s.g.v. is $(2,3,5)$ and $(2,3,5,6)$ respectively. For $n\geq 7$ distributions with different s.g.v. may have the maximal class.

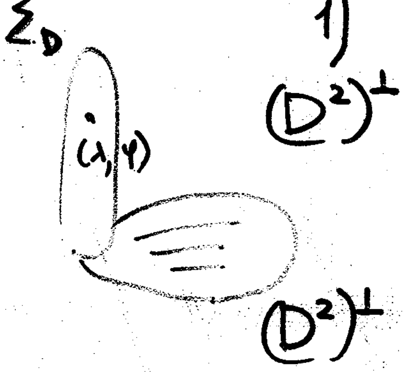
Remarks (on distributions of non-maximal class)

1) $m_D(q)=1 \Leftrightarrow \dim D^3(q)=4$

If D satisfies $\dim D^3(q)=4$ on some open set, then either it is the Goursat distribution or by the factorization of the ambient manifold by the characteristics of D^2 (or series of such factorizations) one can get a distrib. $\widetilde{D}$, satisfying $\dim \widetilde{D}^3=5$. $\Rightarrow$ the case of non-Goursat distributions of const. class 1 can be reduced to the case of distr. of class >1.

2) Do there exist completely nonholonomic rank 2 distributions of constant class $2\leq m\leq n-4$? The answer is negative at least for $m=2,3$.

Sketch of the proof of Theorem



1) The characteristic foliation on $(D^2)^\perp$ can be lifted to the parametrized foliation on Σ_D

$\Gamma_t(\lambda,\varphi) = (\varphi^{-1}(t), \varphi(\cdot) - t)$ - the flow on Σ_D

Let h be a vector field, generating this flow

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2) Let g_1, g_2 be some fundamental vector fields of the principal bundle Σ_D

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3) For the parameterized curve $t \to F_{(n-4)} \circ \varphi^{-1}(t)$ in $\mathbb{P}W_\lambda$ (where $W_\lambda = e(\lambda)^{\angle} / \mathrm{span}(e(\lambda), C(\lambda))$ one can construct the canonical (up to the sign) moving frame $(e_1, e_1', \dots, e_1^{(2n-6)})$ at $t=0$ $F_{(n-4)}$

Try to lift this frame to Σ_D.

To lift e_1 (the "generator" of the flag by differentiations) one needs to normalize 3 coefficients (near g_1, g_2, and e). It is possible for $n \geq 6$. This normalized lift + vector fields [illegible], g_1, g_2, h, [lift e_1, lift e_{2n-6}] - canon. frame

Application to equivalence of variational problems

To the underdetermined ODE

$z'(x) = L(x, y, \dots, y^{(m)})$ one can

associate $(2, m+3)$ distribution D_L

given by $\begin{cases} dy_i - y_{i+1}\,dx = 0, & 0 \le i \le m-1 \\ dz - L(x, y_0, \dots, y_m)\,dx = 0 \end{cases}$

in $R^{m+3} = (x, y_0, \dots, y_m, z)$

Consider two Lagrangians

$\lambda = L\,(x, y, y', \dots, y^{(m)})\,dx$

$\bar{\lambda} = \bar{L}\,(\bar{x}, \bar{y}, \bar{y}', \dots, \bar{y}^{(m)})\,d\bar{x}$

Prop $\overset{m \ge 3}{\vee}$ $\lambda \sim \bar{\lambda}$ up to divergent equivalence and multiplication by constant

$\left(\text{i.e. } \underbrace{\varphi^*}_{\text{the prolongation of a contact transf.}}(\bar{\lambda}) = \underbrace{\alpha}_{\text{constant}}\lambda + D\mu(x, y, \dots, y^{(n-1)}) + \text{contact form} \right)$

$\Longleftrightarrow$ $D_L \sim D_{\bar{L}}$ (as distributions)

$\Downarrow$

$\lambda = (y^{(m)})^2\,dx$ is the most symmetric Lagrangian w.r.t. the given equivalence with $2m+5$-dim. group of symmetries.

Connection with Tanaka theory

Let $pr: T^*M \to \mathbb{P}T^*M$ be the canon. projection to the projectivization of T^*M

$$\mathcal{J}_{(n-4)} \subset \mathcal{J}_{n-5} \subset \dots \subset \mathcal{J}_{(0)} = \mathcal{J}^{(0)} \subset \mathcal{J}^{(1)} \subset \dots \subset \mathcal{J}^{(n-3)}$$

$\cup$

V_{n-4} - vertical plane

$E_D = pr_* \mathcal{J}_{(n-4)}$, $\dim E(\bar\lambda) = 2 \quad \forall \lambda \in \mathbb{P}(D^2)^\perp$

$\oplus$

$$\underbrace{L_1 = pr_* C, \quad L_2 = pr_* V_{n-4}}$$

2 distinguished lines on rank 2 distribution E_D on $\mathbb{P}(D^2)^\perp$.

Prop If D is of maximal class, then E_D has the same symbol at any point of $pr(R_D)$, where $R_D = \{\lambda \in (D^2)^\perp : \nu(\lambda) = n-3\}$

The symbol is isomorphic to the graded Lie algebra, spanned by $(h, \varepsilon_1, \dots, \varepsilon_{2m}, \eta)$

s.t. $\quad \mathfrak{g}_{-1} = \operatorname{span}\{h, \varepsilon_1\}$, $\mathfrak{g}_{-i} = \mathbb{R}\varepsilon_i$, $2 \le i \le 2m$,

$\quad \mathfrak{g}_{-(2m+1)} = \mathbb{R}\eta$

$\quad \varepsilon_i = [h, \varepsilon_{i-1}]$, $[\varepsilon_i, \varepsilon_{2m-i+1}] = (-1)^{i+1}\eta$, where $m = n-3$

The universal prolong. is G_2 for $n=5$ & $gl_2(\mathbb{R}) \ltimes \mathfrak{n}_{2n-5}$, $n>5$