CR-manifolds, Pseudo product structures and 2^{nd} order ODE

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Second order ODE, complex holomorphic $y'' = B(x, y, y')$	3-dim manifolds ($\mathbb{C}^3$) 2 hol. dir. fields Z_1, Z_2 ($\Leftrightarrow$ 2 foliations by hol. curves) non-involutivity $[Z_1, Z_2] \notin \text{span}(Z_1, Z_2)$	CR-manifolds of dim 6, CR-dim=2 $D = \text{span}(Z_1, Z_2)$ $J\|_{Z_1} = \text{i}$, $J\|_{Z_2} = -\text{i}$ special Levi form curvature condition

$(x, y, p = \frac{dy}{dx})$
$Z_1 = \frac{\partial}{\partial p}$
$Z_2 = \frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + B\frac{\partial}{\partial p}$

2^{nd} foliation encodes all structure
$\dot{x} = 1$, $\dot{y} = p$,
$\dot{p} = B(x, y, p)$

Embedding into $\mathbb{C}^4$:
$y = \phi(x, c, d)$
$\bar{w}_2 = \phi(\bar{z}_2, z_1, w_1)$

Mappings

(prolonged) point transformations	mappings that preserve Z_1 and Z_2 up to scale	CR-mappings

Most symmetric objects

$y'' = 0$

$$\begin{array}{ccc} & \mathbb{F}(2,1) & \\ \swarrow & & \searrow \\ \mathbb{CP}(2) & & (\mathbb{CP}(2))^* \end{array}$$

$y = cx + d$

$\frac{w_1 - \bar{w}_2}{2\,\mathrm{i}} = z_1 \bar{z}_2$

(polarisation of $\operatorname{Im} w = |z^2|$)

$\boxed{\mathrm{PSL}(3, \mathbb{C})}$

acts by projective transformations

induced action

acts as polarisation of $\mathrm{SU}(2,1)$

Problem from CR geometry

Sphere $\operatorname{Im} w = |z|^2$ can be characterised by the property that there exist non-trivial automorphisms Φ with $\Phi(0) = 0$ and $\mathrm{d}\,\Phi(0) = \mathrm{id}$, namely

$$z \mapsto \frac{z + aw}{1 - 2\,\mathrm{i}\,\bar{a}z - (r + \mathrm{i}\,|a|^2)w}$$
$$w \mapsto \frac{w}{1 - 2\,\mathrm{i}\,\bar{a}z - (r + \mathrm{i}\,|a|^2)w}$$

Is the analogous statement true for (elliptic) CR manifolds of codimension 2?
What symmetries can appear? Describe manifolds with symmetries.

Easy:

- for given $B \Rightarrow$ (infinitesimal) automorphisms
- for given (infinitesimal) automorphism $\Rightarrow B$

infinitesimal automorphisms: $\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \phi\frac{\partial}{\partial p}$
with $\phi = \frac{\partial \eta}{\partial x} + p\left(\frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial x}\right) - p^2\frac{\partial \xi}{\partial y}$.
Solve

$$\xi\frac{\partial B}{\partial x} + \eta\frac{\partial B}{\partial y} + \phi\frac{\partial B}{\partial p} + \left(2\frac{\partial \xi}{\partial x} + 3p\frac{\partial \xi}{\partial y} - \frac{\partial \eta}{\partial y}\right)B - \frac{\partial^2 \eta}{(\partial x)^2}$$
$$+p\left(\frac{\partial^2 \xi}{(\partial x)^2} - 2\frac{\partial \eta}{\partial x \partial y}\right) + p^2\left(2\frac{\partial^2 \xi}{\partial x \partial y} - \frac{\partial \eta}{(\partial y)^2}\right) + p^3\frac{\partial^2 \xi}{(\partial y)^2} = 0$$

Difficulty: Don't know neither B nor ξ, η.

Power series methods, normal form (adapted to defining equation):

- such CR-manifold is torsion-free
- ϕ is (algebraic) deformation of

$$z_1 \mapsto \frac{z_1}{1 - 2\,\mathrm{i}\,\bar{a} z_1}, \qquad w_1 \mapsto \frac{w_1}{1 - 2\,\mathrm{i}\,\bar{a} z_1}$$
$$z_2 \mapsto z_2 + a w_2, \qquad w_2 \mapsto w_2$$

which corresponds to

$$x \mapsto x + ty, \quad y \mapsto y, \quad p \mapsto \frac{p}{1 + tp}.$$

Cartan geometry, normal form (adapted to symmetry):

- in normal coordinates ϕ is exactly as above
- hence, has the same topology (curve of fixed points $y = p = 0$)

Consequence

$$y\frac{\partial B}{\partial x} - p^2\frac{\partial B}{\partial p} + 3pB = 0$$

Solution

$$B(x, y, p) = F(y, x - \frac{x}{p})p^3$$

Due to regularity,

$$B(x, y, p) = \sum_{j=0}^{3} f_j(y)(y - px)^{3-j}p^j.$$

Question: Can they be equivalent to $B = 0$??

Theorem (Ezhov, S., 2005)

There are local coordinates x, y, p such that B takes the reduced form

$$B = f_0(y)(y - px)^3 + f_1(y)p(y - px)^2.$$

Two reduced forms are equivalent if and only if they are equivalent under

$$x \mapsto \frac{c_1 x}{1 - cy}, \quad y \mapsto \frac{c_2 y}{1 - cy},$$

i.e. the remaining freedom in coordinate choice consists of three complex parameters c_1, c_2, c.

Idea of proof.

Case 1: $y\frac{\partial}{\partial x}$ is the only shear-symmetry. Then preserving reduced form requires preserving $y\frac{\partial}{\partial x}$.
Such mappings satisfy a pair of second order ODE ($\Rightarrow$ 4 parameters).
But one parameter corresponds to the one-parametric shear symmetry group and

$$x \mapsto \frac{c_1 x}{1-cy}, \quad y \mapsto \frac{c_2 y}{1-cy}$$

is known to preserve the reduced form.

Case 2: There is a second shear symmetry. $\Rightarrow$ Study ODE with more symmetries.

Which of those ODE/CR-manifolds have additional automorphisms?

S. Lie classified ODE by (infinitesimal) symmetries

8 symmetries $\Rightarrow y'' = 0$
3 symmetries $\Rightarrow$ short list of ODE
2 symmetries $\Rightarrow y'' = f(y')$ (for $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$)
or $y'' = \frac{f(y')}{x}$ (for $\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$)
1 symmetry $\Rightarrow y'' = f(x, y')$.

Theorem (Ezhov, S.)

$y'' = 0$ and $y'' = (y - xy')^3$ are (up to equivalence) the only ODE with more than one shear symmetry.
$y'' = (y - xy')^3$ is $\mathrm{SL}(2, \mathbb{C})$ invariant.
The ODE with exactly two symmetries with fixed point 0 are equivalent to

$$y'' = y^k (y - xy')^3 \quad \text{or} \quad y'' = y^\ell y' (y - xy')^2 + C y^{2\ell+2} (y - xy')^3.$$

The additional automorphisms are

$$(k+2)x\frac{\partial}{\partial x} - 2y\frac{\partial}{\partial y} \quad \text{resp.} \quad (\ell+2)x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

The corresponding CR manifolds

$$w_1 + w_1^2 \bar{z}_2^2 - (\bar{w}_2 - z_1 \bar{z}_2)^2 = 0$$

$$\bar{w}_2 = z_1 \bar{z}_2 + \sqrt{k+2} w_1 \bar{z}_2 \int \frac{dy}{y^2 \sqrt{1 + w_1^2 \bar{z}_2^{k+2}}}$$

$\int \frac{dy}{y^2 \sqrt{1+w_1^2 \bar{z}_2^{k+2}}}$ is a hypergeometric function, which satisfies non-linear ODE $\Rightarrow$ (apparently new) relation

Shear invariant ODE with a second transitive symmetry:
The examples from above shifted in x-direction (by power series methods, comparison of parameters)
But corresponding coordinate transformations is highly transcendental.
Example: $f_0(y) \equiv 0$ and $f_1(y)$ satisfies

$$\left(\frac{f_1}{3f_1 + yf_1'}\right)'' = -2f_1.$$