

Differential and Variational Calculus in Invariant Frames

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Outline:

- Group invariant contact structure on a jet bundle
- Structure equations for invariant frame
- Invariant variational complex
- Invariant Euler-Lagrange operator
- Invariant Noether correspondence
- Symbolic implementation

The talk is based on:

- *Invariant Euler-Lagrange Equations and the Invariant Variational Bicomplex*, I. Kogan, P. Olver, Acta Appl. Math. 76, 137-193, (2003)
- Ongoing work with Ian Anderson

Group invariant contact structure on a jet bundle

Standard local coframe on $J^\infty(M, p)$

Local coordinates: $x^1, \dots, x^p, u^1, \dots, u^q, u_J^m, m = 1 \dots q, J -$
multi-index.

Basis of horizontal sub-bundle	Basis of vertical sub-bundle
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Cotangent

horizontal one-forms dx^1, \dots, dx^p	contact one-forms $\theta^m = du^m - \sum_{i=1}^p u_i^m dx^i,$ $\theta_J^m = du_J^m - \sum_{i=1}^p u_{Ji}^m dx^i.$
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Tangent

total derivatives: $\frac{d}{dx^i} = \frac{\partial}{\partial x^i} + \sum_{m=1}^q u_i^m \frac{\partial}{\partial u^m}$ $+ \sum_{m,J} u_{Ji}^m \frac{\partial}{\partial u_J^m}$	vertical derivatives $\frac{\partial}{\partial u^m},$ $\frac{\partial}{\partial u_J^m}, m = 1 \dots q$
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Prolongation of the action

$G \curvearrowright M \Rightarrow G \curvearrowright J^k(M, p)$ unique prolongation s.t

- G -action respects projections $J^k \rightarrow J^l, \forall k > l$.
- $g^*(\text{contact form}) = \text{contact form} \forall g \in G$.

Thm. (*Ovsianikov*) $\exists n < \dim G$ s. t. G acts locally free on an open dense subset of $J^n(M, p)$.

Local invariants

$$G \curvearrowright J^\infty \Rightarrow h: \mathfrak{g} \rightarrow \mathcal{X}(J^\infty)$$

$Im(h) = \Gamma$ infinitesimal generators of G -action.

- invariant functions $f(\mathbf{x}, \mathbf{u}^{(n)})$ (differential invariants):
 $\mathcal{L}_v f := v(f) = 0, \forall v \in \Gamma.$
- invariant vector fields: $Y \in \mathcal{X}(J^\infty): \mathcal{L}_v Y = 0, \forall v \in \Gamma.$
- invariant differential forms: $\lambda \in \Lambda^*(J^\infty): \mathcal{L}_v \lambda = 0, \forall v \in \Gamma.$

$\exists G$ - invariant contact structure on $\mathcal{U} \subset J^\infty(M, p)$.

Invariant horizontal basis	Invariant vertical basis
<i>Cotangent</i>	
invariant “horizontal” one-forms $\omega^1, \dots, \omega^p$ $\tilde{H} = \text{span} \{ \omega^i \} \neq H = \text{span} \{ dx^i \}$ unless the action is projectable	invariant contact one-forms ϑ_J^m $\mathcal{C} = \text{span} \{ \vartheta_J^m \} = \text{span} \{ \theta_J^m \}$
<i>Tangent</i>	
invariant total diff. operators $\mathcal{D}_1, \dots, \mathcal{D}_p$ $\text{span} \{ \mathcal{D}_i \} = \text{span} \left\{ \frac{d}{dx_i} \right\}$	invariant vertical diff. operators $V_1, \dots, V_q, V_m^J, m = 1..q$ $\text{span} \{ V_m \} \neq \text{span} \left\{ \frac{\partial}{\partial u^m} \right\}$ unless the action is projectable

Example $SE(2)$ -invariant contact structure on $J^\infty(\mathbb{R}^2, 1)$.

Cotangent

<p>invariant “horizontal” one-form</p> $\omega = ds + \frac{u_x}{\sqrt{1+u_x^2}}\theta, \text{ where } ds = \sqrt{1+u_x^2}dx$	<p>invariant contact one-forms</p> $\vartheta = \frac{\theta}{\sqrt{1+u_x^2}}$ $\vartheta_x = \frac{(1+u_x^2)\theta_x - u_x u_{xx}\theta}{(1+u_x^2)^2}$ <p>...</p>
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Tangent

<p>invariant total diff. operator:</p> $\mathcal{D} = \frac{1}{\sqrt{1+u_x^2}} \frac{d}{dx} = \frac{d}{ds}$	<p>invariant vertical diff. operators</p> $V = -\frac{u_x}{\sqrt{1+u_x^2}} \frac{\partial}{\partial x} + \frac{1}{\sqrt{1+u_x^2}} \frac{\partial}{\partial u}$ $V^x = (1+u_x^2) \frac{\partial}{\partial u_x} + 3u_x u_{xx} \frac{\partial}{\partial u_{xx}}$ <p>...</p>
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Motivation

(S. Lie:)

Most of the symmetric problems can be written in terms of invariants.
It is natural to perform further computations in terms of invariants.

**Example: Lagrangian invariant under
 $SE(2) = SO(2) \times \mathbb{R}^2 \curvearrowright \mathbb{R}^2$ (Euclidean group.)**

$$\mathcal{L}[u] = \int \frac{u_{xx}^2}{2(1+u_x^2)^{5/2}} dx = \int L dx \iff \int \frac{1}{2} \kappa^2 ds$$

$$\downarrow E = \left(\frac{d}{dx}\right)^2 \frac{\partial}{\partial u_{xx}} - \left(\frac{d}{dx}\right) \frac{\partial}{\partial u_x} \quad ?? \quad \downarrow$$

$$E(L) = 0 \iff \kappa_{ss} + \frac{1}{2} \kappa^3 = 0$$

$$E(L) = \frac{1}{2} \frac{30 u_2^3 u_1^2 - 5 u_2^3 - 20 u_2 u_1 u_3 - 20 u_2 u_1^3 u_3 + 2 u_4 + 4 u_4 u_1^2 + 2 u_4 u_1^4}{(1 + u_1^2)^{(9/2)}}.$$

$$(u_1 = u_x, \dots, u_4 = u_{xxxx})$$

Invariantization: moving frame construction

(Fels, Olver 1999)

Local cross-section: submanifold \mathcal{K} of open $\mathcal{U} \subset M$, s. t.

- $T_{\mathbf{z}}\mathcal{K} \oplus T_{\mathbf{z}}\mathcal{O}_{\mathbf{z}} = T_{\mathbf{z}}\mathcal{U}, \quad \forall \mathbf{z} \in \mathcal{K}$
- \mathcal{K} intersects each connected component of $\mathcal{O}_{\mathbf{z}} \cap \mathcal{U}$ at the unique point.

Thm (follows from the Frobenius thm)

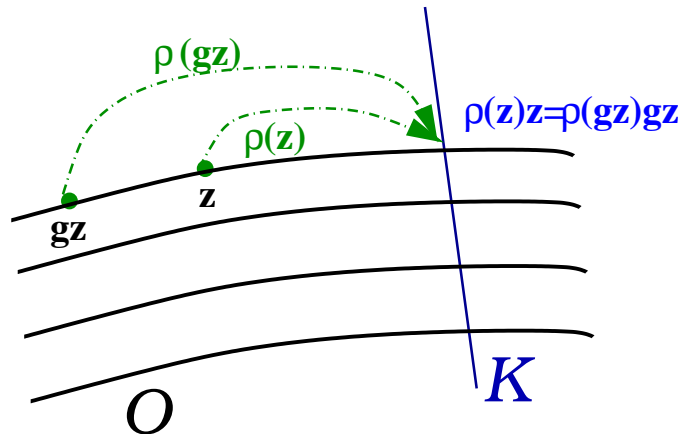
$G \curvearrowright \mathcal{U}$ semi. reg ($\Leftrightarrow \exists r, \forall \mathbf{z} \in \mathcal{U} : \dim \mathcal{O}_{\mathbf{z}} = r.$)

\Downarrow

\exists loc. cross-section on \mathcal{U}

Moving frame map

$\rho : \mathcal{U} \rightarrow G$ is defined by the condition $\rho(\mathbf{z}) \cdot \mathbf{z} \in \mathcal{K}$



$G \curvearrowright M$ free $\Rightarrow \forall \mathbf{z} \in M : \dim \mathcal{O}_{\mathbf{z}} = \dim G = \kappa$.

\Downarrow

$\rho : \mathcal{U} \rightarrow G$ defined by $\boxed{\rho(\mathbf{z}) \cdot \mathbf{z} \in \mathcal{K}}$ smooth, G -equivariant:

$$\rho(g \cdot \mathbf{z}) \cdot (g \cdot \mathbf{z}) = \rho(\mathbf{z}) \cdot \mathbf{z}, \stackrel{\text{freeness}}{\implies} \rho(g \cdot \mathbf{z}) = \rho(\mathbf{z})g^{-1}.$$

Invariantization in terms of $\rho : \mathcal{U} \rightarrow G$.

$$\mathbf{z}_0 = \mathcal{O}_{\mathbf{z}} \cap \mathcal{K} = \rho(\mathbf{z}) \cdot \mathbf{z}$$

Thm: The following maps define a projection on the corresponding invariant subspaces.

- functions: $\iota : \mathcal{F}(\mathcal{U}) \rightarrow \mathcal{F}^G(\mathcal{U}) :$

$$\iota(f)(\mathbf{z}) := f(\rho(\mathbf{z}) \cdot \mathbf{z}) = f(\mathbf{z}_0)$$

- differential forms: $\iota : \Lambda(\mathcal{U}) \rightarrow \Lambda^G(\mathcal{U}), \quad \forall \lambda \in \Lambda^*(\mathcal{U}), \forall \mathbf{z} \in \mathcal{U}:$

$$\iota(\lambda)|_{\mathbf{z}} := \rho(\mathbf{z})^*(\lambda|_{\mathbf{z}_0}).$$

- vector fields: $\iota : \mathcal{X}(\mathcal{U}) \rightarrow \mathcal{X}^G(\mathcal{U}), \quad \forall X \in \mathcal{X}(\mathcal{U}), \forall \mathbf{z} \in \mathcal{U}:$

$$\iota(X)|_{\mathbf{z}} = [\rho(\mathbf{z})^{-1}]_* X|_{\mathbf{z}_0}.$$

Properties of ι :

- ι preserves linear independence of forms and vector fields.
- ι -preserves pairing: $\langle \iota\lambda; \iota X \rangle = \langle \lambda; X \rangle$.
- ι -preserves duality of bases.
- $\forall \mathbf{z} \in \mathcal{K}, \forall \lambda \in \Lambda^*(\mathcal{U}), \forall X \in \mathcal{X}(\mathcal{U})$:

$$\iota(f)|_{\mathbf{z}_0} = f|_{\mathbf{z}_0}, \iota(X)|_{\mathbf{z}_0} = X|_{\mathbf{z}_0}, \iota(\lambda)|_{\mathbf{z}_0} = \lambda|_{\mathbf{z}_0}.$$

- ι preserves contact ideal.

Structure equations for invariantized forms.

$$d\iota(\lambda) = \iota(d\lambda) + \sum_{j=1}^r \rho^*(\mu^j) \wedge \iota[\mathcal{L}_{v_j}(\lambda)]$$

$\dim G = \dim \mathcal{O} = r$

$\hat{\mathbf{v}} = (\hat{v}_1, \dots, \hat{v}_r)^T$ a basis of \mathfrak{g} ;

$\mu = (\mu^1, \dots, \mu^r)$ – dual basis of invariant differential forms on G
(Maurer-Cartan forms)

$\mathbf{v} = (v_1, \dots, v_r)^T$ – basis for infinitesimal generators of G -action

Let zero set $K = (K^1(z), \dots, K^r(z)) : \mathcal{U} \rightarrow \mathbb{R}^r$ define \mathcal{K} .

Transversality condition $\det \mathbf{v}(K) := \mathcal{L}_{v_i}(K^j) \neq 0 \pmod{K}$.

$$\rho^*(\mu) = -\iota[dK \cdot \mathbf{v}(K)^{-1}]$$

- \mathcal{K} – constructive (\exists coordinate cross-section)
- ρ – non-constructive

How much can be done without ρ ? \Rightarrow

Invariantization: implicit approach

Theorem. $\forall f \in \mathcal{F}(\mathcal{U}), \quad \forall \lambda \in \Lambda^*(\mathcal{U}), \quad \forall X \in \mathcal{X}(\mathcal{U})$
 $\exists!$ locally invariant $\iota f \in \mathcal{F}^G(\mathcal{U}), \quad \iota \lambda \in (\Lambda^*)^G(\mathcal{U}), \quad \iota X \in \mathcal{X}^G(\mathcal{U})$ s.t.

$$\iota f|_{\mathcal{K}} = f|_{\mathcal{K}}, \quad \iota X|_{\mathcal{K}} = X|_{\mathcal{K}}, \quad \iota \lambda|_{\mathcal{K}} = \lambda|_{\mathcal{K}}.$$

Structure equations

$$d \iota(\lambda) = \iota [d \lambda - (dK)A \wedge \mathbf{v}(\lambda)]$$

$$\text{where } A = \mathbf{v}(K)^{-1},$$

$$\mathbf{v}(\lambda) = (\mathcal{L}_{v_1} \lambda, \dots, \mathcal{L}_{v_r} \lambda)^T$$

$\exists!$ locally G - invariant frame/coframe on $\mathcal{U} \subset J^\infty(M, p)$ which coincide with the standard frame/coframe on \mathcal{K} .

Invariant horizontal basis	Invariant vertical basis
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Cotangent

invariant “horizontal” one-forms $\omega^1, \dots, \omega^p$	invariant contact one-forms ϑ_J^m $\mathcal{C} = \text{span} \{ \vartheta_J^m \} = \text{span} \{ \theta_J^m \}$
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Tangent

invariant total diff. operators $\mathcal{D}_1 = \iota \frac{d}{dx_1}, \dots, \mathcal{D}_p = \iota \frac{d}{dx_p}$ $\text{span} \{ \mathcal{D}_i \} = \text{span} \left\{ \frac{d}{dx_i} \right\}$	invariant vertical diff. operators $V_1, \dots, V_q, V_m^J, m = 1..q$
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Input:

- **group action:** $\mathbf{v} = (\mathbf{v}_1, \dots, \mathbf{v}_l)^T$ – basis for inf. generators.
- **local cross-section:** \mathcal{K} – zero set $K(\mathbf{z}) = (K^1, \dots, K^r): \mathcal{J} \rightarrow \mathbb{R}^r$,

$$\det \mathbf{v}(K) \neq 0 \text{ mod } K.$$

Output (via linear algebra and differentiation)

- structure equations for invariant coframe and frame \Rightarrow *structure of the differential algebra of invariants*
- prolongations of vector fields
- invariant integration by parts \Rightarrow Euler-Lagrange and Helmholtz operators

Variational Bicomplex:

Gelfand, Tulczyjew, Tsujishita, Vinogradov, Takens, Anderson

Bigrading of exterior differential algebra:

Grading: $\Lambda^* = \bigoplus \Lambda^k$, where $\Lambda^k = \left\{ \underbrace{\sum 1\text{-form} \wedge \cdots \wedge 1\text{-form}}_{k \text{ times}} \right\}$.

$d: \Lambda^k \rightarrow \Lambda^{k+1}$, $d \circ d = 0 \Rightarrow$ de Rham complex:

Bigrading: $\Lambda^* = \bigoplus \Lambda^{s,t}$, where $\Lambda^{s,t} =$

$\left\{ \underbrace{\sum \text{hor. } 1\text{-form} \wedge \cdots \wedge \text{hor. } 1\text{-form}}_{s \text{ times}} \wedge \underbrace{\text{cont. } 1\text{-form} \wedge \cdots \wedge \text{cont. } 1\text{-form}}_{t \text{ times}} \right\}$

$d: \Lambda^{s,t} \rightarrow \Lambda^{s+1,t} \oplus \Lambda^{s,t+1} \Rightarrow d = d_H + d_V \Rightarrow$ Bicomplex:

$d^2 = (d_H + d_V)^2 = 0 \Rightarrow d_H^2 = 0, d_V^2 = 0, d_H \circ d_V = -d_V \circ d_H$

Example: $du = u_x dx + (du - u_x dx) = d_H u + d_V u$

Variational Bicomplex (locally exact)

$$\begin{array}{ccccccccc}
 \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 d_V \uparrow & & d_V \uparrow & & d_V \uparrow & & d_V \uparrow & & d_V \uparrow & & \partial_V \uparrow \\
 \Lambda^{0,2} & \xrightarrow{d_H} & \Lambda^{1,2} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Lambda^{p-1,2} & \xrightarrow{d_H} & \Lambda^{p,2} & \xrightarrow{I} & \mathcal{F}^2 \\
 d_V \uparrow & & d_V \uparrow & & \uparrow & & d_V \uparrow & & d_V \uparrow & & \partial_V \uparrow \\
 \Lambda^{0,1} & \xrightarrow{d_H} & \Lambda^{1,1} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Lambda^{p-1,1} & \xrightarrow{d_H} & \Lambda^{p,1} & \xrightarrow{I} & \mathcal{F}^1 \\
 d_V \uparrow & & d_V \uparrow & & \uparrow & & d_V \uparrow & & \uparrow & & \nearrow \partial_V \\
 \Lambda^{0,0} & \xrightarrow{d_H} & \Lambda^{1,0} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Lambda^{p-1,0} & \xrightarrow{d_H} & \Lambda^{p,0} & &
 \end{array}$$

- $d_V d_H = -d_H d_V$, $d_H^2 = 0$, $d_V^2 = 0$
- $\lambda = L(\mathbf{x}, \mathbf{u}^{(n)}) dx^1 \wedge \dots \wedge dx^p \in \Lambda^{p,0}$ -Lagrangian
- $I: \Lambda^{p,s} \rightarrow \mathcal{F}^s \subset \Lambda^{p,s}$ - int. by parts operator, $I \circ d_H = 0$, $I^2 = I$
- $\partial_V = I \circ d_V$ - variational derivative, $\partial_V \circ d_H = 0$

Integration by parts operator, $\lambda \in \Lambda^{p,s}$:

$$I(\lambda) = \sum_m \frac{1}{s} \theta^m \wedge \left(\left(\frac{\partial}{\partial u^m} \lrcorner \lambda \right) + \sum_J \left(-\frac{d}{dx_J} \right) \left(\frac{\partial}{\partial u_J^m} \lrcorner \lambda \right) \right).$$

Euler-Lagrange operator ($\lambda \in \Lambda^{p,0}$)

$$\lambda = L(\mathbf{x}, \mathbf{u}^{(n)}) d\mathbf{x} \xrightarrow{d_V} \sum_{m,J} \frac{\partial L}{\partial u_J^m} \theta_J^m \wedge d\mathbf{x} \xrightarrow{I} \sum_m EL^m(\mathbf{x}, \mathbf{u}^{(2\mathbf{n})}) \theta^m \wedge d\mathbf{x}$$

$$\partial_V = I \circ d_V \Rightarrow E^m(\mathbf{x}, \mathbf{u}^{(2\mathbf{n})}) = 0, \quad m = 1, \dots, q - \text{Euler-Lagrange eq.}$$

$$d_V(\lambda) - \partial_V(\lambda) = d_H(\mu), \quad \mu \in \Lambda^{p-1,1}$$

$$\Lambda^{p,0} \supset \text{trivial Lagrangians} = \ker \partial_V \supset d_H \Lambda^{p-1,0}$$

$$\text{Local exactness: } \ker \partial_V = d_H \Lambda^{p-1,0}$$

Inverse Problem of Calculus of Variations

Is a given system of equations $\{EL^m(\mathbf{x}, \mathbf{u}^{(2\mathbf{n})}) = 0 | m = 1, \dots, q\}$ equivalent to an Euler-Lagrange system ?

Simplification: for $\beta = \sum_m EL^m(\mathbf{x}, \mathbf{u}^{(2\mathbf{n})})\theta^m \wedge d\mathbf{x} \in \Lambda^{p,1}$
 $\exists? \lambda \in \Lambda^{p,0}$, s. t. $\partial_V(\lambda) = \beta$

Euler-Lagrange forms = $Im(\partial_V^0 : \Lambda^{p,0} \rightarrow \Lambda^{p,1}) \subset Ker(\partial_V^1 : \Lambda^{p,1} \rightarrow \Lambda^{p,2})$

Local exactness: $\Rightarrow \ker \partial_V^1 = Im(\partial_V^0)$

Noether Correspondence.

Symmetries of $\int L(\mathbf{x}, \mathbf{u}^{(n)})d\mathbf{x} \leftrightarrow$ Conservation laws of $E(L)$

generalized divergence symmetry: $Y = \sum_{j=1}^q Q^j(\mathbf{x}, \mathbf{u}^{(n)}) \frac{\partial}{\partial u^j}$ s.t
 $\exists A = (A_1, \dots, A_p): pr^n Y(L) = \text{Div} A$

\Downarrow

conservation law: $P = (P_1, \dots, P_p)$ s.t. $\text{Div} P \equiv 0 \pmod{E(L)}$

E.L.-eq: $E(L) = \left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} L(\mathbf{x}, (\mathbf{u} + \varepsilon f)^{(n)}) - \text{Div} M$

$Q, A, M \Rightarrow P$

$$\begin{array}{ccccccccc}
& \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
d_V \uparrow & & d_V \uparrow & & d_V \uparrow & & d_V \uparrow & & d_V \uparrow & & \partial_V \uparrow \\
\Lambda^{0,2} & \xrightarrow{d_H} & \Lambda^{1,2} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Lambda^{p-1,2} & \xrightarrow{d_H} & \Lambda^{p,2} & \xrightarrow{I} & \mathcal{F}^2 \\
d_V \uparrow & & d_V \uparrow & & \uparrow & & d_V \uparrow & & d_V \uparrow & & \partial_V \uparrow \\
\Lambda^{0,1} & \xrightarrow{d_H} & \Lambda^{1,1} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Lambda^{p-1,1} & \xrightarrow{d_H} & \Lambda^{p,1} & \xrightarrow{I} & \mathcal{F}^1 \\
d_V \uparrow & & d_V \uparrow & & \uparrow & & d_V \uparrow & & \uparrow & & \nearrow \partial_V \\
\Lambda^{0,0} & \xrightarrow{d_H} & \Lambda^{1,0} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Lambda^{p-1,0} & \xrightarrow{d_H} & \Lambda^{p,0} & &
\end{array}$$

- $\lambda = L(\mathbf{x}, \mathbf{u}^{(n)}) dx^1 \wedge \dots \wedge dx^p \in \Lambda^{p,0}$ -Lagrangian
- $d_V \lambda - \partial_V \lambda = d_H \mu, \mu \in \Lambda^{p-1,1}$
- Y is a variational symmetry: $\mathcal{L}_{prY}(\lambda) = d_H(\alpha), \alpha \in \Lambda^{p-1,0}$
- $\pi = Y \lrcorner \mu + \alpha \in \Lambda^{p-1,0}$ conservation law: $d_H \pi = 0 \text{ mod E-L eq.}$

Example:

$$\lambda = \frac{1}{2}\kappa^2 ds \Rightarrow EL = \kappa_{ss} + \frac{1}{2}\kappa^3$$

Generalized symmetry: $Y = \frac{-3u_2^2 u_1 + u_3(1+u_1^2)}{(1+u_1^2)^{\frac{5}{2}}} \frac{\partial}{\partial u} = \kappa_s \sqrt{1+u_1^2} \frac{\partial}{\partial u}$.

Conservation law:

$$\begin{aligned} P &= \frac{1}{4} \frac{u_2^4 + 36u_2^4 u_1^2 - 24u_2^2 u_3 u_1 - 24u_2^2 u_3 u_1^3 + 4u_3^2 + 8u_3^2 u_1^2 + 4u_3^2 u_1^4}{(1+u_1^2)^6} \\ &= \frac{1}{2}\kappa_s^2 + \frac{1}{8}\kappa^4 \end{aligned}$$

Check:

$$\text{Div}P = \frac{dP}{dx} = \kappa_s \sqrt{1+u_1^2} EL$$

conservation law with characteristic

$$Q = \kappa_s \sqrt{1+u_1^2}$$

Invariant Variational Complex

Invariant local coframe/frame on $\mathcal{U} \subset J^\infty(E, p)$.

Invariant horizontal basis	Invariant vertical basis
<i>Invariants moving coframe</i>	
<p>invariant “horizontal” one-forms</p> $\omega^1, \dots, \omega^p$ <p>$\tilde{H} = \text{span} \{ \omega^i \} \neq H = \text{span} \{ dx^i \}$ unless the action is projectable</p>	<p>invariant contact one-forms</p> $\vartheta^m, \vartheta_J^m$ <p>$\text{span} \{ \vartheta^m, \vartheta_J^m \} = \text{span} \{ \theta^m, \theta_J^m \}$</p>
<i>Invariants moving coframe</i>	
<p>invariant total diff. operators</p> $\mathcal{D}_1, \dots, \mathcal{D}_p$ <p>$\text{span} \{ \mathcal{D}_i \} = \text{span} \left\{ \frac{d}{dx_i} \right\}$</p>	<p>invariant vertical diff. operators</p> $V_1, \dots, V_q, V_m^J, m = 1..q$ <p>$\text{span} \{ V_m \} \neq \text{span} \left\{ \frac{\partial}{\partial u^m} \right\}$ unless the action is projectable</p>

Invariant Variational “Bicomplex”.

- variational edge is a complex
- interior rows are locally exact.

$$\begin{array}{ccccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
 & \uparrow d_V & & \uparrow d_{\tilde{V}} & & \uparrow d_{\tilde{V}} & & \uparrow d_{\tilde{V}} & & \uparrow d_{\tilde{V}} & & \uparrow \partial_{\tilde{V}} \\
 & \tilde{\Lambda}^{0,2} & \xrightarrow{d_{\tilde{H}}} & \tilde{\Lambda}^{1,2} & \xrightarrow{d_{\tilde{H}}} & \dots & \xrightarrow{d_{\tilde{H}}} & \tilde{\Lambda}^{p-1,2} & \xrightarrow{d_{\tilde{H}}} & \tilde{\Lambda}^{p,2} & \xrightarrow{\tilde{I}} & \mathcal{F}^2 \\
 & \uparrow d_{\tilde{V}} & & \uparrow d_{\tilde{V}} & & \uparrow & & \uparrow d_{\tilde{V}} & & \uparrow d_{\tilde{V}} & & \uparrow \partial_{\tilde{V}} \\
 & \tilde{\Lambda}^{0,1} & \xrightarrow{d_{\tilde{H}}} & \tilde{\Lambda}^{1,1} & \xrightarrow{d_{\tilde{H}}} & \dots & \xrightarrow{d_{\tilde{H}}} & \tilde{\Lambda}^{p-1,1} & \xrightarrow{d_{\tilde{H}}} & \tilde{\Lambda}^{p,1} & \xrightarrow{\tilde{I}} & \tilde{\mathcal{F}}^1 \\
 & \uparrow d_{\tilde{V}} & & \uparrow d_{\tilde{V}} & & \uparrow & & \uparrow d_V & & \uparrow & & \nearrow \partial_{\tilde{V}} \\
 & \tilde{\Lambda}^{0,0} & \xrightarrow{d_{\tilde{H}}} & \tilde{\Lambda}^{1,0} & \xrightarrow{d_{\tilde{H}}} & \dots & \xrightarrow{d_{\tilde{H}}} & \tilde{\Lambda}^{p-1,0} & \xrightarrow{d_{\tilde{H}}} & \tilde{\Lambda}^{p,0} & &
 \end{array}$$

- $\lambda \in \tilde{\Lambda}^{p,0}$ -Lagrangian $\rightarrow \partial_{\tilde{V}} \lambda = d_{\tilde{V}} \lambda - d_{\tilde{H}} \mu, \mu \in \tilde{\Lambda}^{p-1,1}$ (E.-L. eq.)
- variational symmetry: $Y \lrcorner d_{\tilde{V}} \lambda = d_{\tilde{H}}(\alpha), \alpha \in \tilde{\Lambda}^{p-1,0}$
- $\pi = Y \lrcorner \mu + \alpha \in \tilde{\Lambda}^{p-1,0}$ conservation law: $d_{\tilde{H}} \pi = 0 \text{ mod E.-L. eq.}$

Invariant bigrading: $\Lambda^* = \bigoplus \tilde{\Lambda}^{s,t} = \bigoplus \iota\Lambda^{s,t}$.

$$\tilde{\Lambda}^{s,t} \neq \Lambda^{s,t} (**)$$

$$d\iota(\lambda) = \iota[d\lambda - (dK)A \wedge \mathbf{v}(\lambda)], \text{ where } A = \mathbf{v}(K)^{-1}.$$

$$\lambda \in \Lambda^{s,t} \Rightarrow \iota\lambda \in \tilde{\Lambda}^{s,t} \text{ and}$$

$$\mathcal{L}_v(\lambda) = [\mathcal{L}_v(\lambda)]_0 + [\mathcal{L}_v(\lambda)]_V \in \Lambda^{s,t} \bigoplus \Lambda^{s-1,t+1} (**)$$

$$d = d_{\tilde{H}} + d_{\tilde{V}} + d_W (**)$$

where

$$d_{\tilde{H}} \iota(\lambda) = \iota[d_H \lambda - (d_H K)A \wedge \mathbf{v}(\lambda)_0],$$

$$d_{\tilde{V}} \iota(\lambda) = \iota[d_V \lambda - (d_V K)A \wedge \mathbf{v}(\lambda)_0],$$

$$d_W \iota(\lambda) = \iota[-(d_V K)A \wedge \mathbf{v}(\lambda)_V],$$

(**) unless the action is projectable.

for $s \geq 1$ unless the action is projectable:

$$d: \tilde{\Lambda}^{s,t} \longrightarrow \tilde{\Lambda}^{s+1,t} \oplus \tilde{\Lambda}^{s,t+1} \oplus \tilde{\Lambda}^{s-1,t+2} \Rightarrow d = d_{\tilde{H}} + d_{\tilde{V}} + d_W$$

$$d^2 = (d_{\tilde{H}} + d_{\tilde{V}} + d_W)^2 = 0$$

$$d_{\tilde{H}}^2 = 0, \quad d_{\tilde{V}}^2 + d_{\tilde{H}} d_W + d_W d_{\tilde{H}} = 0, \quad d_{\tilde{H}} \circ d_{\tilde{V}} = -d_{\tilde{V}} \circ d_{\tilde{H}}, \quad d_W^2 = 0$$

$$\partial_{\tilde{V}}^2 = 0$$

Euler-Lagrange operator in the invariant bicomplex

**Example: Lagrangian invariant under
 $SE(2) = SO(2) \times \mathbb{R}^2 \curvearrowright \mathbb{R}^2$ (Euclidean group.)**

$$\mathcal{L}[u] = \int \frac{u_{xx}^2}{2(1+u_x^2)^{5/2}} dx = \int L dx \iff \int \frac{1}{2} \kappa^2 ds$$

$$\downarrow E = \left(\frac{d}{dx}\right)^2 \frac{\partial}{\partial u_{xx}} - \left(\frac{d}{dx}\right) \frac{\partial}{\partial u_x} \quad ?? \quad \downarrow$$

$$E(L) = 0 \iff \kappa_{ss} + \frac{1}{2} \kappa^3 = 0$$

$$E(L) = \frac{1}{2} \frac{30 u_2^3 u_1^2 - 5 u_2^3 - 20 u_2 u_1 u_3 - 20 u_2 u_1^3 u_3 + 2 u_4 + 4 u_4 u_1^2 + 2 u_4 u_1^4}{(1 + u_1^2)^{(9/2)}}.$$

$$(u_1 = u_x, \dots, u_4 = u_{xxxx})$$

G -invariant variational problems for curves in \mathbb{R}^2 .

$$\int \tilde{\lambda} = \int \tilde{L}(\kappa, \kappa_1, \dots, \kappa_n) \omega,$$

κ – fundamental invariant under G – action

ω – invariant arc-length

$\kappa_i = \mathcal{D}^i \kappa$ – i -the derivative of κ with respect to invariant arc-length

\mathcal{D} derivative with respect to ω

The invariant Euler-Lagrange operator:

$$\boxed{\mathcal{A}^* \mathcal{E} - \mathcal{B}^* \mathcal{H}}$$

$$\mathcal{E}(\tilde{L}) = \sum_{i=0}^n (-\mathcal{D})^i \frac{\partial \tilde{L}}{\partial \kappa_i}, \quad \mathcal{H}(\tilde{L}) = \sum_{i>j \geq 0}^n \kappa_{i-j} (-\mathcal{D})^j \frac{\partial \tilde{L}}{\partial \kappa_i} - \tilde{L}.$$

$d_{\tilde{V}}(\omega) = \mathcal{B}(\vartheta_0) \wedge \omega$, $d_{\tilde{V}}(\kappa) = \mathcal{A}(\vartheta_0)$ – depend on G , computable by diff. and linear algebra.

for multi-dimensional formula see Kogan, Olver 2003

Invariant integration by parts for plane curves.

$G \curvearrowright J(\mathbb{R}^2, 1)$, ω – invariant differential form, \mathcal{D} – dual invariant differential operator, κ – generating invariant, $\kappa_i = (\mathcal{D})^i \kappa$.

$$\tilde{\lambda} = \tilde{L}(\kappa, \kappa_1, \dots, \kappa_n) \omega$$

$$d_{\tilde{V}} \tilde{\lambda} = d_{\tilde{V}} \tilde{L} \wedge \omega + \tilde{L} d_{\tilde{V}} \omega = \sum_{i=0}^n \frac{\partial \tilde{L}}{\partial \kappa_i} (d_{\tilde{V}} \kappa_i) \wedge \omega + \tilde{L} d_{\tilde{V}} \omega \equiv (\text{mod } d_{\tilde{H}})$$

$$(d_{\tilde{V}} \kappa_i) \wedge \omega = (d_{\tilde{V}} \mathcal{D} \kappa_{i-1}) \wedge \omega = (d_{\tilde{V}} d_{\tilde{H}} \kappa_{i-1}) - \kappa_i d_{\tilde{V}} \omega = - (d_{\tilde{H}} d_{\tilde{V}} \kappa_{i-1}) - \kappa_i d_{\tilde{V}} \omega$$

$$\equiv \sum_{i=0}^n d_{\tilde{H}} \left(\frac{\partial \tilde{L}}{\partial \kappa_i} \right) (d_{\tilde{V}} \kappa_{i-1}) - \left(\sum_{i=0}^n \frac{\partial \tilde{L}}{\partial \kappa_i} \kappa_i - \tilde{L} \right) d_{\tilde{V}} \omega$$

$$= \sum_{i=0}^n (-\mathcal{D}) \left(\frac{\partial \tilde{L}}{\partial \kappa_i} \right) (d_{\tilde{V}} \kappa_{i-1}) \wedge \omega - (\dots) d_{\tilde{V}} \omega.$$

repeat!

Symbolic: recursively implement integration by parts, **not** the formulas.

$$d_{\tilde{V}} \tilde{\lambda} \equiv \mathcal{E}(\tilde{L}) d_{\tilde{V}} \kappa \wedge \omega - \mathcal{H}(\tilde{L}) d_{\tilde{V}} \omega.$$

$$\mathcal{E}(\tilde{L}) = \sum_{i=0}^n (-\mathcal{D})^i \frac{\partial \tilde{L}}{\partial \kappa_i}, \quad \mathcal{H}(\tilde{L}) = \sum_{i>j \geq 0}^n \kappa_{i-j} (-\mathcal{D})^j \frac{\partial \tilde{L}}{\partial \kappa_i} - \tilde{L}.$$

$$\lambda = L(x, u, u_1, \dots, u_m) dx \quad \leftrightarrow \quad \tilde{\lambda} = \tilde{L}(\kappa, \kappa_1, \dots, \kappa_n) \omega$$

$$d_V(dx) = 0$$

$$d_{\tilde{V}}(\omega) = \mathcal{B}(\vartheta_0) \wedge \omega$$

$$d_V(u) = \theta_0$$

$$d_{\tilde{V}}(\kappa) = \mathcal{A}(\vartheta_0)$$

$$d_{\tilde{V}} \tilde{\lambda} \equiv \left[\mathcal{A}^* \mathcal{E}(\tilde{L}) - \mathcal{B}^* \mathcal{H}(\tilde{L}) \right] \vartheta_0 \wedge \omega.$$

Examples:

- **Euclidean group:** $SE(2) = SO(2) \times R^2$

$$\mathcal{A}^* = \left(\frac{d}{ds}\right)^2 + \kappa^2$$

$$\mathcal{B}^* = -\kappa$$

- **Affine group:** $SA(2) = SL(2) \times R^2$

$$\mathcal{A}^* = \left(\frac{d}{ds}\right)^4 + \frac{5}{3}\kappa \left(\frac{d}{ds}\right)^2 + \frac{5}{3}\kappa_s \left(\frac{d}{ds}\right) + \frac{1}{3}\kappa_{ss} + \frac{4}{9}\kappa^2$$

$$\mathcal{B}^* = \frac{1}{3} \left(\frac{d}{ds}\right)^2 - \frac{2}{9}\kappa$$

**Example: Lagrangian invariant under
 $SE(2) = SO(2) \times \mathbb{R}^2 \curvearrowright \mathbb{R}^2$ (Euclidean group.)**

$$\mathcal{L}[u] = \int \frac{u_{xx}^2}{2(1+u_x^2)^{5/2}} dx = \int L dx \iff \int \frac{1}{2} \kappa^2 ds$$

$$\left\downarrow E = \left(\frac{d}{dx}\right)^2 \frac{\partial}{\partial u_{xx}} - \left(\frac{d}{dx}\right) \frac{\partial}{\partial u_x} \right. \quad \left. \left(\left(\frac{d}{ds}\right)^2 + \kappa^2\right) \mathcal{E} + \kappa \mathcal{H} \right\downarrow$$

$$E(L) = 0 \iff \kappa_{ss} + \frac{1}{2} \kappa^3 = 0$$

$$E(L) = \frac{1}{2} \frac{30 u_2^3 u_1^2 - 5 u_2^3 - 20 u_2 u_1 u_3 - 20 u_2 u_1^3 u_3 + 2 u_4 + 4 u_4 u_1^2 + 2 u_4 u_1^4}{(1 + u_1^2)^{(9/2)}}.$$

$$(u_1 = u_x, \dots, u_4 = u_{xxxx})$$

Noether correspondence in the invariant bicomplex

Noether correspondence. Inductive approach.

Classical computation:

- find (generalized) infinitesimal symmetries of a Lagrangian (Lie's method),
 - compute corresponding conservation laws.
-

Inductive approach

- reduce a Lagrangian by its group of point symmetries G ,
- find (generalized) infinitesimal symmetries of the reduced Lagrangian (Lie's method),
- compute **corresponding** conservation laws (invariant with respect to G).

Infinitesimal symmetry condition for $\lambda \in \tilde{\Lambda}^{p,0}$.

$$\mathcal{L}_{prY} \lambda \equiv d_{\tilde{H}} \alpha \text{ mod } \tilde{\Lambda}^{p-1,1}$$

\Updownarrow

$$prY \lrcorner d_{\tilde{V}} \lambda = d_{\tilde{H}} \alpha, \text{ for } \alpha \in \tilde{\Lambda}^{p-1,0}.$$

Noether correspondence:

$d_{\tilde{V}} \lambda - \partial_{\tilde{v}} \lambda = d_{\tilde{H}} \mu$ for $\mu \in \tilde{\Lambda}^{p-1,1} \implies d_{\tilde{H}} P = 0 \text{ mod } EL$, where

$\boxed{\pi = prY \lrcorner \mu + \alpha}$ is a conservation law.

- $SE(2) \curvearrowright \mathbb{R}^2$ -inv. Lagrangian: $\lambda = \frac{1}{2}\kappa^2\omega$, $\omega = ds + \frac{u_1}{\sqrt{1+u_x^2}}\theta$.

$$d_{\tilde{V}}\lambda = EL\vartheta \wedge \omega + d_H M, \text{ where } M = \kappa_1\vartheta_1 - \kappa\vartheta_2.$$

- Euler-Lagrange equation: $EL = \kappa_{ss} + \frac{1}{2}\kappa^3$.

- $Y = \xi(\kappa, \kappa_s)V$. symmetry condition $\implies \xi = \kappa_1 f(\kappa^2 + 4\kappa_1^2)$.

Take $Y = \kappa_1 V$

$$prY \lrcorner d_{\tilde{V}}\lambda = d_{\tilde{H}}A \text{ where } A = \kappa\kappa_2 + \frac{1}{4}\kappa^4 - \kappa_1^2.$$

- Conservation laws: $P = Y \lrcorner M + A$

$$P = \kappa_s^2 + \frac{1}{4}\kappa^4, \text{ with } \frac{d}{ds}P = \kappa_s(\kappa_{ss} + \frac{1}{2}\kappa^3) \equiv 0 \text{ mod } EL.$$

- $SA(2) \curvearrowright \mathbb{R}^2$ -inv. Lagrangian: $\lambda = \mu \omega$, $\mu = \frac{u_2 u_4 - \frac{5}{3} u_3^2}{u_2^8/3}$,
 $\omega = u_2^{1/3} dx + \frac{u_3}{3 u_2^{5/3}} \theta$.

$$d_{\tilde{V}} \lambda = EL \vartheta \wedge \omega + d_H M, \text{ where } M = \frac{2}{3} \vartheta_1 - \frac{2}{3} \vartheta_2 - \vartheta_4.$$

- Euler-Lagrange equation: $-\left(\frac{2}{3} \mu_2 + \frac{2}{9} \mu^2\right)$.

- $Y = \xi(\kappa, \kappa_s)V$. symmetry condition $\implies \xi = \kappa_1 f\left(\frac{2}{9} \kappa^3 + \kappa_1^2\right)$.
 Take $Y = \kappa_1 V$

$$prY \lrcorner d_{\tilde{V}} \lambda = d_{\tilde{H}} A \text{ where } A = \mu_4 + 2\mu\mu_2 + \frac{2}{27} \mu^3.$$

- Conservation laws: $P = Y \lrcorner M + A$

$$P = \frac{2}{27} \mu^3 + \frac{1}{3} \mu_1^2$$

Implemented relative to invariant frames

IVBSTRUCTURE and IVARCALC programmed in VESSIOT environment.

(for maple worksheet contact iakogan@ncsu.edu)

- integration by parts \Rightarrow E.L equations, Helmholtz conditions.
- vector field prolongation
- exactness of interior rows of variational bicomplex
- Noether correspondence (non divergence symmetries)

Thank you!