Symmetries, Associated Conservation Laws and Double Reductions of PDEs

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When a differential equation admits a Noether symmetry, a conservation law is associated with this symmetry, and a double reduction can be achieved as a result of this association [2, 3, 4]. The association of conservation laws with Noether symmetries was extended to Lie Bäcklund symmetries [1] and nonlocal symmetries [5, 6, 7] recently. This opened the door to the extension of the theory on double reductions to partial differential equations (PDEs) that do not have a Lagrangian and therefore do not possess Noether symmetries.

We present a theorem to effect a double reduction of PDEs with two independent variables. Such a double reduction is possible when a PDE (or system of PDEs) admits a symmetry which is associated with a conservation law. Some examples are given.

Notation and Basics

We denote a q^{th} order ($q \geq 1$) system of p PDEs of 2 independent variables $x = (x^1, x^2)$ with components x^i and n dependent variables $u = (u^1, u^2, \ldots, u^n)$ with components u^α by

$$F^\beta\left(x, u, u_{(1)}, \ldots, u_{(q)}\right) = 0, \quad \beta = 1, \ldots, p. \qquad (1)$$

The system (1) admits a **Lie point symmetry** with generator

$$X = \xi^i(x,u)\frac{\partial}{\partial x^i} + \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

if $XF^\beta = 0$ on the solution space of (1).

Canonical coordinates (or similarity variables) of X are variables r, s, w in which $X = \frac{\partial}{\partial s}$. They are found by solving

$$Xs(x,u) = 1, Xr(x,u) = Xw(x,u) = 0.$$

Solutions of (1) invariant under X satisfy an ODE

$$G(r, w(r), w_r, w_{rr}, \ldots, w_{r^q}) = 0$$

which is found by the transformation of (1) into canonical coordinates. This is the **first reduction** [2, 4, 3].

The vector

$$T = (T^1(x, u, u_{(1)}, \ldots, u_{(q-1)}), T^2(x, u, u_{(1)}, \ldots, u_{(q-1)}))$$

is a **conserved vector** of (1) if

$$D_i T^i = 0 \qquad (2)$$

on the solution space of (1). The expression (2) is a **conservation law** of (1). Here D_i is the total derivative to x^i.

Association Criteria [1]

If X and T satisfy

$$X(T^i) + T^i D_j(\xi^j) - T^j D_j(\xi^i) = 0, \quad i = 1, 2, \qquad (3)$$

then X is **associated** with T.

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The Conservation Law in Similarity Variables

Let $(x^1, x^2) = (t, x)$. We define a nonlocal variable v by $T^t = v_x$, $T^x = -v_t$.

In the similarity variables $T^r = v_s, T^s = -v_r$, so that the conservation law is rewritten as

$$D_r T^r + D_s T^s = 0$$

with

$$T^s = \frac{T^t D_t(s) + T^x D_x(s)}{D_t(r)D_x(s) - D_x(r)D_t(s)}, \qquad (4)$$

and

$$T^r = \frac{T^t D_t(r) + T^x D_x(r)}{D_t(r)D_x(s) - D_x(r)D_t(s)}. \qquad (5)$$

The components T^x, T^t depend upon $(x, t, u, u_{(1)}, u_{(2)}, \ldots, u_{(q-1)})$ which means that T^s, T^r depend upon $(s, r, w, w_r, w_{rr}, \ldots, w_{r^{q-1}})$ for solutions invariant under X. Therefore $D_s T^s + D_r T^r = 0$ becomes $\frac{\partial T^s}{\partial s} + D_r T^r = 0$ or

$$T^r = \int \frac{\partial T^s}{\partial s} dr + f(s).$$

For T associated with X we have $XT^r = 0$ and $XT^s = 0$. Thus T^r and T^s are invariant under X [9]. This means

$$\frac{\partial}{\partial s}T^r = 0 \quad \text{and} \quad \frac{\partial}{\partial s}T^s = 0.$$

The conservation law in canonical coordinates becomes

$$D_r T^r = 0. \qquad (6)$$

Double Reduction - The main result

A PDE $F = 0$ of order q with two independent variables, which admits a Lie point symmetry X that is associated with a conserved vector T, **is reduced to an ODE of order $q - 1$**, namely $T^r = k$, where T^r is given by (5) for solutions invariant under X.

EXAMPLES

The **linear heat equation** $u_t = u_{xx}$ admits the symmetry generator

$$X = 2x\frac{\partial}{\partial x} + 4t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u}$$

which is associated with the conserved vector

$$T = \left(\frac{f'u}{t^{\frac{1}{4}}}, \frac{f''u}{t^{\frac{3}{4}}} - \frac{f'u_x}{t^{\frac{1}{4}}}\right)$$

where $f\left(y = x/\sqrt{t}\right)$ satisfies $4f'' = 2yf' - f$ in terms of Bessel functions of the second kind.

In **canonical coordinates**, $X = \partial/\partial s$ with

$$r = \frac{x}{\sqrt{t}}, \quad s = \frac{1}{4}\ln t, \quad v(r) = t^{\frac{1}{4}}u(t,x).$$

From (4) and (5),

$$T^s = -f'v, \quad T^r = (2f'r - 4f'')v + 4f'v_r.$$

Thus the **twice reduced** equation is

$$(2f'r - 4f'')v + 4f'v_r = k.$$

Integration gives the invariant solution

$$u = ct^{-\frac{1}{4}}f' \exp\left(-\frac{x^2}{4t} + \int \frac{k}{4f'}dr\right)\Big|_{r=x/\sqrt{t}}.$$

The **sine-Gordon equation** $z_{xy} = \sin z$ admits the scaling symmetry

$$X = \frac{\partial}{\partial x} + c\frac{\partial}{\partial y}$$

associated with $D_x(\cos z) + D_y\left(z_x^2/2\right) = 0$. The use of canonical coordinates $s = x, r = cx - y, z$ yield the **double reduction**

$$z_r = \frac{\pm\sqrt{2}}{c}\sqrt{k_1 + c\cos z}.$$

A solution invariant under X is thus

$$\pm\int \frac{cdz}{\sqrt{k_1 + c\cos z}} = \sqrt{2}(cx - y) + k_2.$$

The **BBM equation** $u_t = u_{txx} + uu_x$ has three conserved vectors

$T_1 = \left(u, -u_{tx} - \frac{1}{2}u^2\right)$,
$T_2 = \left(\frac{1}{2}u^2 + \frac{1}{2}u_x^2, -uu_{tx} - \frac{1}{3}\right)$ and
$T_3 = \left(\frac{1}{3}u^3, u_t^2 - u_{tx}^2 - u^2u_{tx} - \frac{1}{4}u^4\right)$

associated with the scaling symmetry

$$X = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}.$$

In the canonical coordinates $s = t, r = ct - x, u$ the r^{th} components of T_i lead to **double reductions** of the BBM equation

$T_1^r = cu - cu_{rr} - \frac{1}{2}u^2 = k_1$,
$T_2^r = \frac{1}{3}\left(cu^2 - u^3\right) + \frac{1}{2}c^3u_r^2 + cuu_{rr} = k_2$ and
$T_3^r = \frac{c}{3}u^3 - \frac{1}{4}u^4 + u^2 - c^2u_{rr}^2 + cu^2u_{rr} = k_3$.

Integration of these give solutions of the BBM equation

$$\int \exp\left(\left(\frac{u^3}{6c} - \frac{1}{2}u^2 + \frac{k_1}{c}u\right)\right) du = c_1(ct - x) + c_2,$$

$$\int \frac{c(c^2+2)(c^2+3)du}{\tilde{c}_1u^{-c^2} + 2(c^2+2)u^3 - 2(c^2+3)u^2 + 2\tilde{k}_2} = ct - x + c_2 \quad \text{and}$$

$$\int \left[u^3 \pm 6\int\sqrt{cu^3/3 + u^2 - k_3}du + k_4\right]^{-\frac{1}{2}} du = \pm\sqrt{3c}(x - ct) + k_5.$$

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