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Multiplication of solutions for systems of PDE's

$$B^{(1)}dV^{(1)} + \cdots + B^{(m)}dV^{(m)} = 0$$

S – solution set

$* : S \times S \longrightarrow S$ bi-linear

Cauchy-Riemann equations

$$\begin{cases} \dfrac{\partial V}{\partial x} = \dfrac{\partial \tilde{V}}{\partial y} \\[2ex] \dfrac{\partial V}{\partial y} = -\dfrac{\partial \tilde{V}}{\partial x} \end{cases} \qquad (CR)$$

Multiplication on the solution set

$$(U + i\tilde{U})(V + i\tilde{V}) = (UV - \tilde{U}\tilde{V}) + i(U\tilde{V} + \tilde{U}V)$$

$$(U, \tilde{U}) * (V, \tilde{V}) = (UV - \tilde{U}\tilde{V},\ U\tilde{V} + \tilde{U}V)$$

Construct complicated solutions of CR from simple solutions with $*$

$$(V, \tilde{V}) = \sum_r a_r \underbrace{(x,y) * \cdots * (x,y)}_{r \text{ factors}} = \sum_r a_r (x,y)_*^r$$

<u>Q.</u> Are there other systems of PDE's with multiplication on the solution set?

Cofactor pair systems

$$\begin{cases} \ddot{q}^h + \Gamma^h_{ij}\dot{q}^i\dot{q}^j = F^h, \quad \text{where} \\ F = -(\operatorname{Cof} J)^{-1}\nabla V = -(\operatorname{Cof}\tilde{J})^{-1}\nabla\tilde{V} \end{cases} \qquad \text{(CPS)}$$

$$\operatorname{Cof} J = (\det J)\, J^{-1}$$

$J, \tilde{J}$ Special Conformal Killing (SCK) tensors

$$\left(\nabla_h J_{ij} = \tfrac{1}{2}(\alpha_i g_{jh} + \alpha_j g_{ih}) \right)$$

$$\boxed{(\operatorname{Cof} J)^{-1}\nabla V = (\operatorname{Cof}\tilde{J})^{-1}\nabla\tilde{V}}$$

Multiplication on solution set (H. Lundmark)

<u>$n=2$</u>:

$$(U,\tilde{U}) * (V,\tilde{V}) = \left(UV - \det(\tilde{J}^{-1}J)\tilde{U}\tilde{V},\ U\tilde{V} + \tilde{U}V - \operatorname{tr}(\tilde{J}^{-1}J)\tilde{U}\tilde{V}\right)$$

Jacobi family of separable potentials

$$J = \begin{bmatrix} -x^2+\lambda_1 & -xy \\ -xy & -y^2+\lambda_2 \end{bmatrix} \qquad \tilde{J} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\ddot{q} = -(\operatorname{cof} J)^{-1} \nabla V = -\nabla \tilde{V}$$

$$(V_k, \tilde{V}_k) := \left((0,1)_*^2 + (\lambda_1\lambda_2, \lambda_1+\lambda_2) \right) * (0,1)_*^{k-1}$$

Infinite sequence of separable potentials

$$\tilde{V}_1 = x^2 + y^2$$
$$\tilde{V}_2 = (x^2+y^2)^2 - (\lambda_1 x^2 + \lambda_2 y^2)$$
$$\tilde{V}_3 = (x^2+y^2)^3 - 2(x^2+y^2)(\lambda_1 x^2 + \lambda_2 y^2) + (\lambda_1^2 x^2 + \lambda_2^2 y^2)$$
$$\vdots$$

$$\boxed{A_\mu dV_\mu \equiv 0 \pmod{Z_\mu}}$$

Q – n-dim. differentiable mfd

$$Z_\mu = Z^{(0)} + \mu Z^{(1)} + \dots + \mu^{m-1} Z^{(m-1)} + \mu^m \qquad Z^{(i)} \in C^\infty(Q)$$

$$A_\mu = A^{(0)} + \mu A^{(1)} + \dots + \mu^k A^{(k)} \qquad A^{(i)} \text{ (1,1)-tensor}$$

$$V_\mu = V^{(0)} + \mu V^{(1)} + \dots + \mu^{m-1} V^{(m-1)} \qquad V^{(i)} \text{ unknown fcn}$$

3 parameters – (n, m, k)

Ex. $(n,m,k) = (n,2,1)$

$$(A + \mu\tilde{A})\, d(V + \mu\tilde{V}) = AdV + \mu(Ad\tilde{V} + \tilde{A}dV) + \underbrace{\mu^2}_{\equiv -Z-\mu\tilde{Z}} \tilde{A}d\tilde{V} \equiv$$

$$\equiv (AdV - Z\tilde{A}d\tilde{V}) + \mu(Ad\tilde{V} + \tilde{A}dV - \tilde{Z}\tilde{A}d\tilde{V}) \equiv 0$$

$$\Longleftrightarrow \begin{cases} AdV = Z\tilde{A}d\tilde{V} \\ \tilde{A}dV = -Ad\tilde{V} + \tilde{Z}\tilde{A}d\tilde{V} \end{cases}$$

$$\left.\begin{array}{l} Z_\mu = 1 + \mu^2 \\ A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \\ \tilde{A} = Id \end{array}\right\} \Longrightarrow (CR)$$

$$A_\mu dV_\mu \equiv \alpha^{(0)} + \mu\alpha^{(1)} + \cdots + \mu^{m-1}\alpha^{(m-1)} \equiv 0$$

$$\Longleftrightarrow \quad \alpha^{(0)} = \ldots = \alpha^{(m-1)} = 0$$

nm 1:st order linear PDE's

m unknown fcns $V^{(j)}$

n independent variables q^i

In general <u>overdetermined</u>.

$*$ - multiplication

$$U_\mu V_\mu = Q_\mu Z_\mu + R_\mu , \quad \deg R_\mu < \deg Z_\mu = m$$

$$\boxed{U_\mu * V_\mu := R_\mu}$$

<u>Thm.</u> $*$ is an operation on the solution set iff

$$A_\mu d Z_\mu \equiv 0 \pmod{Z_\mu}$$

<u>proof.</u> Suppose $A_\mu d U_\mu \equiv A_\mu d V_\mu \equiv 0$

$$A_\mu d (U_\mu * V_\mu) = A_\mu d (U_\mu V_\mu - Q_\mu Z_\mu)$$

$$= V_\mu A_\mu d U_\mu + U_\mu A_\mu d V_\mu - Z_\mu A_\mu d Q_\mu - Q_\mu A_\mu d Z_\mu$$

$$\equiv - Q_\mu A_\mu d Z_\mu$$

▨

Can use $*$ to generate non-trivial solutions from trivial (constant) solutions.

$$\sum_r a_r \mu * \cdots * \mu = \sum_r a_r \mu_*^r$$

Matrix notation

$$P_\mu = P^{(0)} + P^{(1)}\mu + \cdots + P^{(a)}\mu^a \equiv [1\ \mu \cdots \mu^{m-1}] P_c e_1$$

$$P_c = P^{(0)}C^0 + P^{(1)}C + \cdots + P^{(a)}C^a$$

$$C = \begin{bmatrix} 0 & 0 & \cdots & -z^{(0)} \\ 1 & 0 & & -z^{(1)} \\ & \ddots & & \vdots \\ & & 1 & -z^{(m-1)} \end{bmatrix} \qquad e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$V_\mu = V^{(0)} + \mu V^{(1)} + \cdots + \mu^{m-1} V^{(m-1)} \longleftrightarrow \begin{bmatrix} V^{(0)} \\ \vdots \\ V^{(m-1)} \end{bmatrix} = V_c e_1$$

$$A_\mu dV_\mu \equiv 0 \longleftrightarrow \sum_{i=0}^{k} C^i V' A^{(i)} = 0$$

$$V' = \begin{bmatrix} \partial_1 V^{(0)} & \cdots & \partial_n V^{(0)} \\ \vdots & & \vdots \\ \partial_1 V^{(m-1)} & \cdots & \partial_n V^{(m-1)} \end{bmatrix}$$

$$V_\mu * U_\mu \longleftrightarrow V_c U_c e_1$$

$$\sum_r a_r \mu_*^r \longleftrightarrow \sum_r a_r C^r e_1$$

Power series

$$\mu \longleftrightarrow \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = Ce_1$$

$$\mu_*^r \longleftrightarrow C^r e_1$$

$$\sum_{r=0}^{\infty} a_r C^r e_1 = \sum_{r=0}^{\infty} a_r (TJT^{-1})^r e_1$$

$$= T\left(\sum_{r=0}^{\infty} a_r J^r\right) T^{-1} e_1$$

Convergence when $|\lambda_i| < R := 1 / \overline{\lim_r} |a_r|^{1/r}$

λ_i - eigenvalues of C = roots of Z_μ

__Thm.__ $\sum_{r=0}^{\infty} a_r \mu_*^r$ is a solution if $|\lambda_i| \leq R - \varepsilon$.

<u>Ex.</u> $(n,m,k) = (2,2,1)$

$$A_\mu dV_\mu \equiv 0 \iff \begin{cases} -\dfrac{\partial V}{\partial y} = x\dfrac{\partial \tilde{V}}{\partial x} \\ x\dfrac{\partial V}{\partial x} + y\dfrac{\partial V}{\partial y} = x\dfrac{\partial \tilde{V}}{\partial y} \end{cases}$$

Multiplication:

$$\begin{bmatrix} U \\ \tilde{U} \end{bmatrix} * \begin{bmatrix} V \\ \tilde{V} \end{bmatrix} = \begin{bmatrix} UV - x\tilde{U}\tilde{V} \\ U\tilde{V} + \tilde{U}V - y\tilde{U}\tilde{V} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}^2 = \begin{bmatrix} -x \\ -y \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}^3 = \begin{bmatrix} xy \\ -x+y^2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}^4 = \begin{bmatrix} x^2 - xy^2 \\ 2xy - y^3 \end{bmatrix}$$

Power series: $\quad C = \begin{bmatrix} 0 & -x \\ 1 & -y \end{bmatrix}, \quad \begin{cases} x = \lambda_1\lambda_2 \\ y = -\lambda_1 - \lambda_2 \end{cases}$

$$\sum_r a_r C^r e_1 = T\left(\sum a_r D^r\right)T^{-1}e_1$$

$$= \frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} \lambda_2 & \lambda_1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} \sum a_r\lambda_1^r & 0 \\ 0 & \sum a_r\lambda_2^r \end{bmatrix}\begin{bmatrix} -1 & -\lambda_1 \\ 1 & \lambda_2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\lambda_1 - \lambda_2}\begin{bmatrix} \lambda_1 \sum a_r\lambda_2^r - \lambda_2 \sum a_r \lambda_1^r \\ \sum a_r\lambda_1^r - \sum a_r\lambda_2^r \end{bmatrix}$$

General solution:

$$\begin{bmatrix} V \\ \tilde{V} \end{bmatrix} = \frac{1}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_1 \Psi(\lambda_2) - \lambda_2 \varphi(\lambda_1) \\ \varphi(\lambda_1) - \Psi(\lambda_2) \end{bmatrix}$$

Characterization of systems with *

$$\boxed{A_\mu dZ_\mu \equiv 0 \pmod{Z_\mu}}$$

- Complicated non-linear sys. of PDE's
- For fixed Z_μ it is easy to find all solutions for A_μ.

Special case $(n,m,k) = (n,n,1)$

$$\begin{cases} A_\mu = X_\mu = X + \mu I \\ Z_\mu = \det X_\mu \end{cases}$$

$$A_\mu dZ_\mu \equiv 0$$

$$\Longleftrightarrow$$

$$X_\mu \, d(\det X_\mu) = (\det X_\mu)\, d(\operatorname{tr} X_\mu)$$

Known solutions

1. $X = \tilde{J}^{-1} J$, $J, \tilde{J}$ SCK tensors
2. $N_X = 0$

1. $X = \tilde{J}^{-1} J$ – Cofactor pair systems

J SCK if $\nabla_h J_{ij} = \frac{1}{2}\left(\alpha_i g_{jh} + \alpha_j g_{ih}\right)$

CPS:

$$\ddot{q}^h + \Gamma^h_{ij}\dot{q}^i\dot{q}^j = F^h, \quad \text{where}$$

$$F = -(\mathrm{Cof}\,J)^{-1}\nabla V = -(\mathrm{Cof}\,\tilde{J})^{-1}\nabla\tilde{V}, \quad \mathrm{Cof}\,J = (\det J)\,J^{-1}$$

$$\boxed{(\mathrm{Cof}\,J)^{-1}\nabla V = (\mathrm{Cof}\,\tilde{J})^{-1}\nabla\tilde{V}}$$

Multiplication $*$ on solution set (H. Lundmark)

Same compatibility conditions

$$\left(\mathrm{Cof}\,(J + \mu\tilde{J})\right)^{-1}\nabla V_\mu = (\mathrm{Cof}\,\tilde{J})^{-1}\nabla\tilde{V}$$

$$\Longleftrightarrow$$

$$X_\mu \nabla V_\mu \equiv 0 \pmod{\det X_\mu}, \quad X_\mu = \tilde{J}^{-1}J + \mu I$$

$$X_\mu \nabla Z_\mu \equiv 0$$

2. $N_X = 0$

N_X - the Nijenhuis torsion of X, a $(1,2)$-tensor

$$N_X(u,v) = [Xu, Xv] - X[Xu,v] - X[u,Xv] + X^2[u,v]$$

Thm.

$$N_X = 0 \implies X_\mu\, d(\det X_\mu) = (\det X_\mu)\, d(\operatorname{tr} X_\mu)$$

proof.

- $2\big(X\,d(\det X) - (\det X)\,d(\operatorname{tr} X)\big)_i = (N_X)^h_{ij}\,(\operatorname{Cof} X)^j_h$
- $N_{X+\mu I} = N_X$