# Invariant Prolongation and Detour Complexes

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**Dedicated to the memory of Tom Branson!**

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## TWO PROBLEMS:

I: Given a suitable linear overdetermined PD operator $D$, can one construct an equivalent first order prolonged system that is actually a connection. In fact we want more. The connection should be invariant in the sense that it is canonical; it should depend only on $D$ – essentially.

II. $\exists$ a construction of (elliptic) conformal **complexes** on curved conformal manifolds? (If yes then these will generalise conformal elliptic operators)

Moral: I & II are linked

## Finite type PDE

An example. In Riemannian geometry a tangent field $k$ is an infinitesimal isometry if Lie differentiation along its flow preserves the metric $g$, that is $\mathcal{L}_k g = 0$. I.e.

$$\nabla_a k_b + \nabla_b k_a = 0 \Leftrightarrow \nabla_a k_b = \mu_{ab} \in \Lambda^2.$$

Differentiate and use

$$(S^2 \otimes \Lambda^1) \cap (\Lambda^1 \otimes \Lambda^2) = 0$$

to obtain a prolonged system which is actually a connection:

$$\nabla_a^D \begin{pmatrix} k_b \\ \mu_{bc} \end{pmatrix} := \begin{pmatrix} \nabla_a k_b - \mu_{ab} \\ \nabla_a \mu_{bc} - R_{bc}{}^d{}_a k_d \end{pmatrix} = 0.$$

$R$ is the curvature of $\nabla$. Solutions of the original equation are in 1-1 correspondence with sections of $\mathbb{T} := \Lambda^1 \oplus \Lambda^2$ that are parallel for $\nabla^D$. It follows that the original equation has at most rank$(\mathbb{T}) = n(n+1)/2$ solutions. The curvature of $\nabla^D$ obstructs solutions and, in particular, the maximal number of solutions is achieved only if the connection $\nabla^D$ is flat.

If $\nabla$ is any connection on $\Lambda^1$ then note that the equation $\nabla_a k_b + \nabla_b k_a = 0$ is invariant under the transformations

$$k_b \mapsto \widehat{k}_b = e^{2\omega} k_b \quad \text{and} \quad \nabla \mapsto \widehat{\nabla}$$

where, as an operator on 1-forms,

$$\widehat{\nabla}_a u_b = \nabla_a u_b - \Upsilon_a u_b - \Upsilon_b u_a \quad \text{with} \quad \Upsilon = d\omega \ .$$

Restricting to torsion-free connections, these show that the equation is in fact **projectively invariant**. It is well-defined on manifolds that have only an equivalence class of connections.

Q: Does the connection $\nabla^D$ have this property?

A: Yes! – it is an **invariant curvature adjustment** of the **normal projective tractor connection**.

A more general class of equations are those for conformal Killing forms: For a $p$-form $\kappa$ the equation is that for any tangent vector field $u$ we have

$$\nabla_u \kappa = \varepsilon(u)\tau + \iota(u)\rho \quad \star$$

where, on the right-hand side $\tau$ is a $(p-1)$-form, $\rho$ is a $(p+1)$-form, and $\varepsilon(u)$ and $\iota(u)$ indicate, respectively, the exterior multiplication and (its formal adjoint) the interior multiplication of $g(u, \ )$.

A prolonged system and connection equivalent to $\star$ was given by Uwe Semmelmann. A general method of constructing such prolonged systems was developed by Branson, Čap, Eastwood & G. But neither of these treatments touched the issue of invariance. The system $\star$ is conformally invariant. It turns out that for that system there is a conformally invariant connection $\nabla^D$ math.DG/0601751 G. & Šilhan.

## DETOUR COMPLEXES.

First **BGG sequences:** A large class of Riemannian differential operators with good **conformal behaviour** are organised into sequences – like and including the de Rham complex.

metric **conformally flat**: $\Rightarrow$ BGG complexes

$$\mathcal{B}^0 \to \mathcal{B}^1 \to \cdots \to \mathcal{B}^n$$

- $\mathcal{B}^i$ are irreducible tensor-spinor bundles. On sphere each such resolves an irreducible rep. $\mathbb{V}$ for the group $G = SO(p+1, q+1)$ of conformal motions

There are curved analogues of these as sequences (Eastwood,Rice,Baston,G., Slovak, Čap, Souček . . ) – but **not as complexes**.

In **even dimensions** there are also "long operators" $\mathcal{B}^k \to \mathcal{B}^{n-k}$ for $k = 1, \cdots, n/2 - 1$ Boe-Collingwood. Thus there are conformal **detour** sequences of the form

$$\mathcal{B}^0 \xrightarrow{D_0} \mathcal{B}^1 \xrightarrow{D_1} \cdots \xrightarrow{D_{k-1}} \mathcal{B}^k \xrightarrow{L_k} \mathcal{B}^{n-k} \xrightarrow{D_{n-k}} \cdots \xrightarrow{D_{n-1}} \mathcal{B}^n ,$$

**complexes if conformally flat**.

**CURVED ANALOGUES?':** Yes if "de Rham" :

**Theorem 1 (Branson & G.)** *In even dimensions there are conformally invariant differential operators*

$L_k : \Lambda^k \to \Lambda_k \qquad\qquad L_k = (\delta d)^{n/2-k} +$ lower ord. .

*so that*

$$\Lambda^0 \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^{k-1} \xrightarrow{d} \Lambda^k \xrightarrow{L_k} \Lambda_k \xrightarrow{\delta} \Lambda_{k-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Lambda_0$$

*is a conformal complex. In Riemannian signature the complex is elliptic.*

$L_{n/2-1}$ is the usual Maxwell operator $\delta d$, while $L_0$ is the critical conformal Laplacian of GJMS. For $k \neq 0$ the complex may be viewed as a differential form analogue of the critical GJMS operator. For $0 \leq k \leq n/2-2$ the $L_k$ have the form $\delta Q_{k+1} d$ where the $Q_\ell$ are operators on closed forms that generalise the $Q$-curvature; for example they give conformal pairings that descend to pairings on de Rham cohomology math.DG/0511311 [Br. & G]

For other case it looks hopeless! The translation principle of Eastwood et al ER,B,CSS . . uses tractor connection twistings of e.g. de Rham:

$$\Lambda^0(\mathcal{V}) \stackrel{d^\nabla}{\to} \Lambda^1(\mathcal{V}) \stackrel{d^\nabla}{\to} \Lambda^2(\mathcal{V})$$

but $d^\nabla \circ d^\nabla$ is the curvature of $\nabla$ acting on $\mathcal{V}$. Decompose by differential splitting operators according to central character to get other complexes.

In c-flat dimensions $n \geq 5$ the initial part of the deformation complex is,

$$T \stackrel{\mathsf{K}_0}{\to} S^2_0[2] \stackrel{\mathsf{C}}{\to} \Lambda^{2,2}[2] \stackrel{\mathsf{Bi}}{\to} \Lambda^{3,2}[2] \to \cdots$$

where $T$ is the tangent bundle. Here C is the linearisation, at a conformally flat structure, of the Weyl curvature as an operator on conformal structure; Bi is a conformal integrability condition arising from the Bianchi identity; the operator $\mathsf{K}_0$ is the conformal Killing operator, viz the operator which takes infintesimal deformations to their action on conformal structure. $H^1$ is formal tangent space to the moduli space of conformally flat structures.

So no curved generalisation. But a way around difficulties is to look at $\boldsymbol{k = 1}$ **detours**.

The (Fefferman-Graham) **obstruction tensor** $\mathcal{B}_{ab} = \Delta^{n/2-2}\delta^\nabla\delta^\nabla C + lots$ – trace-free conformal conformal 2-tensor, generalises the Bach tensor. Let us write $B$ for the linearisation of $g \mapsto \mathcal{B}^g$. By taking the Lie derivative of $\mathcal{B}_{ab}$, and using the fact [Graham,Hirachi] $\mathcal{B}_{ab}$ is the total metric variation of $\int Q$ we get:

**Theorem 2 (Branson & G)** *On even dimensional pseudo-Riemannian manifolds with the Fefferman-Graham obstruction tensor vanishing everywhere, the sequence of operators*

$$T \stackrel{K_0}{\to} S^2_0 \stackrel{B}{\to} S^2_0 \stackrel{K_0^*}{\to} T$$

*is a formally self-adjoint complex of conformally invariant operators. In Riemannian signature the complex is elliptic.*

$H^1$ is formal tangent space to the moduli space of obstruction-flat structures.

**Lots of complexes:** Riemannian $n$-manifolds. The simplest of detours is the Maxwell:

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{\delta d} \Lambda_1 \xrightarrow{\delta} \Lambda_0.$$

Twist with a connection $(\tilde{\nabla}, \mathcal{V}) \Rightarrow$ not a cx. Write $F := curvature(\tilde{\nabla})$ and $F\cdot$ for action twisted 1-forms, $F\cdot : \Lambda^1(V) \to \Lambda_1(V)$ by

$$(F\cdot\varphi)_a := F_a{}^b \varphi_b,$$

Then

$$M^{\tilde{\nabla}} : \Lambda^1(V) \to \Lambda_1(V)$$

by

$$M^{\tilde{\nabla}}\varphi = \delta^{\tilde{\nabla}} d^{\tilde{\nabla}}\varphi - F\cdot\varphi.$$

has $M^{\tilde{\nabla}} d^{\tilde{\nabla}}$ an exterior algebraic action by the "Yang-Mills current" $\delta^{\tilde{\nabla}} F$, thus

**Theorem 3 (G, Somberg & Souček)** *The sequence of operators,*

$$\Lambda^0(V) \stackrel{d^{\tilde{\nabla}}}{\to} \Lambda^1(V) \stackrel{M^{\tilde{\nabla}}}{\to} \Lambda_1(V) \stackrel{\delta^{\tilde{\nabla}}}{\to} \Lambda_0(V)$$

*is a complex if and only if the curvature $F$ of the connection $\tilde{\nabla}$ satisfies the (pure) Yang-Mills equation*

$$\delta^{\tilde{\nabla}} F = 0.$$

*In addition:*

• *If $\tilde{\nabla}$ is an orthogonal or unitary connection then the sequence is formally self-adjoint.*

• *In Riemannian signature the sequence is elliptic.*

• *In dimension 4 the complex is conformally invariant.*

Lots of examples: E.g. harmonic curvature Riemannian manifolds . .

We may use the Yang-Mills detour complex Theorem to construct more differential complexes. Suppose that there are vector bundles $\mathcal{B}^0$, $\mathcal{B}^1$, $\mathcal{B}_1$ and $\mathcal{B}_0$ and differential operators $L_0$, $L_1$, $L^1$, $L^0$, $\mathcal{D}$ and $\overline{\mathcal{D}}$ which act as indicated in the following diagram:

$$\text{[D]}\qquad \begin{array}{ccccccc} \Lambda^0(V) & \xrightarrow{d^{\tilde{\nabla}}} & \Lambda^1(V) & \xrightarrow{\quad M^{\tilde{\nabla}}\quad} & \Lambda_1(V) & \xrightarrow{\delta^{\tilde{\nabla}}} & \Lambda_0(V) \\ {\scriptstyle L_0}\uparrow & & {\scriptstyle L_1}\uparrow & & {\scriptstyle L^1}\downarrow & & {\scriptstyle L^0}\downarrow \\ \mathcal{B}^0 & \xrightarrow{\mathcal{D}} & \mathcal{B}^1 & \xrightarrow{\quad M^{\mathcal{B}}\quad} & \mathcal{B}_1 & \xrightarrow{\overline{\mathcal{D}}} & \mathcal{B}_0 \end{array}$$

the operator $M^{\mathcal{B}} : \mathcal{B}^1 \to \mathcal{B}_1$ is defined to be the composition $L^1 M^{\tilde{\nabla}} L_1$. Suppose that the squares at each end commute. Then on $\mathcal{B}^0$ we have

$$M^{\mathcal{B}}\mathcal{D} = L^1 M^D L_1 \mathcal{D} = L^1 M^D d^D L_0 = L^1 \epsilon(\delta^D F) L_0,$$

and similarly $\overline{\mathcal{D}} M^{\mathcal{B}} = -L^0 \iota(\delta^D F) L_0$. Thus if $\tilde{\nabla}$ is Yang-Mills then the lower sequence, viz.

$$\mathcal{B}^0 \xrightarrow{\mathcal{D}} \mathcal{B}^1 \xrightarrow{M^{\mathcal{B}}} \mathcal{B}_1 \xrightarrow{\overline{\mathcal{D}}} \mathcal{B}_0\ , \qquad (1)$$

is a complex.

**NB:** if the connection $\tilde{\nabla}$ preserves a Hermitian or metric structure on $V$ then we need only the single commuting square $d^{\tilde{\nabla}} L_0 = L_1 \mathcal{D}$ on $\mathcal{B}^0$; – by taking formal adjoints we obtain a second commuting square.

**Example:** Conformal transformation of the Schouten tensor is controlled by the equation

$$D\sigma := \mathsf{TF}(\nabla_a \nabla_b \sigma + P_{ab}\sigma) = 0.$$

A metric $\sigma^{-2}\boldsymbol{g}$ is Einstein if and only if the scale $\sigma \in \mathcal{E}[1]$ is non-vanishing and satisfies this. We want to see this operator arise in a detour complex.

$D\sigma = 0$ is a **finite type** PDE. We prolong to get a connection "equivalent" to the equation.

It is easy to see at the outset that the prolonged system $\mathcal{T}$ is a bundle extension of $J^1\mathcal{E}[1]$ by densities $\mathcal{E}[-1]$, that is

$$0 \to \mathcal{E}[-1] \to \mathcal{T} \to J^1\mathcal{E}[2] \to 0 \ .$$

We may compute explicitly in terms of weighted tensor bundles by using the connection on $\mathcal{E}[1]$ and the Levi-Civita connection. In an obvious way these give a splitting of $J^2\mathcal{E}[1]$ into $\mathcal{E}[1] \oplus \Lambda^1[1] \oplus S^2[1]$ and $D$ may be written as the first order system

$$\nabla_a \sigma - \mu_a = 0$$

$$\nabla_a \mu_b + \boldsymbol{g}_{ab}\rho + P_{ab}\sigma = 0 \ .$$

Differentiating again and using explicitly this observation that there are no completely symmetric 3-tensors that are pure trace in the last pair of indices we obtain that

$$\nabla_a \rho - P_{ab}\mu^b = 0.$$

So the prolonged system is

$$\mathcal{T} \stackrel{g}{=} \mathcal{E}[1] \oplus \Lambda^1[1] \oplus \mathcal{E}[-1],$$

and we obtain the connection

$$\nabla_a^D \begin{pmatrix} \sigma \\ \mu_b \\ \rho \end{pmatrix} := \begin{pmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + g_{ab}\rho + P_{ab}\sigma \\ \nabla_a \rho - P_{ab}\mu^b \end{pmatrix}$$

So that solutions of $D$ are exactly the parallel sections of $\nabla^D$. This is exactly the construction from Bailey, Eastwood, G. (1994) of the **normal conformal tractor connection**. It is equivalent to the normal Cartan connection.

Also we have

$$0 \to S_0^2\mathcal{E}[1] \to J^2\mathcal{E}[1] \to \mathcal{T} \to 0$$

(and the connection splits this). So we have a canonical operator $\mathbb{D} : \mathcal{E}[1] \to \Lambda^0(\mathcal{T})$ which is the composition of $j^2 : \mathcal{E}[1] \to J^2\mathcal{E}[1]$ followed by the projection $J^2\mathcal{E}[1] \to \mathcal{T}$. In terms of a choice of metric and its Levi-Civita connection, this is given by

$$\sigma \mapsto (\sigma, \nabla_a \sigma, -\frac{1}{n}(\Delta + J)\sigma).$$

This is a called a differential splitting operator, since through the jet projections there is conformally invariant surjection $X : \mathcal{E}(\mathcal{T}) \to \mathcal{E}[1]$ which inverts $\mathbb{D}$.

There is also a conformal splitting operator

$$E : S^2_0[1] \to \mathcal{E}^1(\mathcal{T})$$

by

$$\psi_{ab} \mapsto (0, \psi_{ab}, -(n-1)^{-1}\nabla^b\psi_{ab}).$$

and

**Proposition 4 (G, Somberg, Souček)** *As differential operators on $\mathcal{E}[1]$, we have*

$$\nabla^D \mathbb{D} = ED \ .$$

*For $\sigma \in \mathcal{E}[1]$, $\mathbb{D}\sigma$ is parallel if and only if $P\sigma = 0$.*

We have our commuting square – and the also the far square as the tractor connection preserves a metric.

The curvature $\Omega_{ab}{}^{C}{}_{E}$ of the tractor connection satisfies

$$\nabla_a^D \Omega^a{}_b{}^C{}_E = \begin{pmatrix} 0 & 0 & 0 \\ B^c{}_b & (n-4)A_b{}^c{}_e & 0 \\ 0 & -B_{eb} & 0 \end{pmatrix}$$

where $A$ is the Cotton tensor and $B$ is the Bach tensor. Say that a *pseudo-Riemannian* manifold is *semi-harmonic* if its tractor curvature is Yang-Mills – this matrix vanishes.

**Lemma 5** *In dimension 4 the normal conformal tractor connection is a Yang-Mills connection if and only if the structure is Bach-flat.*

Write $M^{\mathcal{T}}$ for the composition $E^* M^{\nabla} E$, From the diagram [D] we have.

**Theorem 6 (GSS)** *The sequence*

$$\mathcal{E}^0[1] \xrightarrow{P} \mathcal{E}^{1,1}[1] \xrightarrow{M^{\mathcal{T}}} \mathcal{E}_{1,1}[-1] \xrightarrow{P^*} \mathcal{E}_0[-1] \qquad (2)$$

*has the following properties.*
*It is a formally self-adjoint sequence of differential operators and, for* $\sigma \in \mathcal{E}[1]$

$$(M^{\mathcal{T}} D\sigma)_{ab} = -TFS(B_{ab}\sigma - (n-4)A_{abc}\nabla^c\sigma), \qquad (3)$$

*where* $TFS(\cdots)$ *indicates the trace-free symmetric part of the tensor concerned. In particular it is a complex on semi-harmonic manifolds.*
*In the case of Riemannian signature the complex is elliptic.*
*In dimension 4, (2) is sequence of conformally invariant operators and it is a complex if and only if the conformal structure is Bach-flat.*

**Corollary 7** *Einstein 4-manifolds are Bach-flat.*

**Proof:** If a non-vanishing density $\sigma$ is an Einstein scale then, calculating in that scale, we have $M^{\mathcal{T}} D\sigma = -B\sigma$, where $B$ is the Bach tensor. On the other hand if $\sigma$ is an Einstein scale then $D\sigma = 0$.

**Twistor example:** We write Tw for the so-called twistor bundle, that is the subbundle of $\Lambda \otimes \mathbb{S}[1/2]$ consisting of form spinors $u_a$ such that $\gamma^a u_a = 0$, where $\gamma_a$ is the usual Clifford symbol. We use $\mathbb{S}$ and Tw also for the section spaces of these bundles. The *twistor operator* is the conformally invariant Stein-Weiss gradient

$$\mathbf{T} : \mathbb{S}[1/2] \to \mathsf{Tw}$$

given explicitly by

$$\psi \mapsto \nabla_a \psi + \frac{1}{n} \gamma_a \gamma^b \nabla_b \psi \ .$$

This completes to a differential complex as follows.

**Theorem 8 (G ,Somberg, Souček)** *On semi-harmonic pseudo-Riemannian $n$-manifolds $n \geq 4$ we have a differential complex*

$$\mathbb{S}[1/2] \stackrel{\mathbf{T}}{\to} \mathsf{Tw} \stackrel{\mathbf{N}}{\longrightarrow} \overline{\mathsf{Tw}} \stackrel{\mathbf{T}^*}{\to} \overline{\mathbb{S}}[-1/2], \qquad (4)$$

*where $\mathbf{T}$ twistor operator, $\mathbf{T}^*$ its formal adjoint, and $\mathbf{N}$ is third order. The sequence is formally self-adjoint and in the case of Riemannian signature the complex is elliptic.*

*In dimension 4 the sequence (4) is conformally invariant and it is a complex if and only if the conformal structure is Bach-flat.*