(Wünsch's) nearly invariant calculus for parabolic geometries

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joint work with Andreas Čap
(and others over years — M.G. Eastwood, V. Souček, A.R. Gover)

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Structure

1. Bibliography

2. Parabolic Geometries and Weyl connections
   - Cartan connections as analogies to affine geometry on manifolds
   - Weyl structures, connections and Rho–tensors

3. Nearly invariant calculus
   - Invariant and fundamenta differentials

Structure

Important sources

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- P. Gauduchon, Structures de Weyl-Einstein, espaces de twisteurs et variétés de type $S^1 \times S^3$, J. reine angew. Math. **469** (1995), 1–50.

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Cartan connection

Definition

Cartan geometries of type G/P are deformations of the homogeneous space $G \to G/P$ with the Maurer–Cartan form $\omega \in \Omega^1(G; \mathfrak{g})$:
absolute parallelism $\omega \in \Omega^1(\mathcal{G}, \mathfrak{g})$ on a principal fiber bundle $\mathcal{G} \to M$ with structure group P, enjoying suitable invariance properties with respect to the principal action of P:

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- $\omega_{|T_u\mathcal{G}} : T_u\mathcal{G} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{G}$ (the absolute parallelism condition).

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Morphisms of the parabolic geometries are principal fiber bundle morphisms $\varphi : \mathcal{G} \to \mathcal{G}'$ with $\varphi^*(\omega') = \omega$.

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- *Grading element* is the unique $E \in \mathfrak{g}_0$ with $\operatorname{ad} E_{|\mathfrak{g}_i} = i \cdot \operatorname{id}_{\mathfrak{g}_i}$ for all $i = -k, \dots, k$.

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$$\begin{aligned}\kappa(u)(X, Y) &= K(\omega^{-1}(X)(u), \omega^{-1}(Y)(u)) \\ &= [X, Y] - \omega(u)([\omega^{-1}(X), \omega^{-1}(Y)]).\end{aligned}$$

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- Curvature is a horizontal 2–form.

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On the underlying manifold M we obtain:

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- γ is a linear connection form on M, $\theta + \gamma$ is the affine Weyl connection
- P is a one–form on M valued in T^*M

Natural bundles and tractor bundles

Natural bundles

P–representation λ on a vector space $\mathbb{V}$ provides the homogeneous vector bundle $G \times_P \mathbb{V}$ and,
more generally, the associated vector bundles

$$\mathcal{V} = \mathcal{G} \times_P \mathbb{V}$$

with standard fiber $\mathbb{V}$ over all manifolds with a parabolic geometry of the type G/P. These are the *natural bundles* $\mathcal{V}$.

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Tractor bundles

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Note: The adjoint representation $\mathcal{G}$ provides the *adjoint tractor bundles* $\mathcal{A}$, the standard representation of a matrix group on $\mathbb{R}^n$ provides the *standard tractors* $\mathcal{T}$.

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Change of Weyl connections

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Here $\underline{i}$ is a multiindex $(i_1, \dots, i_k)$ with $i_j \geq 0$.
We put $\underline{i}! = i_1! \cdots i_k!$ and $\|\underline{i}\| = i_1 + 2i_2 + \dots + ki_k$, while $(-1)^{\underline{i}} = (-1)^{i_1+\dots+i_k}$

Tranformations – continued

Change of Rho–tensors

If $\hat{\sigma} = \sigma \cdot \exp \Upsilon$, then:

$$\hat{\mathsf{P}}_i(\xi) = \sum_{\|\underline{j}\|+\ell=i} \frac{(-1)^{\underline{j}}}{\underline{j}!} \mathrm{ad}(\Upsilon_k)^{j_k} \circ \cdots \circ \mathrm{ad}(\Upsilon_1)^{j_1}(\xi_\ell) +$$

$$\sum_{m=1}^{k} \sum_{\substack{\|\underline{j}\|+m=i \\ j_1=\cdots=j_{m-1}=0}} \frac{(-1)^{\underline{j}}}{\underline{j}!(j_m+1)} \mathrm{ad}(\Upsilon_k)^{j_k} \circ \cdots \circ \mathrm{ad}(\Upsilon_m)^{j_m}(\nabla_\xi \Upsilon_m) +$$

$$\sum_{\|\underline{j}\|+\ell=i} \frac{(-1)^{\underline{j}}}{\underline{j}!} \mathrm{ad}(\Upsilon_k)^{j_k} \circ \cdots \circ \mathrm{ad}(\Upsilon_1)^{j_1}(\mathsf{P}_\ell(\xi)).$$

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The derivatives

Invariant derivative

In analogy to the covariant derivatives we define $\nabla^\omega : C^\infty(\mathcal{G}, \mathbb{V}) \to C^\infty(\mathcal{G}, \mathfrak{g}_-^* \otimes \mathbb{V})$, $\quad \nabla s(u)(X) = \omega^{-1}(X)(u)\cdot s \in \mathbb{V}$. This operation **does not** transform sections into sections, in general!

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The extension of the invariant derivative to argument $X \in \mathfrak{g}$ $D^\omega : C^\infty(\mathcal{G}, \mathbb{V}) \to C^\infty(\mathcal{G}, \mathfrak{g}^* \otimes \mathbb{V})$, $\quad \nabla s(u)(X) = \omega^{-1}(X)(u) \cdot s \in \mathbb{V}$ is and invariant differential operator $\mathcal{A}^* \otimes \mathcal{V} \to \mathcal{V}$ called the *fundamental derivative*.

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- $D^\omega_{\xi+\zeta} s = \nabla_\xi s + \mathsf{P}(\xi) \bullet s - \zeta \bullet s$, ξ a vector in TM, ζ a vertical vector on $\mathcal{G}_0$.

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- on all tractor bundles, the fundamental derivative is related to the invariant linear connection $\nabla^{\mathcal{V}}_\xi s = \nabla_\xi s + \mathsf{P}(\xi) \bullet s + \xi \bullet s$.

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Both invariant and fundamental derivatives allow iteration!

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1st order invariant jets

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defines a natural operator (action given on $J^1\mathbb{V}$).

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higher orders

$C^\infty(\mathcal{G}, \mathbb{V})^P \ni s \mapsto \bar{j}_\omega^k s = (s, \nabla^\omega s, \dots, (\nabla^\omega)^k s) \in$
$C^\infty(\mathcal{G}, \mathbb{V} \oplus \cdots \oplus (\otimes^k \mathfrak{g}_-^* \otimes \mathbb{V}))^P$
is an invariant operator valued in semi–holonomic jets.

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Both invariant and fundamental derivatives allow iteration!

1st order invariant jets

$C^\infty(\mathcal{G}, \mathbb{V}) \ni s \mapsto (s, \nabla^\omega s) \in C^\infty(\mathcal{G}, \mathbb{V} \oplus (\mathfrak{g}_-^* \otimes \mathbb{V}))$
defines a natural operator (action given on $J^1\mathbb{V}$).

higher orders

$C^\infty(\mathcal{G}, \mathbb{V})^P \ni s \mapsto \bar{j}^k_\omega s = (s, \nabla^\omega s, \dots, (\nabla^\omega)^k s) \in$
$C^\infty(\mathcal{G}, \mathbb{V} \oplus \cdots \oplus (\otimes^k \mathfrak{g}_-^* \otimes \mathbb{V}))^P$
is an invariant operator valued in semi–holonomic jets.

Note: Symmetrization provides similar formulae for holonomic jets $j^k\omega s$, but these are **not equivariant**!

Bianchi and Ricci identities

The Bianchi identity in terms of the invariant differnetial is

$$0 = \sum_{\text{cycl}}\Big([\kappa(X,Y),Z]+\kappa([X,Y],Z)-\kappa(\kappa(X,Y),Z)-\nabla^{\omega}_{Z}\kappa(X,Y)\Big)$$

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while the Ricci identity reads

$$\nabla^{\omega}_{X} \nabla^{\omega}_{Y} s - \nabla^{\omega}_{Y} \nabla^{\omega}_{X} s = \nabla^{\omega}_{[X,Y]} - \nabla^{\omega}_{\kappa_{-}(X,Y)} s + \kappa_{\geq 0}(X, Y) \bullet s$$

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Note: also available in terms of the fundamental derivative.

The expansion

Proposition

Whatever expression we build of the invariant semiholonomic jets of sections and the Cartan curvatures in terms of the invariant derivatives (or the fundamental derivatives) will expand in expressions in the iterated covariant derivatives by means of the Weyl connections, the iterated covariant derivatives of the curvature tensor of the Cartan connections and of the iterated covariant derivatives of the P tensors.

Example:

$$\nabla^\omega_X \nabla^\omega_Y \tilde{s}(u) = (D^\omega)^2 \tilde{s}(u)(X, Y) = (\nabla_X + \mathsf{P}(v)(X))(\nabla s(v))(Y) =$$
$$(T\sigma \cdot \gamma^{-1}(u)(X) - \omega^{-1}(\mathsf{P}(v)(X))(u)) \cdot (\nabla^\omega \tilde{s})(u)(Y) =$$
$$\nabla_X \nabla_Y s(v) + \nabla_{[\mathsf{P}(v)(X), Y]_{\mathfrak{g}_-}} s(v) - [\mathsf{P}(v)(X), Y]_{\mathfrak{p}} \bullet s(v)$$

Main theorem

Proposition

The transformation properties of any expansion of a formula in invariant derivatives and Cartan curvatures are algebraic in Υ's.

Theorem

Every expression in affine invariants of the Weyl connections with algebraic transformation rules in Υ's is an expansion of some expression in invariant derivatives and the Cartan curvature.