MOVING FRAMES FOR LIE PSEUDOGROUPS

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What is New?

- Direct constructive algorithms for:
  - Invariant Maurer–Cartan forms
  - Structure equations
  - Moving frames
  - Differential invariants
  - Invariant differential forms
  - Invariant differential operators
  - Syzygies and recurrence formulas
- Basis Theorem for differential invariants

Further applications:

- Symmetry groups of differential equations
- Vessiot method of group splitting
- Congruence of curves, surfaces, etc. in homogeneous spaces
- Invariant variational bicomplex:
  - Calculus of variations
  - Gauge theories
  - Riemannian submanifolds
  - Characteristic classes of foliations, Gelfand-Fuks cohomology

Diffeomorphism Pseudogroup

M^m	m dimensional manifold
$\mathcal{D} = \mathcal{D}(M)$	pseudogroup of local diffeomorphisms of M
$\mathcal{D}^n \subset J^n(M, M)$	bundle of nth order jets, $0 \leq n \leq \infty$
$\sigma(j_z^n \varphi) = z$	source map
$\tau(j_z^n \varphi) = \varphi(z)$	target map

$\mathcal{D}^\infty$ inherits the structure $\Omega^{r,s}$ of a *variational bicomplex* from $J^\infty(M, M)$:

Coordinates on $\mathcal{D}^\infty$: $(z^a, Z^b, Z^b_{c_1}, Z^b_{c_1 c_2}, \ldots)$, where z^a, Z^b are local coordinates of M about the source and the target, and $Z^b_{c_1}$, $Z^b_{c_1 c_2}$, $\ldots$ stand for the derivative variables.

Horizontal forms:	dz^a
Contact forms:	$\theta^b_{c_1 \cdots c_p} = dZ^a_{c_1 \cdots c_p} - \sum_i^m Z^a_{c_1 \cdots c_p c_{p+1}} dz^{c_{p+1}}$

Split the exterior derivative $d = d_H + d_V$ into horizontal and vertical components.

Horizontal differential d_H:

$$d_H z^a = dz^a, \qquad d_H Z^b_{c_1\cdots c_p} = Z^a_{c_1\cdots c_p c_{p+1}} dz^{c_{p+1}},$$
$$d_H(dz^a) = 0, \qquad d_H \theta^b_{c_1\cdots c_p} = dz^{c_{p+1}} \wedge \theta^b_{c_1\cdots c_p c_{p+1}}.$$

Vertical differential d_V:

$$d_V z^a = 0, \qquad d_V Z^b_{c_1\cdots c_p} = \theta^b_{c_1\cdots c_p},$$
$$d_V(dz^a) = 0, \qquad d_V \theta^b_{c_1\cdots c_p} = 0.$$

Groupoid Structure: $\quad j^n_{\varphi(z)}(\psi) \cdot j^n_z(\varphi) = j^n_z(\psi \circ \varphi)$

Important submanifolds of $\mathcal{D}^n$:

$\mathcal{D}^n{}_{|z} = \sigma^{-1}(z) \subset \mathcal{D}^n$,

$\mathcal{F}^n = \cup_{z\in M} \mathcal{D}^n_z = \cup_{z\in M} \{ g^n \in \mathcal{D}^n \mid \sigma(g^n) = \tau(g^n) = z \}$.

$\mathcal{F}^n \to M$ is a principal fiber bundle with the structure group $\mathrm{GL}^n(m)$.

Maurer-Cartan forms for $\mathcal{D}^\infty$

These are represented by invariant contact forms on $\mathcal{D}^\infty$.

$\mathcal{D}$ acts on $\mathcal{D}^n$, $n \geq 0$, from both left and right by

$$L_\psi j^n_z \varphi = j^n_z(\psi \circ \varphi), \qquad R_\psi j^n_z \varphi = j^n_{\psi^{-1}(z)}(\varphi \circ \psi).$$

These actions preserve the bicomplex structure of $\mathcal{D}^\infty$ and so they commute with the horizontal and vertical differentials d_H, d_V.

Construction of right invariant forms on $\mathcal{D}^\infty$:

The target coordinate Z^b invariant under $R_\psi \implies$

$$\omega^b = d_H Z^b = \sum_c Z^b_c dz^c, \qquad \mu^b = d_V Z^b = dZ^b - Z^b_c dz^c,$$

are also invariant under R_ψ.

Operators of invariant differentiation:

$\mathbb{D}_{Z^a} = W_a^b \mathbb{D}_{z^b}$, where

$$\mathbb{D}_{z^b} = \frac{\partial}{\partial z^b} + \sum_{p \geq 0} Z^c_{c_1 \cdots c_p b} \frac{\partial}{\partial Z^c_{c_1 \cdots c_p}} \quad \text{and } W = (\nabla Z)^{-1}.$$

Right invariant coframe on $\mathcal{D}^\infty$:

$$\omega^a, \qquad \mu^a_{b_1 \cdots b_p} = \mathcal{L}_{\mathbb{D}_{Z^{b_1}}} \cdots \mathcal{L}_{\mathbb{D}_{Z^{b_p}}} \mu^a, \quad p \geq 0.$$

Example: $M = \mathbb{R}$. As a coordinate space

$$\mathcal{D}^\infty(\mathbb{R}) = \{z, Z, Z_z, Z_{zz}, \ldots, Z_{z^n}, \ldots\}.$$

Now

$$\mathbb{D}_z = \frac{\partial}{\partial z} + Z_z \frac{\partial}{\partial Z} + Z_{zz} \frac{\partial}{\partial Z_z} + \cdots .$$

Basic right invariant horizontal form $\omega = d_H Z = Z_z dz$.

The dual total differentiation $\mathbb{D}_Z = \dfrac{1}{Z_z} \mathbb{D}_z$ commutes with the group action.

Right invariant Maurer-Cartan forms:

$$\mu = \theta, \quad \mu_Z = \mathcal{L}_{\mathbb{D}_Z}\mu = (Z_z)^{-1}\theta_z,$$
$$\mu_{ZZ} = \mathcal{L}^2_{\mathbb{D}_Z}\mu = (Z_z)^{-3}(Z_z\theta_{zz} - Z_{zz}\theta_z), \ldots .$$

Similarly, the source coordinates z^a are invariant under L_ψ

$$\Longrightarrow \quad \tau^a = dz^a, \quad \vartheta^a = \sum_{c=1}^{m} W^a_c \theta^c \quad \text{are invariant under } L_\psi.$$

Left invariant coframe on $\mathcal{D}^\infty$:

$$dz^a, \quad \vartheta^a_{b_1\cdots b_p} = \mathcal{L}_{\mathbb{D}_{z^{b_1}}} \cdots \mathcal{L}_{\mathbb{D}_{z^{b_p}}} \vartheta^a, \quad p \geq 0.$$

Example: $M = \mathbb{R}$ again. Basic horizontal form $\tau = dz$. The total derivative $\mathbb{D}_z$ commutes with the left action of $\mathcal{D}(\mathbb{R})$. Left-invariant Maurer-Cartan forms

$$\vartheta = (Z_z)^{-1}\theta, \qquad \vartheta_z = (Z_z)^{-2}(Z_z\theta_z - Z_{zz}\theta),$$
$$\vartheta_{zz} = (Z_z)^{-3}(Z_z^2\theta_{zz} - 2Z_zZ_{zz}\theta_z - (Z_zZ_{zzz} - 2Z_{zz}^2)\theta), \ldots$$

Structure Equations

Taylor series method: Write

$$Z^a[\![h]\!] = \sum_{|J|\geq 0} \frac{1}{|J|!} Z^a_J h^J,$$

$$\theta^a[\![h]\!] = \sum_{|J|\geq 0} \frac{1}{|J|!} \theta^a_J h^J,$$

$$\mu^\alpha[\![H]\!] = \sum_{|J|\geq 0} \frac{1}{|J|!} \mu^a_J H^J.$$

Then

$$\mu[\![Z[\![h]\!] - Z]\!] = \theta[\![h]\!].$$

$$\Big(f(z+h) = f \circ \varphi^{-1}(\varphi(z) + (\varphi(z+h) - \varphi(z))), \quad \text{and}$$

$$\frac{\partial}{\partial z^a}(f \circ \varphi^{-1}(z)) = (\mathbb{D}_{Z^a} f) \circ \varphi^{-1}(z). \Big)$$

Write $H = Z[\![h]\!] - Z$. Then

$$dH = (\nabla_h Z[\![h]\!] dz - \omega) + (\theta[\![h]\!] - \theta),$$
$$d\theta[\![h]\!] = -\nabla\theta[\![h]\!] \wedge dz.$$

Apply d and eliminate $h \Longrightarrow$

Diffeomorphism Pseudogroup Structure Equations:

$$d\mu[\![H]\!] = \nabla_H \mu[\![H]\!] \wedge (\mu[\![H]\!] - dZ),$$
$$d\omega = -\omega \wedge \mu[\![0]\!].$$

Example: $M = \mathbb{R}$; invariant coframe ω, μ, μ_Z, ...

$$d\omega = -\omega \wedge \mu_Z,$$
$$d\mu_{Z^n} = -\mu_{Z^{n+1}} \wedge \omega + \sum_{i=0}^{n-1} \binom{n}{i} \mu_{Z^{i+1}} \wedge \mu_{Z^{n-i}}.$$

Pseudogroups

$\mathcal{G} \subset \mathcal{D}$ is a *pseudogroup* if

1. $\mathrm{id} \in \mathcal{G}$,
2. $\varphi,\, \psi \in \mathcal{G} \Rightarrow,\, \varphi \circ \psi \in \mathcal{G}$ where defined,
3. $\varphi \in \mathcal{G} \Rightarrow \varphi^{-1} \in \mathcal{G}$.

$\mathcal{G}$ is a *Lie* (or continuous) *pseudogroup* if, in addition, for all $n \geq N$,

4. $\mathcal{G}^n \subset \mathcal{D}^n$ is a subbundle,
5. $\mathcal{G}^{N+k} = \mathrm{pr}^k\, \mathcal{G}^N, \quad k \geq 1$,
6. $\varphi \in \mathcal{G} \Longleftrightarrow j_z^n \varphi \in \mathcal{G}^n$.

Remark: It follows that $\mathcal{G}_z^n \subset \mathcal{D}_z^n$ is a (finite dimensional) Lie group.

Examples of Lie pseudogroups.

1. Symmetry groups of Euler, Navier-Stokes, boundary layer, quasi-geostrophic equations and various other equations arising in fluid mechanics, magnetohydrodynamics, meteorology and geophysics.
2. Symmetry groups in gauge and field theories – Maxwell, Yang-Mills, conformal field theories, general relativity. Current/loop groups.
3. Symmetry groups of integrable equations in 2+1 dimensions – KP, Davey-Stewartson, and their variants.
4. Canonical transformations in Hamiltonian mechanics.
5. Configuration spaces:
   a) $Diff(\Omega) \to$ compressible fluid flow
   b) $Diff_{\mathrm{vol}}(\Omega) \to$ incompressible fluid flow
   c) Canonical transformations $\to$ Maxwell-Vlasov

6. Transformations preserving a geometric structure:
   a) Foliations
   b) Symplectic/Poisson structures
   c) Contact structures (quantomorphisms)
   d) Complex manifolds/real hypersurfaces
   e) G-structures
7. Image recognition – shape representation.
8. Finite dimensional Lie group actions.

Infinitesimal generators

A local vector field $\mathbf{v} \in \mathcal{X}(M)$ is a $\mathcal{G}$ *vector field* if the flow $\Phi_t^{\mathbf{v}} \in \mathcal{G}$ for all fixed t on some interval about 0.

Let $\mathcal{G}^n$ be given locally by $F_\alpha(z, Z^{(n)}) = 0$. Then a $\mathcal{G}$ vector field $\mathbf{v}$ satisfies

$$F_\alpha(z, \Phi_t^{\mathbf{v}\,(n)}) = 0 \quad \Longrightarrow \quad L_\alpha(z, j_z^n \mathbf{v}) = 0.$$

These are the *infinitesimal determining equations for* $\mathcal{G}$.

Invariant coframe for $\mathcal{G}^\infty$: Simply pull back ω^i, μ^a_J defined on $\mathcal{D}^\infty$ to $\mathcal{G}^\infty$.

Relations: On $\mathcal{G}^\infty$, the Maurer-Cartan forms μ^a_J satisfy the right invariant infinitesimal determining equations

$$L_\alpha(Z, \mu^a_J) = 0. \qquad (1)$$

Structure equations for $\mathcal{G}^\infty$ $\backsim$ structure equations for $\mathcal{D}^\infty$ modulo the relations (1).

The proof is based on the *Replacement Principle*:

If Ω is a right (or left) invariant form on $\mathcal{D}^\infty$ and

$$\Omega_{j^\infty_z \mathrm{id}} = \sum_a g_i(z) dz^a + \sum_{a,J} h^J_a(z) dZ^a_J,$$

then

$$\Omega = \sum_a g_i(Z) \omega^a + \sum_{a,J} h^J_a(Z) \mu^a_J.$$

$$(\text{or } \Omega = \sum_a g_i(z) dz^a + \sum_{a,J} h^J_a(z) \vartheta^a_J.)$$

EXAMPLE: The general symmetry transformation $\Psi_{F,G,H}$ of the KP equation

$$(u_t + \tfrac{3}{2}uu_x + \tfrac{1}{4}u_{xxx})_x + \tfrac{3}{4}\epsilon u_{yy} = 0, \qquad \epsilon = \pm 1,$$

is given by

$$T = F(t), \qquad Y = yF'(t)^{2/3} + G(t),$$

$$X = xF'(t)^{1/3} - \frac{2}{9}\epsilon\, y^2 \frac{F''(t)}{F'(t)^{2/3}} - \frac{2}{3}\epsilon\, y \frac{G'(t)}{F'(t)^{1/3}} + H(t),$$

$$U = \frac{u}{F'(t)^{2/3}} + \frac{2}{9}x\frac{F''(t)}{F'(t)^{5/3}} + \frac{4}{27}\epsilon\, y^2\left(\frac{4}{3}\frac{F''(t)^2}{F'(t)^{8/3}} - \frac{F'''(t)}{F'(t)^{5/3}}\right)$$
$$+ \frac{4}{9}\epsilon\, y\left(\frac{G'(t)F''(t)}{F'(t)^{7/3}} - \frac{G''(t)}{F'(t)^{4/3}}\right) + \frac{2}{3}\frac{H'(t)}{F'(t)} + \frac{2}{9}\epsilon\frac{G'(t)^2}{F'(t)^2}\,,$$

with the composition rule $\Psi_{\hat{F},\hat{G},\hat{H}} = \Psi_{F,G,H} \circ \Psi_{f,g,h}$, where

$$\hat{F} = F \circ f, \qquad \hat{G} = g(F' \circ f)^{2/3} + G \circ f,$$

$$\hat{H} = h(F' \circ f)^{1/3} - \frac{2}{9}\epsilon\frac{g^2\, F'' \circ f}{(F' \circ f)^{2/3}} - \frac{2}{3}\epsilon\frac{g\, G' \circ f}{(F' \circ f)^{1/3}} + H \circ f.$$

The infinitesimal generators

$$\mathbf{v} = \tau(t,x,y,u)\partial_t + \xi(t,x,y,u)\partial_x + \eta(t,x,y,u)\partial_y + \varphi(t,x,y,u)\partial_u,$$

where

$$\tau = f(t), \qquad \xi = \tfrac{1}{3}xf'(t) - \tfrac{2}{9}\epsilon\, y^2 f''(t) - \frac{2}{3}\epsilon\, yg'(t) + h(t),$$

$$\eta = \tfrac{2}{3}yf'(t) + g(t),$$

$$\varphi = -\tfrac{2}{3}uf'(t) + \tfrac{2}{9}xf''(t) - \tfrac{4}{27}\epsilon\, y^2 f'''(t) - \tfrac{4}{9}\epsilon\, yg''(t) + \tfrac{2}{3}h'(t),$$

satisfy

$$\tau_x = 0, \quad \tau_y = 0, \quad \tau_u = 0, \quad \xi_x - \tfrac{1}{3}\tau_t = 0, \quad \xi_y + \tfrac{2}{3}\,\epsilon\,\eta_t = 0,$$

$$\xi_u = 0, \quad \eta_x = 0, \quad \eta_y - \tfrac{2}{3}\tau_t = 0, \quad \eta_u = 0, \quad \varphi - \tfrac{2}{3}\,\xi_t + \tfrac{2}{3}\,u\tau_t = 0,$$

along with all their differential consequences.

The Maurer-Cartan forms satisfy the "lifted infinitesimal determining equations"

$$\mu^t_X = 0, \quad \mu^t_Y = 0, \quad \mu^t_U = 0, \quad \mu^x_X - \tfrac{1}{3}\mu^t_T = 0, \quad \mu^x_Y + \tfrac{2}{3}\epsilon\,\mu^y_T = 0,$$

$$\mu^x_U = 0, \quad \mu^y_X = 0, \quad \mu^y_Y - \tfrac{2}{3}\mu^t_T = 0, \quad \mu^y_U = 0,$$

$$\mu^u - \tfrac{2}{3}\mu^x_T + \tfrac{2}{3}U\mu^t_T = 0, \quad \ldots$$

Basis of linearly independent Maurer–Cartan forms:

$$\omega^1 := \mu^t, \quad \omega^2 := \mu^x, \quad \omega^3 := \mu^y, \quad \omega^4 := \mu^u,$$
$$\omega^5 := \mu^t_T = \mu^t_{1,0,0,0}, \quad \omega^6 := \mu^y_T = \mu^y_{1,0,0,0},$$
$$\alpha^i := \mu^u_{i,0,0,0}, \quad \beta^i := \mu^u_{i-1,1,0,0}, \quad \gamma^i := \mu^u_{i-1,0,1,0}.$$

Structure equations for the invariant horizontal forms

$$d\sigma^t = \omega^5 \wedge \sigma^t,$$
$$d\sigma^x = (\tfrac{3}{2}\omega^4 + U\omega^5) \wedge \sigma^t + \tfrac{1}{3}\omega^5 \wedge \sigma^x - \tfrac{2}{3}\epsilon\,\omega^6 \wedge \sigma^y,$$
$$d\sigma^y = \omega^6 \wedge \sigma^t + \tfrac{2}{3}\omega^5 \wedge \sigma^y,$$
$$d\sigma^u = \alpha^1 \wedge \sigma^t + \beta^1 \wedge \sigma^x + \gamma^1 \wedge \sigma^y - \tfrac{2}{3}\omega^5 \wedge \sigma^u.$$

Structure equations for the Maurer-Cartan forms

$$
\begin{aligned}
d\omega^1 &= -\,\omega^5\wedge\sigma^t,\\
d\omega^2 &= -\,(\tfrac{3}{2}\omega^4+U\omega^5)\wedge\sigma^t-\tfrac{1}{3}\omega^5\wedge\sigma^x+\tfrac{2}{3}\epsilon\,\omega^6\wedge\sigma^y,\\
d\omega^3 &= -\,\omega^6\wedge\sigma^t-\tfrac{2}{3}\omega^5\wedge\sigma^y,\\
d\omega^4 &= -\alpha^1\wedge\sigma^t-\beta^1\wedge\sigma^x-\gamma^1\wedge\sigma^y+\tfrac{2}{3}\omega^5\wedge\sigma^u,\\
d\omega^5 &= -\,\tfrac{9}{2}\beta^1\wedge\sigma^t,\\
d\omega^6 &= -\,\tfrac{1}{3}\omega^5\wedge\omega^6+\tfrac{9}{4}\epsilon\,\gamma^1\wedge\sigma^t-3\beta^1\wedge\sigma^y,\\
d\alpha^1 &= -\,\tfrac{3}{2}\omega^4\wedge\beta^1-\tfrac{5}{3}\omega^5\wedge\alpha^1-U\omega^5\wedge\beta^1-\omega^6\wedge\gamma^1\\
&\qquad -\alpha^2\wedge\sigma^t-\beta^2\wedge\sigma^x-\gamma^2\wedge\sigma^y+3\beta^1\wedge\sigma^u,\\
d\beta^1 &= -\,\omega^5\wedge\beta^1-\beta^2\wedge\sigma^t,\\
d\gamma^1 &= -\,\tfrac{4}{3}\omega^5\wedge\gamma^1+\tfrac{2}{3}\epsilon\,\omega^6\wedge\beta^1+\tfrac{4}{3}\epsilon\,\beta^2\wedge\sigma^y-\gamma^2\wedge\sigma^t,\\
d\alpha^2 &= -\,3\omega^4\wedge\beta^2-\tfrac{8}{3}\omega^5\wedge\alpha^2-2U\omega^5\wedge\beta^2-2\omega^6\wedge\gamma^2+9\alpha^1\wedge\beta^1\\
&\qquad +3\beta^2\wedge\sigma^u-\alpha^3\wedge\sigma^t-\beta^3\wedge\sigma^x-\gamma^3\wedge\sigma^y,\\
d\beta^2 &= -\,2\omega^5\wedge\beta^2-\beta^3\wedge\sigma^t,\\
d\gamma^2 &= -\,\tfrac{7}{3}\omega^5\wedge\gamma^2+2\epsilon\,\omega^6\wedge\beta^2-\tfrac{9}{2}\beta^1\wedge\gamma^1-\gamma^3\wedge\sigma^t+\tfrac{4}{3}\epsilon\,\beta^3\wedge\sigma^y,\\
&\ \vdots
\end{aligned}
$$

Fix the values of the target coordinates T, X, Y, U to get

$$
\begin{aligned}
d\omega^1 &= -\,\omega^1 \wedge \omega^5, \\
d\omega^2 &= -\,\tfrac{3}{2}\omega^1 \wedge \omega^4 - U\omega^1 \wedge \omega^5 - \tfrac{1}{3}\omega^2 \wedge \omega^5 + \tfrac{2}{3}\epsilon\,\omega^3 \wedge \omega^6, \\
d\omega^3 &= -\,\omega^1 \wedge \omega^6 - \tfrac{2}{3}\omega^3 \wedge \omega^5, \\
d\omega^4 &= -\,\omega^1 \wedge \alpha^1 - \omega^2 \wedge \beta^1 - \omega^3 \wedge \gamma^1 + \tfrac{2}{3}\omega^4 \wedge \omega^5, \\
d\omega^5 &= -\,\tfrac{9}{2}\omega^1 \wedge \beta^1, \\
d\omega^6 &= \tfrac{9}{4}\epsilon\,\omega^1 \wedge \gamma^1 - 3\omega^3 \wedge \beta^1 - \tfrac{1}{3}\omega^5 \wedge \omega^6, \\
d\alpha^1 &= -\,\omega^1 \wedge \alpha^2 - \omega^2 \wedge \beta^2 - \omega^3 \wedge \gamma^2 + \tfrac{3}{2}\omega^4 \wedge \beta^1 - \tfrac{5}{3}\omega^5 \wedge \alpha^1 \\
&\qquad - U\omega^5 \wedge \beta^1 - \omega^6 \wedge \gamma^1, \\
d\beta^1 &= -\,\omega^1 \wedge \beta^2 - \omega^5 \wedge \beta^1, \\
d\gamma^1 &= -\,\omega^1 \wedge \gamma^2 + \tfrac{4}{3}\epsilon\,\omega^3 \wedge \beta^2 - \tfrac{4}{3}\omega^5 \wedge \gamma^1 + \tfrac{2}{3}\epsilon\,\omega^6 \wedge \beta^1, \\
d\alpha^2 &= -\,\omega^1 \wedge \alpha^3 - \omega^2 \wedge \beta^3 - \omega^3 \wedge \gamma^3 - \tfrac{8}{3}\omega^5 \wedge \alpha^2 - 2U\omega^5 \wedge \beta^2 \\
&\qquad - 2\omega^6 \wedge \gamma^2 + 9\alpha^1 \wedge \beta^1, \\
d\beta^2 &= -\,\omega^1 \wedge \beta^3 - 2\omega^5 \wedge \beta^2, \\
d\gamma^2 &= -\,\omega^1 \wedge \gamma^3 + \tfrac{4}{3}\epsilon\,\omega^3 \wedge \beta^3 - \tfrac{7}{3}\omega^5 \wedge \gamma^2 + 2\epsilon\,\omega^6 \wedge \beta^2 - \tfrac{9}{2}\beta^1 \wedge \gamma^1.
\end{aligned}
$$

These correspond to the structure equations for the KP symmetry algebra obtained by Reid, Lisle, Boulton by Taylor series expansions.

Extended Jet Bundles

$J^n = J^n(M) = \{n\text{-jets of } p\text{-dimensional submanifolds of } M\}$.

Local coordinates on J^n: $\quad (x^i, u^\alpha, u^\alpha_{i_1}, \ldots, u^\alpha_{i_1 \cdots i_n})$.

$\mathcal{D}$ acts on J^n in the usual fashion, and this action factors into an action of $\mathcal{D}^n$ on J^n.

Let $\mathcal{E}^n$ be the pull-back of $\mathcal{D}^n \to M$ under $\widetilde{\pi}^n_o \colon J^n \to M$; hence $\mathbf{g}^n \in \mathcal{E}^n$ consists of a pair $\mathbf{g}^n = (z^n, g^n)$, where both $z^n \in J^n$, $g^n \in \mathcal{D}^n$ are based at the same point $z \in M$.

Source and *target maps*

$$\sigma(\mathbf{g}^n) = z^n, \quad \tau(\mathbf{g}^n) = g^n \cdot z^n.$$

$\mathcal{D}$ acts on $\mathcal{E}^n$ from the left by

$$L_\psi \mathbf{g}^n = (j^n_z \psi \cdot z^n, g^n \cdot j^n_{\psi(z)} \psi^{-1}).$$

Then $\tau(L_\psi \mathbf{g}^n) = \tau(\mathbf{g}^n)$ so that the target coordinates are $\mathcal{D}$ invariant.

Moving Frames

Let $\mathcal{H}^n \subset \mathcal{E}^n$ be the subbundle determined by the jets of transformations in a Lie pseudogroup $\mathcal{G}$.

A *moving frame* of order n is a $\mathcal{G}$-equivariant section

$$\rho\colon \mathcal{V} \to \mathcal{H}^n$$

defined on some open $\mathcal{V} \subset J^n$.

Hence $\sigma(\rho^n(z^n)) = z^n$ and $\rho^n(\psi \cdot z^n) = L_\psi \rho(z^n),\ \psi \in \mathcal{G}$.

Existence of moving frames: *Isotropy subgroup* of z^n:

$$\mathcal{I}^n_{z^n} = \{ g^n \in \mathcal{G}^n_z \mid g^n \cdot z^n = z^n \}.$$

$\mathcal{G}$ *acts freely* at z^n if $\mathcal{I}^n_{z^n} = \{\mathrm{id}^n_z\}$ and *locally freely* at z^n if $\mathcal{I}^n_{z^n}$ is a discrete subgroup of $\mathcal{G}^n_z$.

Remark: This is a slight generalization of the usual freeness of Lie group actions.

Example: The group $(x, u) \to (x + a, u + bx^2 + cx + d)$ acts freely on $J^n(\mathbb{R}^2, 1)$ for $n \geq 0$ in the above sense, but in the usual Lie group terminology, the action is free only when $n \geq 2$.

Write $r_n = \dim \mathcal{G}^n{}_{|z}$. If $\mathcal{G}$ acts freely at order n, then $r_n \leq \dim J^n$.

Alternate growth condition to the one provided by Spencer cohomology!

Theorem. *A local moving frame of order n exists in a neighborhood of* $z^{(n)} \in J^n(M)$ *if and only if* $\mathcal{G}$ *acts locally freely at* $z^{(n)}$.

Theorem. *If* $\mathcal{G}^n$ *acts locally freely at* $z^n \in J^n$*, then* $\mathcal{G}^l$ *acts locally freely at any* $z^l \in J^l$ *with* $\pi^l_n(z^l) = z^n$ *for* $l > n$.

Construction: Choose a cross-section K for the action of $\mathcal{G}$ on J^n. Define $\rho(z^n)$ by the condition $\tau(\rho(z^n)) \in K$.

By the invariance of τ, the components of $\tau \circ \rho$ contain a complete set of (local) differential invariants for the action of $\mathcal{G}$ on J^n.

INVARIANTIZATION

Decompose $\Omega^*(\mathcal{E}^\infty) = \oplus_{i,j,k}\Omega^{i,j,k}$, where

i is the horizontal degree

j is the contact degree in J^∞

k is the contact degree in $\mathcal{D}^\infty$

Let π_J be the projection $\pi_J\colon \Omega^* \to \oplus_{i,j}\Omega^{i,j,0}$.
π_J preserves $\mathcal{D}$ invariance.

The *lift* of $\omega \in \Omega^*(J^\infty)$ is $\lambda(\omega) = \pi_J(\tau^*(\omega))$.

Let ρ be a moving frame for a pseudogroup $\mathcal{G}$.
The *invariantization* $\iota(\omega)$ of ω is $\iota(\omega) = \rho^*\lambda(\omega)$.

Theorem. *The invariantization of a (local) coframe on J^∞ produces an $\mathcal{G}$* invariant *(local) coframe on J^∞.*

Structure equations: Formally

$$d\iota(\omega) = \iota(d\omega + \mathcal{L}_{\mathrm{pr}\,\mathbf{v}}\omega). \qquad (*)$$

$\big($Write $\mathcal{L}_{\mathrm{pr}\,\mathbf{v}}\omega = \zeta^a_{,b_1\cdots b_r}\omega_a^{b_1\cdots b_r}$. Then

$$\iota(\mathcal{L}_{\mathrm{pr}\,\mathbf{v}}\omega) = \rho^*(\mu^a_{b_1\cdots b_r}) \wedge \iota(\omega_a^{b_1\cdots b_r}).\big)$$

The invariant Maurer-Cartan forms $\mu^a_{b_1\cdots b_r}$ above are subject to the IDE for $\mathcal{G}$. This can be exploited in analyzing the structure of invariant objects for $\mathcal{G}$.

Recurrence formulas.

Fix a coordinate cross section. Write

$$H^i = \iota(x^i), \qquad I^\alpha_{j_1\cdots j_r} = \iota(u^\alpha_{j_1\cdots j_r})$$

for the normalized differential invariants. Call an invariant *phantom* if it is a constant.

Let

$$\omega^i = \pi_H \iota(dx^i),$$
$$\beta^i = \pi_H \iota(\xi^i), \quad \psi^\alpha_{j_1\cdots j_r} = \pi_H \iota(\mathcal{L}_{\mathrm{pr}\,\mathbf{v}} u^\alpha_{j_1\cdots j_r}),$$

and let $\mathcal{D}_i$ be total differential operators dual to ω^i.

The horizontal component of (*) yields

$$(\mathcal{D}_j H^i)\omega^j = \omega^i + \beta^i,$$
$$(\mathcal{D}_{j_{r+1}} I^\alpha_{j_1\cdots j_r})\omega^{j_{r+1}} = I^\alpha_{j_1\cdots j_r j_{r+1}}\omega^{j_{r+1}} + \psi^\alpha_{j_1\cdots j_r}.$$

The above equations for phantom invariants can be solved for the independent horizontal invariantized Maurer-Cartan forms. Substitute the expression for these into the above equations for non-phantom invariants to derive the recurrence formulas

$$\mathcal{D}_j H^i = \delta^i_j + P^i_j,$$
$$\mathcal{D}_{j_{r+1}} I^\alpha_{j_1\cdots j_r} = I^\alpha_{j_1\cdots j_r j_{r+1}} + M^\alpha_{j_1\cdots j_r, j_{r+1}}.$$

Syzygies.

Theorem. *A complete system of basic syzygies is given by*

$$\begin{aligned}
&\mathcal{D}_{j_1\cdots j_r} I^\alpha_{k_1\cdots k_s} = c^\alpha_{j_1\cdots j_r k_1\cdots k_s} + M^\alpha_{k_1\cdots k_s, j_1\cdots j_r},\\
&\mathcal{D}_{j_1\cdots j_r} I^\alpha_{k_1\cdots k_s l_1\cdots l_t} - \mathcal{D}_{k_1\cdots k_s} I^\alpha_{j_1\cdots j_r l_1\cdots l_t}\\
&\qquad\qquad = M^\alpha_{k_1\cdots k_s l_1\cdots l_t, j_1\cdots j_r} - M^\alpha_{j_1\cdots j_r l_1\cdots l_t, k_1\cdots k_s},
\end{aligned}$$

where $c^\alpha_{j_1\cdots j_r k_1\cdots k_s}$ *is a phantom differential invariant and* $I^\alpha_{j_1\cdots j_r l_1\cdots l_t}$, $I^\alpha_{k_1\cdots k_s l_1\cdots l_t}$ *are normalized differential invariants with* $\{j_1\cdots j_r\}\cap\{k_1\cdots k_s\}=\emptyset$.

KP SYMMETRIES CONTINUED:

Horizontal invariantized Maurer-Cartan forms

$$\alpha_{ijkl} = \pi_H \iota(\tau_{,ijkl}), \qquad \beta_{ijkl} = \pi_H \iota(\xi_{,ijkl}),$$
$$\gamma_{ijkl} = \pi_H \iota(\eta_{,ijkl}), \qquad \zeta_{ijkl} = \pi_H \iota(\varphi_{,ijkl}),$$

satisfy the lifted infinitesimal determining equations.

$\Longrightarrow$

A basis for these is provided by

$$\alpha_{T^n}, \quad \beta_{T^n}, \quad \gamma_{T^n}, \quad n \geq 0.$$

Consequently,

$$\alpha_{T^nX^pY^qU^r} = 0, \qquad \text{if } (p,q,r) \neq (0,0,0);$$

$$\beta_{T^nX} = \tfrac{1}{3}\alpha_{T^{n+1}}, \quad \beta_{T^nY} = -\tfrac{2}{3}\epsilon\,\gamma_{T^{n+1}},$$

$$\beta_{T^nYY} = -\tfrac{4}{9}\epsilon\,\beta_{T^{n+2}}, \quad \gamma_{T^nY} = \tfrac{1}{3}\alpha_{T^{n+1}},$$

$$\beta_{T^nX^pY^qU^r} = 0, \quad \gamma_{T^nX^pY^qU^r} = 0, \quad \text{for all other choices of } (p,q,r);$$

$$\zeta_{T^n} = \tfrac{2}{3}\beta_{T^{n+1}} - \tfrac{2}{3}\sum_{s=0}^{n}\binom{n}{s} I_{s00}\alpha_{T^{n-s+1}},$$

$$\zeta_{T^nX} = \tfrac{2}{9}\alpha_{T^{n+2}} - \tfrac{2}{3}\sum_{s=0}^{n}\binom{n}{s} I_{s10}\alpha_{T^{n-s+1}},$$

$$\zeta_{T^nY} = -\tfrac{4}{9}\epsilon\,\gamma_{T^{n+2}} - \tfrac{2}{3}\sum_{s=0}^{n}\binom{n}{s} I_{s01}\alpha_{T^{n-s+1}},$$

$$\zeta_{T^nYY} = -\tfrac{4}{27}\,\alpha_{T^{n+3}} - \tfrac{2}{3}\sum_{s=0}^{n}\binom{n}{s} I_{s02}\alpha_{T^{n-s+1}},$$

$$\zeta_{T^nX^pY^q} = -\tfrac{2}{3}\sum_{s=0}^{n}\binom{n}{s} I_{spq}\alpha_{T^{n-s+1}}, \quad \text{for all other choices of } (p,q),$$

$$\zeta_{T^nU} = -\tfrac{2}{3}\alpha_{T^{n+1}}, \quad \zeta_{T^nX^pY^qU^r} = 0, \qquad \text{if } r \geq 2.$$

Correction terms:

$$
\begin{aligned}
\psi_{pqr} = {} & \tfrac{2}{9}\delta_{q1}\delta_{r0}\alpha_{T^{p+2}} - \tfrac{4}{9}\delta_{q0}\delta_{r1}\epsilon\,\gamma_{T^{p+2}} - \tfrac{8}{27}\delta_{q0}\delta_{r2}\epsilon\,\alpha_{T^{p+3}} \\
& - \sum_{s=0}^{p}\binom{p}{s}\left(\frac{2+q+2r}{3} + \frac{p-s}{s+1}\right) I_{p-s,q,r}\alpha_{T^{s+1}} \\
& + \tfrac{2}{9}\epsilon\, r(r-1)\sum_{s=0}^{p}\binom{p}{s} I_{p-s,q+1,r-2}\alpha_{T^{s+2}} \\
& - \sum_{s=1}^{p}\binom{p}{s} I_{p-s,q+1,r}\beta_{T^{s}} - \sum_{s=1}^{p}\binom{p}{s} I_{p-s,q,r+1}\gamma_{T^{s}} \\
& + \tfrac{2}{3}\epsilon\, r\sum_{s=0}^{p}\binom{p}{s} I_{p-s,q+1,r-1}\gamma_{T^{s+1}}
\end{aligned}
$$

Recurrence formulas:

$$d_H H^1 = \omega^t + \alpha, \qquad d_H H^2 = \omega^x + \beta, \qquad d_H H^3 = \omega^y + \gamma,$$

$$d_H I_{000} = I_{100}\omega^t + I_{010}\omega^x + I_{001}\omega^y - \tfrac{2}{3} I_{000}\alpha_T + \tfrac{2}{3}\beta_T,$$

$$d_H I_{100} = I_{200}\omega^t + I_{110}\omega^x + I_{101}\omega^y - \tfrac{5}{3} I_{100}\alpha_T - \tfrac{2}{3} I_{000}\alpha_{TT} - I_{010}\beta_T + \tfrac{2}{3}\beta_{TT} - I_{001}\gamma_T,$$

$$d_H I_{010} = I_{110}\omega^t + I_{020}\omega^x + I_{011}\omega^y - I_{010}\alpha_T + \tfrac{2}{9}\alpha_{TT},$$

$$d_H I_{001} = I_{101}\omega^t + I_{011}\omega^x + I_{002}\omega^y - \tfrac{4}{3} I_{001}\alpha_T + \tfrac{2}{3}\,\epsilon I_{010}\gamma_T - \tfrac{4}{9}\,\epsilon\gamma_{TT},$$

$$d_H I_{200} = I_{300}\omega^t + I_{210}\omega^x + I_{201}\omega^y - \tfrac{8}{3} I_{200}\alpha_T - \tfrac{7}{3} I_{100}\alpha_{TT} - \tfrac{2}{3} I_{000}\alpha_{TTT} - 2I_{110}\beta_T - I_{010}\beta_{TT} + \tfrac{2}{3}\beta_{TTT} - 2I_{101}\gamma_T - I_{001}\gamma_{TT},$$

$$d_H I_{110} = I_{210}\omega^t + I_{120}\omega^x + I_{111}\omega^y - 2I_{110}\alpha_T - I_{010}\alpha_{TT} + \tfrac{2}{9}\alpha_{TTT} - I_{020}\beta_T - I_{011}\gamma_T,$$

$$d_H I_{101} = I_{201}\omega^t + I_{111}\omega^x + I_{102}\omega^y - \tfrac{7}{3} I_{101}\alpha_T - \tfrac{4}{3} I_{001}\alpha_{TT} - I_{011}\beta_T + (\tfrac{2}{3}\,\epsilon I_{110} - I_{002})\gamma_T + \tfrac{2}{3}\,\epsilon I_{010}\gamma_{TT} - \tfrac{4}{9}\,\epsilon\gamma_{TTT},$$

$$d_H I_{020} = I_{120}\omega^t + I_{030}\omega^x + I_{021}\omega^y - \tfrac{4}{3} I_{020}\alpha_T,$$

$$d_H I_{011} = I_{111}\omega^t + I_{021}\omega^x + I_{012}\omega^y - \tfrac{5}{3} I_{011}\alpha_T + \tfrac{2}{3}\,\epsilon I_{020}\gamma_T,$$

$$\vdots$$

We impose normalizations of the invariants so that the resulting phantom recurrence relations can be solved for the basis of invariantized Maurer-Cartan forms. For this we let

$$
\begin{array}{llll}
H^1 \longmapsto 0, & H^2 \longmapsto 0, & H^3 \longmapsto 0, & \\
I_{000} \longmapsto 0, & I_{100} \longmapsto 0, & I_{010} \longmapsto 0, & \\
I_{001} \longmapsto 0, & I_{200} \longmapsto 0, & I_{101} \longmapsto 0, & \\
I_{020} \longmapsto 1, & I_{011} \longmapsto 0, & I_{002} \longmapsto 0, & \\
I_{i,0,0} \longmapsto 0, & I_{i-1,0,1} \longmapsto 0, & I_{i-2,0,2} \longmapsto 0, & \text{for all } \ i \geq 3.
\end{array}
$$

These yield the following expressions for the basic invariantized Maurer-Cartan forms:

$$\alpha = -\omega^t, \qquad \beta = -\omega^x, \qquad \gamma = -\omega^y,$$

$$\alpha_T = \tfrac{3}{4}(I_{120}\omega^t + I_{030}\omega^x + I_{021}\omega^y), \quad \alpha_{TT} = \tfrac{9}{2}(I_{110}\omega^t + \omega^x),$$

$$\alpha_{TTT} = \tfrac{27}{8}\,\epsilon(I_{012}\omega^x + I_{003}\omega^y), \ldots,$$

$$\beta_T = 0, \quad \beta_{TT} = -\tfrac{3}{2}I_{110}\omega^x, \quad \beta_{TTT} = -\tfrac{3}{2}I_{210}\omega^x, \ldots,$$

$$\gamma_T = -\tfrac{3}{2}\epsilon(I_{111}\omega^t + I_{021}\omega^x + I_{012}\omega^y), \quad \gamma_{TT} = 0,$$

$$\gamma_{TTT} = \tfrac{9}{4}\epsilon(-I_{110}I_{111}\omega^t + (I_{111} - I_{110}I_{021})\omega^x - I_{110}I_{012}\omega^y), \ldots,$$

The higher order invariantized Maurer–Cartan forms can be recursively determined from the equations

$$\alpha_{T^{p+3}} = \tfrac{27}{8}\epsilon(I_{p12}\omega^x + I_{p03}\omega^y) + \tfrac{3}{2}\sum_{s=0}^{p-1}\binom{p}{s} I_{p-2,1,0}\alpha_{T^{s+2}}$$

$$- \tfrac{27}{8}\epsilon\sum_{s=1}^{p}\binom{p}{s} I_{p-s,1,2}\beta_{T^s} + \tfrac{9}{2}\sum_{s=0}^{p-1}\binom{p}{s} I_{p-s,1,1}\gamma_{T^{s+1}}$$

$$- \tfrac{27}{8}\epsilon\sum_{s=1}^{p}\binom{p}{s} I_{p-s,0,3}\gamma_{T^s},$$

$$\beta_{T^{p+1}} = -\tfrac{3}{2}I_{p10}\omega^x + \tfrac{3}{2}\sum_{s=1}^{p-1}\binom{p}{s} I_{p-s,1,0}\beta_{T^s},$$

$$\gamma_{T^{p+2}} = \tfrac{9}{4}\epsilon I_{p11}\omega^x - \tfrac{9}{4}\epsilon\sum_{s=1}^{p-1}\binom{p}{s} I_{p-s,1,1}\beta_{T^s} + \tfrac{3}{2}\sum_{s=0}^{p-1}\binom{p}{s} I_{p-s,1,0}\gamma_{T^{s+1}}.$$

Recurrence formulas between the differentiated and normalized invariants:

$$\mathcal{D}_t I_{110} = I_{210} - \tfrac{3}{2} I_{110} I_{120},$$
$$\mathcal{D}_x I_{110} = I_{120} - \tfrac{3}{2} I_{110} I_{030} + \tfrac{3}{4} \epsilon I_{012},$$
$$\mathcal{D}_y I_{110} = I_{111} - \tfrac{3}{2} I_{110} I_{021} + \tfrac{3}{4} \epsilon I_{003},$$
$$\mathcal{D}_t I_{210} = I_{310} - \tfrac{9}{4} I_{210} I_{120} + \tfrac{3}{2} \epsilon I_{111}^2 + \tfrac{9}{8} I_{111} I_{003} + 12 I_{110}^2,$$
$$\mathcal{D}_x I_{210} = I_{220} - \tfrac{9}{4} I_{210} I_{030} + \tfrac{3}{4} \epsilon I_{112} + \tfrac{3}{2} \epsilon I_{111} I_{021} + \tfrac{9}{8} I_{003} I_{021} + \tfrac{27}{2} I_{110},$$
$$\mathcal{D}_y I_{210} = I_{211} - \tfrac{9}{4} I_{210} I_{021} + \tfrac{3}{2} \epsilon I_{111} I_{012} + \tfrac{3}{4} \epsilon I_{103} + \tfrac{9}{8} I_{012} I_{003},$$
$$\mathcal{D}_t I_{120} = I_{220} + \tfrac{3}{2} \epsilon I_{111} I_{021} - \tfrac{7}{4} I_{120}^2 + 6 I_{110},$$
$$\mathcal{D}_x I_{120} = I_{130} - \tfrac{7}{4} I_{120} I_{030} + \tfrac{3}{2} \epsilon I_{021}^2 + 6,$$
$$\mathcal{D}_y I_{120} = I_{121} - \tfrac{7}{4} I_{120} I_{021} + \tfrac{3}{2} \epsilon I_{021} I_{012},$$
$$\mathcal{D}_t I_{111} = I_{211} - \big(3 I_{120} - \tfrac{3}{2} \epsilon I_{012}\big) I_{111},$$
$$\mathcal{D}_x I_{111} = I_{121} - \big(I_{120} - \tfrac{3}{2} \epsilon I_{012}\big) I_{021} - 2 I_{111} I_{030},$$
$$\mathcal{D}_y I_{111} = I_{112} - \big(I_{120} - \tfrac{3}{2} \epsilon I_{012}\big) I_{012} - 2 I_{111} I_{021},$$
$$\vdots$$

Syzygies for I_{110}, I_{030}, I_{021}, I_{012}, I_{003}:

$$\mathcal{D}_y I_{012} - \mathcal{D}_x I_{003} + \tfrac{3}{4} I_{012} I_{021} - 2 I_{030} I_{003} = 0,$$

$$\mathcal{D}_x I_{021} - \mathcal{D}_y I_{030} + \tfrac{5}{4} I_{021} I_{030} = 0,$$

$$\mathcal{D}_y I_{021} - \mathcal{D}_x I_{012} - \tfrac{1}{2} I_{021}^2 - \tfrac{3}{4} I_{012} I_{030} - 2\epsilon = 0,$$

$$\begin{aligned}&\mathcal{D}_x \mathcal{D}_x I_{110} - \mathcal{D}_t I_{030} - \tfrac{3}{4}\epsilon \mathcal{D}_y I_{021} + \tfrac{3}{2} I_{110} \mathcal{D}_x I_{030} + 2 I_{030} \mathcal{D}_x I_{110}\\ &\qquad - \tfrac{9}{8}\epsilon I_{021}^2 + \tfrac{3}{16}\epsilon I_{030} I_{012} + \tfrac{3}{4} I_{030}^2 I_{110} - \tfrac{9}{2} = 0,\end{aligned}$$

$$\begin{aligned}&\mathcal{D}_x \mathcal{D}_y I_{110} - \mathcal{D}_t I_{021} + I_{030} \mathcal{D}_y I_{110} + I_{021} \mathcal{D}_x I_{110} + \tfrac{3}{2} I_{110} \mathcal{D}_y I_{030}\\ &\qquad - \tfrac{3}{4}\epsilon \mathcal{D}_x I_{003} - \tfrac{9}{8} I_{110} I_{030} I_{021} - \tfrac{3}{4}\epsilon I_{030} I_{003} - \tfrac{9}{8}\epsilon I_{021} I_{012} = 0,\end{aligned}$$

$$\begin{aligned}&\mathcal{D}_x \mathcal{D}_y I_{110} - \mathcal{D}_t I_{021} + I_{030} \mathcal{D}_y I_{110} + I_{021} \mathcal{D}_x I_{110} + \tfrac{3}{2} I_{110} \mathcal{D}_x I_{021}\\ &\qquad + \tfrac{3}{4} I_{110} I_{030} I_{021} - \tfrac{3}{4}\epsilon \mathcal{D}_x I_{003} - \tfrac{9}{8}\epsilon I_{021} I_{012} - \tfrac{3}{4}\epsilon I_{030} I_{003} = 0,\end{aligned}$$

$$\begin{aligned}&\mathcal{D}_y \mathcal{D}_y I_{110} - \mathcal{D}_t I_{012} + \tfrac{3}{2} I_{021} \mathcal{D}_y I_{110} - \tfrac{3}{4} I_{012} \mathcal{D}_x I_{110}\\ &\qquad + \left(\tfrac{3}{2} \mathcal{D}_x I_{012} + \tfrac{3}{4} I_{021}^2 + \epsilon\right) I_{110} - \tfrac{3}{4}\epsilon \mathcal{D}_y I_{003} - \tfrac{15}{16}\epsilon I_{012}^2 = 0,\end{aligned}$$

$$\begin{aligned}&\mathcal{D}_y \mathcal{D}_y \mathcal{D}_y I_{110} - \mathcal{D}_t \mathcal{D}_x I_{003} - 2 I_{030} \mathcal{D}_t I_{003} + 3 I_{021} \mathcal{D}_t I_{012} - 2 I_{003} \mathcal{D}_t I_{030}\\ &+ \left(2\mathcal{D}_y I_{021} - \tfrac{7}{4} I_{021}^2\right) \mathcal{D}_y I_{110} + \left(\tfrac{21}{16} I_{012} I_{021} - \tfrac{5}{4} \mathcal{D}_y I_{012}\right) \mathcal{D}_x I_{110}\\ &+ \left(\tfrac{3}{2} \mathcal{D}_y \mathcal{D}_y I_{021} - \tfrac{15}{4} I_{021} \mathcal{D}_y I_{021} - \tfrac{15}{8} \mathcal{D}_y I_{012} I_{030} + \tfrac{63}{32} I_{021} I_{012} I_{030}\right.\\ &\left.+ \tfrac{3}{4} I_{021}^3 + 6\epsilon I_{021}\right) I_{110} - \tfrac{3}{4}\epsilon \mathcal{D}_y \mathcal{D}_y I_{003} + \tfrac{9}{8}\epsilon I_{021} \mathcal{D}_y I_{003} - \tfrac{57}{16}\epsilon I_{012} \mathcal{D}_y I_{012}\\ &+ \tfrac{3}{4}\epsilon I_{003} \mathcal{D}_y I_{021} - \tfrac{3}{8}\epsilon I_{021}^2 I_{003} - \tfrac{3}{2} I_{003} + \tfrac{9}{64}\epsilon I_{021} I_{012}^2 = 0.\end{aligned}$$

Theorem. *The differential invariants I_{110}, I_{021}, I_{003} form a generating set for the algebra of differential invariants for the KP symmetry pseudogroup.*

Specifically,

$$I_{110} = u_{xx}^{-3/2}\big(u_{tx} + \tfrac{3}{2}uu_{xx} + \tfrac{3}{2}u_x^2 + \tfrac{3}{4}\epsilon u_{yy}\big),$$

$$I_{021} = u_{xx}^{-5/2}(u_{xx}u_{xxy} - u_{xy}u_{xxx}),$$

$$I_{003} = u_{xx}^{-5}(u_{xx}^3 u_{yyy} - 3u_{xx}^2 u_{xy}u_{xyy} + 3u_{xx}u_{xy}^2 u_{xxy} - u_{xy}^3 u_{xxx}).$$

Invariant differential operators:

$$\mathcal{D}_t = u_{xx}^{-3/4}D_t + \tfrac{3}{4}u_{xx}^{-11/4}(2uu_{xx}^2 - \epsilon u_{xy}^2)D_x + \tfrac{3}{2}\epsilon u_{xy}u_{xx}^{-7/4}D_y,$$

$$\mathcal{D}_x = u_{xx}^{-1/4}D_x,$$

$$\mathcal{D}_y = -u_{xx}^{-3/2}u_{xy}D_x + u_{xx}^{-1/2}D_y.$$