

# Discrete symmetries and Lie algebra automorphisms

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Find *all* point symmetries  $\Gamma : (x, y) \rightarrow (\hat{x}(x, y), \hat{y}(x, y))$  of a given ODE

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The point symmetries are prolonged as follows:

$$\hat{y}' = \frac{D_x \hat{y}}{D_x \hat{x}}, \quad \hat{y}'' = \frac{D_x \hat{y}'}{D_x \hat{x}},$$

where  $D_x$  is the total derivative w.r.t.  $x$ :

$$D_x = \partial_x + y' \partial_y + y'' \partial_{y'} + \dots$$

**A little more detail:**

The symmetry condition (2) amounts to

$$\frac{\{(\hat{x}_x + y'\hat{x}_y)(\hat{y}_{xx} + 2y'\hat{y}_{xy} + y'^2\hat{y}_{yy})\} - \{\hat{x} \leftrightarrow \hat{y}\} + (\hat{x}_x\hat{y}_y - \hat{x}_y\hat{y}_x)\omega(x, y, y')}{(\hat{x}_x + y'\hat{x}_y)^3} = \omega\left(\hat{x}, \hat{y}, \frac{\hat{y}_x + y'\hat{y}_y}{\hat{x}_x + y'\hat{x}_y}\right). \quad (3)$$

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**Problem:** The overdetermined system may not be easy to solve (Reid *et al.* 1993). Things are worse for problems that are not so 'simple' (systems, PDEs, higher order, higher symmetries, ...).

## What can be done (easily)?

Lie point symmetries can be found by linearizing the symmetry condition as follows:

$$\hat{x} = x + \epsilon\xi(x, y) + O(\epsilon^2), \quad \hat{y} = y + \epsilon\eta(x, y) + O(\epsilon^2).$$

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For  $|\epsilon|$  sufficiently small, Lie point symmetries are obtained by exponentiation:

$$\hat{x} = e^{\epsilon X}x, \quad \hat{y} = e^{\epsilon X}y$$

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However, after factoring out Lie symmetries, inequivalent discrete symmetries remain to be found.

**First message:** Given a differential equation whose Lie point symmetries are known (and nontrivial), one can find the remaining discrete point symmetries with little extra effort.

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For simplicity, restrict attention to real finite-dimensional Lie algebras

$$\mathcal{L} = \text{Span}(X_1, \dots, X_R).$$

**A useful observation:**

Let  $\Gamma : (x, y) \rightarrow (\hat{x}, \hat{y})$  be a point symmetry. If

$$X = \xi(x, y)\partial_x + \eta(x, y)\partial_y$$

generates the Lie point symmetries  $\Gamma_\epsilon = e^{\epsilon X}$  then

$$\hat{X} = \xi(\hat{x}, \hat{y})\partial_{\hat{x}} + \eta(\hat{x}, \hat{y})\partial_{\hat{y}}$$

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Thus  $\{\hat{X}_1, \dots, \hat{X}_R\}$  is a basis for  $\mathcal{L}$ , so

$$X_i = b_i^j \hat{X}_j; \tag{4}$$

the matrix  $B = (b_i^j)$  is constant and nonsingular.

## Necessary conditions for discrete symmetries

Some information about  $(\hat{x}, \hat{y})$  arises directly from (4):

$$X_i(\hat{x}) = b_i^l \xi_l(\hat{x}, \hat{y}), \quad X_i(\hat{y}) = b_i^l \eta_l(\hat{x}, \hat{y}). \quad (5)$$

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**Problem:** What are the constants  $b_i^l$ ? (Ignorance is not bliss.)

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$$[X_i, X_j] = c_{ij}^k X_k \quad \Rightarrow \quad [\hat{X}_i, \hat{X}_j] = c_{ij}^k \hat{X}_k. \quad (6)$$

In other words,  $\Gamma$  induces a Lie algebra automorphism.

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Consequently the constants  $b_i^l$  satisfy

$$c_{lm}^n b_i^l b_j^m = c_{ij}^k b_k^n. \quad (7)$$

Solve (7) to simplify the matrix  $B$ .

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- 4 the Cartan-Killing metric  $g_{ij} = c_{ik}^l c_{jl}^k$ ; consequently

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Furthermore, (7) implies the linear constraints

$$\gamma_j b_i^j = \gamma_i, \quad \text{where } \gamma_i = c_{ki}^k. \quad (9)$$

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If  $X_j$  is not a central element then define the matrix  $C(j)$  by  $(C(j))_i^k = c_{ij}^k$  and let  $A(j) = \exp\{\epsilon_j C(j)\}$ .

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Central Lie symmetries should be factored out after the symmetry condition has been solved.

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- 5 Factor out the remaining central Lie symmetries.

The second step is the most difficult, but it depends only on the abstract structure of  $\mathcal{L}$ . A table of automorphisms would be handy – this would also be useful in other applications.

## Decomposability

A Lie algebra  $\mathcal{L}$  is *decomposable* if there exist nontrivial subalgebras  $\mathcal{M}_i$  such that

$$\mathcal{L} = \mathcal{M}_1 \dot{+} \cdots \dot{+} \mathcal{M}_S, \quad [\mathcal{M}_i, \mathcal{M}_j] = 0, \quad i \neq j.$$

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**Theorem:** If  $\mathcal{L}$  is decomposable then every Lie algebra automorphism of  $\mathcal{L}$  is the product of

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- automorphisms of each  $\mathcal{M}_i$  (acting trivially on the other components),
- permutations of isomorphic components,
- automorphisms that add a central element to each  $X \notin [\mathcal{L}, \mathcal{L}]$ .

**Second message:** All automorphisms of a decomposable Lie algebra can be constructed from the automorphisms of its components.

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Consequently, the list can be restricted to indecomposable  $\mathcal{L}$ .

## Two- and three-dimensional Lie algebras (cf. Patera *et al.*)

Name	Nonzero $c_{ij}^k$ ( $i < j$ )	Outer Der.	Discrete Gen.	Block Diagonal
$A_{2,1}$	$c_{12}^1 = 1$	—	$p_1$	$(1, 1, 1)$
$A_{3,1}$	$c_{23}^1 = 1$ (nilpotent)	—	—	$(\det(G_{23}), G_{23})$
$A_{3,2}$	$c_{13}^1 = c_{23}^1 = c_{23}^2 = 1$	—	—	$(a, a, 1)$
$A_{3,3}$	$c_{13}^1 = c_{23}^2 = 1$	—	$p_1$	$(S_{12}, 1)$
$A_{3,4}$	$c_{13}^1 = 1, c_{23}^2 = -1$	—	$p_1, (X_2, X_1, -X_3)$	$(1, a, 1)$
$A_{3,5}^u$	$c_{13}^1 = 1, c_{23}^2 = u$ ( $0 <  u  < 1$ )	—	$p_1$	$(1, a, 1)$
$A_{3,6}$	$c_{13}^2 = -1, c_{23}^1 = 1$	—	$p_{23}$	$( a ,  a , 1)$
$A_{3,7}^u$	$c_{13}^1 = c_{23}^2 = u, c_{13}^2 = -1, c_{23}^1 = 1$ ( $u > 0$ )	—	—	$(T(u), T(u), 1)$
$A_{3,8}$	$c_{12}^1 = c_{23}^3 = 1, c_{13}^2 = -2$ ( $\mathfrak{sl}(2, \mathbb{R})$ )	—	$p_{13}, (X_3, -X_2, X_1)$	$(1, 1, 1)$
$A_{3,9}$	$c_{12}^3 = c_{23}^1 = 1, c_{13}^2 = -1$ ( $\mathfrak{so}(3)$ )	—	—	$(1, 1, 1)$

**Code:**  $a, b \in \mathbb{R} \setminus \{0\}$ ;  $G/S$  – general/special linear block;  $p$  – parity switch;

$T(u) = e^t, t \in [0, 2\pi u)$ .

**Example:** The Chazy equation,

$$y''' = 2yy'' - 3y'^2 + \lambda(6y' - y^2)^2,$$

has the Lie algebra  $\mathfrak{sl}(2)$ :

$$X_1 = \partial_x, \quad X_2 = x\partial_x - y\partial_y, \quad X_3 = -x^2\partial_x + (2xy + 6)\partial_y.$$

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From the table, there are four inequivalent automorphisms, generated by

$$\begin{aligned} \Gamma_1 : (\hat{X}_1, \hat{X}_2, \hat{X}_3) &= (-X_1, X_2, -X_3), \\ \Gamma_2 : (\hat{X}_1, \hat{X}_2, \hat{X}_3) &= (X_3, -X_2, X_1). \end{aligned}$$

Solving the necessary condition (5) with  $\Gamma_1$  gives

$$(\hat{x}, \hat{y}) \in \{(-x, -y), (-x - 6/y, y)\}.$$

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However, only the first of these satisfies the symmetry condition.

Similarly  $\Gamma_2$  yields two solutions, of which only

$$(\hat{x}, \hat{y}) = (1/x, -x^2y - 6x)$$

is a symmetry.

## Four-dimensional Lie algebras

Name	Nonzero $c_{ij}^k$ ( $i < j$ )	Out. Der.	Discr. Gen.	Block Diag.
$A_{4,1}$	$c_{24}^1 = c_{34}^2 = 1$ (nilpotent)	$E_3^1, E_4^3$	—	$(ab^2, ab, a, b)$
$A_{4,2}^u$	$c_{14}^1 = u, c_{24}^2 = c_{34}^3 = c_{34}^1 = 1$ ( $u \notin \{0, 1\}$ )	—	—	$(a, b, b, 1)$
$A_{4,2}^1$	$c_{14}^1 = c_{24}^2 = c_{34}^3 = c_{34}^1 = 1$	$E_1^2, E_3^4$	—	$(a, b, b, 1)$
$A_{4,3}$	$c_{14}^1 = c_{34}^2 = 1$	$E_4^3$	—	$(a, b, b, 1)$
$A_{4,4}$	$c_{14}^1 = c_{24}^1 = c_{24}^2 = c_{34}^2 = c_{34}^3 = 1$	$E_3^1$	—	$(a, a, a, 1)$
$A_{4,5}^{uv}$	$c_{14}^1 = 1, c_{24}^2 = u, c_{34}^3 = v$ ( $uv \neq 0, -1 \leq u < v < 1$ )	—	$P_1$	$(1, a, b, 1)$
$A_{4,5}^{uu}$	$c_{14}^1 = 1, c_{24}^2 = c_{34}^3 = u$ ( $u \neq 0, -1 \leq u < 1$ )	—	$P_1$	$(1, G_{23}, 1)$
$A_{4,5}^{u1}$	$c_{14}^1 = u, c_{24}^2 = c_{34}^3 = 1$ ( $u \neq 0, -1 \leq u < 1$ )	—	$P_1$	$(1, G_{23}, 1)$
$A_{4,5}^{11}$	$c_{14}^1 = c_{24}^2 = c_{34}^3 = 1$	—	$P_1$	$(S_{123}, 1)$
$A_{4,6}^{uv}$	$c_{14}^1 = u, c_{24}^2 = c_{34}^3 = v, c_{24}^3 = -1, c_{34}^2 = 1$ ( $u \neq 0, v \geq 0$ )	—	—	$(1, T(v), T(v), 1)$
$A_{4,7}$	$c_{14}^1 = 2, c_{23}^2 = c_{24}^2 = c_{34}^2 = c_{34}^3 = 1$	—	—	$(a^2, a, a, 1)$
$A_{4,8}$	$c_{23}^1 = c_{24}^2 = 1, c_{34}^3 = -1$	$E_4^1$	$P_{12},$ $(-X_1, X_3, X_2, -X_4)$	$(a, 1, a, 1)$
$A_{4,9}^u$	$c_{14}^1 = u+1, c_{23}^1 = c_{24}^2 = 1, c_{34}^3 = u$ ( $0 <  u  < 1$ )	—	$P_{12}$	$(a, 1, a, 1)$
$A_{4,9}^0$	$c_{14}^1 = c_{23}^1 = c_{24}^2 = 1$	$E_2^1$	$P_{12}$	$(a, 1, a, 1)$
$A_{4,9}^1$	$c_{14}^1 = 2, c_{23}^1 = c_{24}^2 = c_{34}^3 = 1$	—	$P_{12}$	$(1, S_{23}, 1)$
$A_{4,10}$	$c_{23}^1 = c_{34}^2 = 1, c_{24}^3 = -1$	$E_4^1$	$P_{124}$	$(a^2,  a ,  a , 1)$
$A_{4,11}^u$	$c_{14}^1 = 2u, c_{23}^1 = c_{24}^2 = 1, c_{24}^3 = c_{34}^3 = u, c_{24}^1 = -1$ ( $u > 0$ )	—	—	$((T(u))^2, T(u), T(u), 1)$
$A_{4,12}$	$c_{13}^1 = c_{23}^2 = c_{24}^2 = 1, c_{14}^1 = -1$	—	$P_{24}$	$(1, 1, 1, 1)$

**Third message:** If you want a list of the automorphisms of five-dimensional indecomposable Lie algebras, see me afterwards!

**The End**