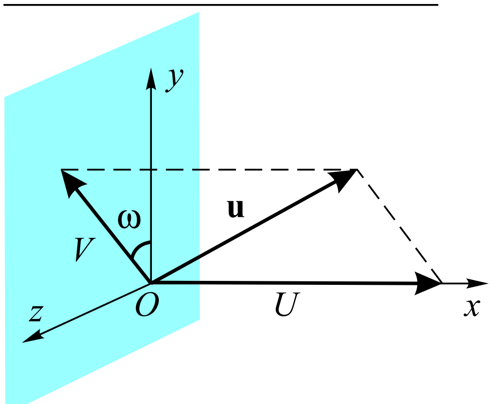
PARTIALLY INVARIANT SOLUTIONS TO IDEAL MAGNETOHYDRODYNAMICS

Sergey Golovin
Queen's University
Kingston, Ontario K7L 4A5, Canada
e-mail: sergey@mast.queensu.ca; sergey@hydro.nsc.ru

The goal of the presentation is to give an analytical description of new solutions [2]–[4] of the ideal magnetohydrodynamics equations generalizing classical one-dimensional continua motions with planar and spherical waves. Investigation of solutions of this type was originated in works [5], [6] for the ideal gas dynamics. The solution are partially invariant with respect to subgroups of admissible Euclidean group.

Flow with planar waves

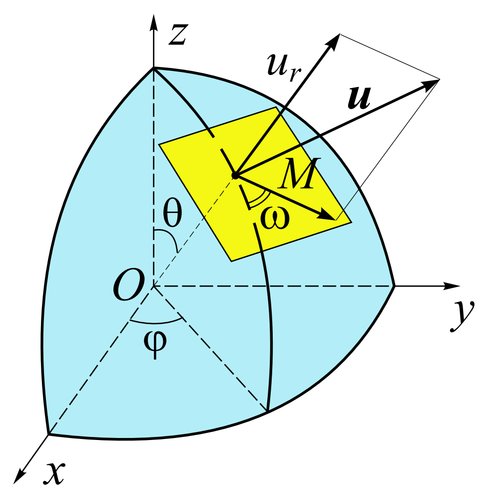


$$U = U(t,x), \quad V = V(t,x),$$
$$\omega = \omega(t,x,y,z).$$

Group of invariance:

$$\{\partial_y, \quad \partial_z,$$
$$z\partial_y - y\partial_z + w\partial_v - v\partial_w\}$$

Flow with spherical waves



$$u_r = U(t,r), \quad M = M(t,r),$$
$$\omega = \omega(t,r,\theta,\varphi).$$

Group of invariance: $O(3)$

We will refer to planes $x = \text{const}$ and spheres $r = \text{const}$ as to the *level surfaces* of the corresponding solutions. The common properties of the solutions are

- Absolute values of normal and tangential projections of vector field $\mathbf{u}$ to the level surface are invariant functions.
- Rotation angle of the vector field $\mathbf{u}$ about the normal to the level surface is non-invariant function.

These solutions generalize the well-known one-dimensional motions with planar and spherical waves, which correspond to $V \equiv 0$ in planar case or $M \equiv 0$ in spherical case. The representation of the solutions above are prescribed by the condition of partially invariance [1] of solution with respect to the admissible group of one rotation and two translations in the planar case, and group $O(3)$ in the spherical case.

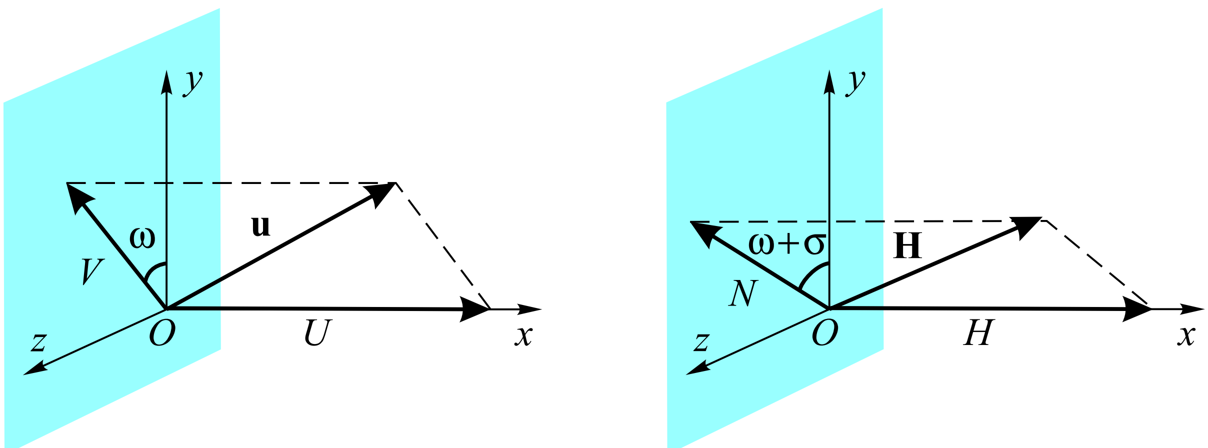
We are going to apply the representations above for construction of exact solutions to ideal magnetohydrodynamics equations

$$\begin{aligned}
&D\,\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0,\\
&D\,\mathbf{u} + \rho^{-1}\nabla p + \rho^{-1}\mathbf{H}\times\mathrm{rot}\,\mathbf{H} = 0,\\
&D\,p + A(p,\rho)\,\mathrm{div}\,\mathbf{u} = 0,\\
&D\,\mathbf{H} + \mathbf{H}\,\mathrm{div}\,\mathbf{u} - (\mathbf{H}\cdot\nabla)\mathbf{u} = 0,\\
&\mathrm{div}\,\mathbf{H} = 0, \quad D = \partial_t + \mathbf{u}\cdot\nabla.
\end{aligned} \tag{1}$$

Here $\mathbf{u}$ is the velocity vector, p is the pressure, ρ is the density, and $\mathbf{H}$ is the magnetic field vector. All functions depend on time t and coordinates $\mathbf{x} = (x, y, z)$. Function $A(p,\rho) = \rho c^2$ ($c = \sqrt{\partial p/\partial\rho}$ is a thermodynamical speed of sound) is determined by the gas state equation $p = f(\rho, S)$, with entropy S.

Solutions with planar waves

In magnetohydrodynamics there are two vector fields at each particle, namely, its velocity $\mathbf{u}$ and its magnetic field vector $\mathbf{H}$. We will assume, that both vector fields satisfy the above dependencies. In the planar case we have the following representation of solution



$$
\begin{aligned}
&u = U(t,x), && H^1 = H(t,x),\\
&v = V(t,x)\cos\omega(t,x,y,z), && H^2 = N(t,x)\cos\big(\omega(t,x,y,z)+\sigma(t,x)\big),\\
&w = V(t,x)\sin\omega(t,x,y,z), && H^3 = N(t,x)\sin\big(\omega(t,x,y,z)+\sigma(t,x)\big),\\
&p = p(t,x), \quad \rho = \rho(t,x), \quad S = S(t,x). &&
\end{aligned}
\tag{2}
$$

Here function ω is non-invariant as long as it depends on all independent variables. All the rest of functions are invariant ones depending only on invariant variables t and x. Substitution of representation (2) into (1) gives the following. From the first (continuity) equation of the system (1) we have

$$
\rho_t + U\rho_x + \rho(U_x - V\sin\omega\,\omega_y + V\cos\omega\,\omega_z) = 0. \tag{3}
$$

This equation implies, that function h

$$h(t,x) = -V^{-1}(\rho_t + U\rho_x + \rho U_x)$$

is invariant one, i.e. it depends only on t and x. Another terms of equation (3) give an equation for ω

$$\sin\omega\,\omega_y - \cos\omega\,\omega_z + h = 0. \tag{4}$$

Besides, we have the following three equations, which relate only invariant functions:

$$\begin{aligned}
&\widetilde{D}\,U + \rho^{-1}p_x + \rho^{-1}NN_x = 0,\\
&\widetilde{D}\,H + hHV = 0,\\
&\widetilde{D}\,p + A(p,\rho)(U_x + hV) = 0.
\end{aligned}$$

Here and further $\widetilde{D} = \partial_t + U\partial_x$. The rest of equations of system (1) produce the following overdetermined system for non-invariant function ω:

$$\begin{aligned}
&\rho V\omega_t + \big(\rho UV - HN\cos\sigma\big)\,\omega_x + \big(\rho V^2\cos\omega - N^2\cos\sigma\cos(\omega+\sigma)\big)\,\omega_y\\
&\quad + \big(\rho V^2\sin\omega - N^2\cos\sigma\sin(\omega+\sigma)\big)\,\omega_z - H(N_x\sin\sigma + N\cos\sigma\sigma_x) = 0.
\end{aligned} \tag{5}$$

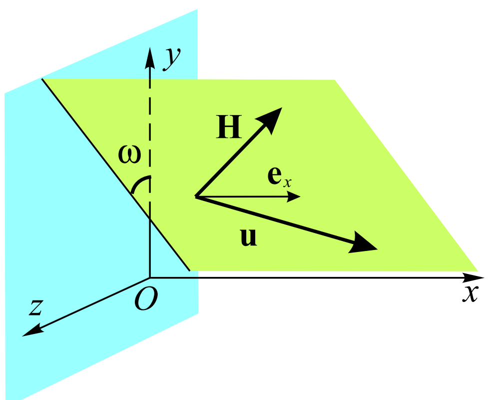
$$\begin{aligned}
&HN\sin\sigma\,\omega_x + \frac{1}{2}N^2\sin(\omega+2\sigma)\,\omega_y - \frac{1}{2}N^2\cos(\omega+2\sigma)\,\omega_z\\
&\qquad + \rho\,\widetilde{D}V - H(N_x\cos\sigma - N\sin\sigma\,\sigma_x) + \frac{1}{2}\,hN^2 = 0.
\end{aligned} \tag{6}$$

$$\begin{aligned}
&N\omega_t + (NU - HV\cos\sigma)\,\omega_x + VN\sin\sigma\sin(\omega+\sigma)\,\omega_y\\
&\qquad - VN\sin\sigma\cos(\omega+\sigma)\,\omega_z + N\widetilde{D}\sigma + HV_x\sin\sigma = 0.
\end{aligned} \tag{7}$$

$$\begin{aligned}
&HV\sin\sigma\,\omega_x + NV\cos\sigma\sin(\omega+\sigma)\,\omega_y\\
&\qquad - NV\cos\sigma\cos(\omega+\sigma)\,\omega_z - \widetilde{D}N + HV_x\cos\sigma - NU_x = 0.
\end{aligned} \tag{8}$$

$$N\big(\sin(\omega+\sigma)\omega_y - \cos(\omega+\sigma)\,\omega_z\big) - H_x = 0. \tag{9}$$

System (4)–(9) for function ω needs to be investigated for compatibility. At that we will look for solutions with functional arbitrariness in determination of function ω. This condition is referred to as "irreducibility condition", which means that if it is satisfied then the partially invariant solution does not coincide with any invariant one.



Theorem 1. *Partially invariant solution (2) is irreducible if and only if $\sigma \equiv 0$ or $\sigma \equiv \pi/2$. This means, that in irreducible solution velocity vector $\mathbf{u}$ at any particle coplanar with its magnetic field vector $\mathbf{H}$ and directional vector of Ox axis.*

At that, equations for ω take the following form

$$\begin{aligned}
&\sin\omega\,\omega_y - \cos\omega\,\omega_z + h = 0,\\
&\omega_t + U\omega_x + V\cos\omega\,\omega_y + V\sin\omega\,\omega_z = 0,\\
&H\omega_x + N\cos\omega\,\omega_y + N\sin\omega\,\omega_z = 0
\end{aligned} \tag{10}$$

Theorem 2. *In the main case $h \neq 0$ the invariant functions are determined from the following system of equations*

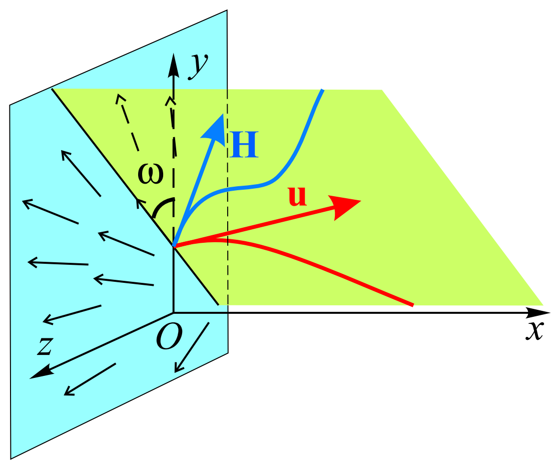
$$\begin{aligned}
&\widetilde{D}\,\rho + \rho(U_x + hV) = 0, && \widetilde{D}\,U + \rho^{-1}p_x + \rho^{-1}NN_x = 0,\\
&\widetilde{D}\,V - \rho^{-1}H_0hN_x = 0, && \widetilde{D}\,p + A(p,\rho)(U_x + hV) = 0,\\
&\widetilde{D}\,N + NU_x - H_0hV_x + hNV = 0, && H = H_0h,\\
&\widetilde{D}\,h + Vh^2 = 0, && H_0h_x + hN = 0.
\end{aligned} \tag{11}$$

Function ω satisfies the implicit equation

$$F(y - h^{-1}\cos\omega,\ z - h^{-1}\sin\omega) = 0 \tag{12}$$

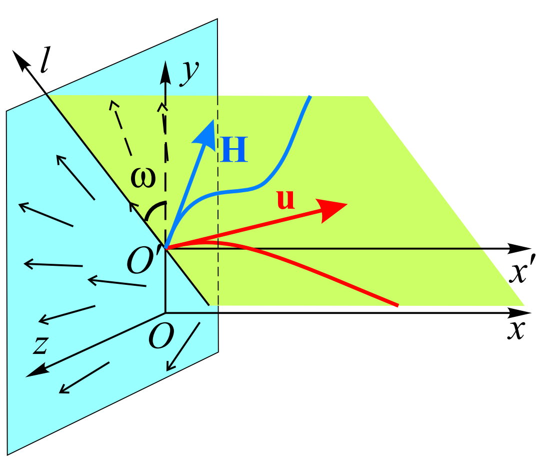
with arbitrary smooth function F.

From equations (10) it follows that ω conserves along particle trajectories and along magnetic field lines. This implies that these curves are flat in 3D-space.



- Particle trajectories and magnetic field lines are located in planes, which are orthogonal to Oyz-plane and turned on angle ω about Ox-axis.
- All particles originating from the same plane x = const moves along the similar curves in 3D-space.
- Position of each trajectory in 3D-space is defined by vector field in plane x = const determined in accordance with equation (12).

Let us introduce Cartesian frame $O'x'l$ in the plane of particle's motion.



It is required to solve Cauchy problem

$$\frac{dx}{dt} = U(t, x), \quad x(t_0) = x_0. \quad (13)$$

The dependence $x = x(t, x_0)$ allows one to determine $l = l(t)$ as

$$l(t) = \int_{t_0}^{t} V(t, x(t, x_0))dt. \quad (14)$$

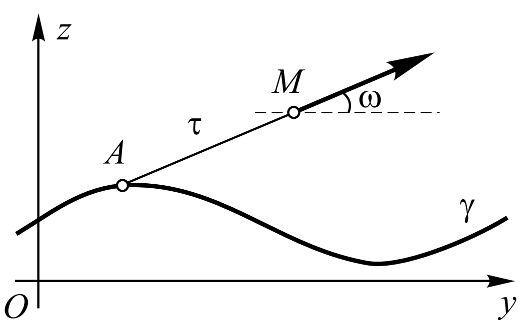
In the frame $Oxyz$ particle's trajectory has the form:

$$x = x(t, x_0), \quad y = y_0 + l(t)\cos\omega_0, \quad z = z_0 + l(t)\sin\omega_0. \quad (15)$$

Magnetic field line, which passes at $t = t_0$ through (x_0, y_0, z_0) is

$$y = y_0 + \cos\omega_0 \int\limits_{x_0}^{x} \frac{N(t_0, s)}{H(t_0, s)} ds, \ z = z_0 + \sin\omega_0 \int\limits_{x_0}^{x} \frac{N(t_0, s)}{H(t_0, s)} ds. \quad (16)$$

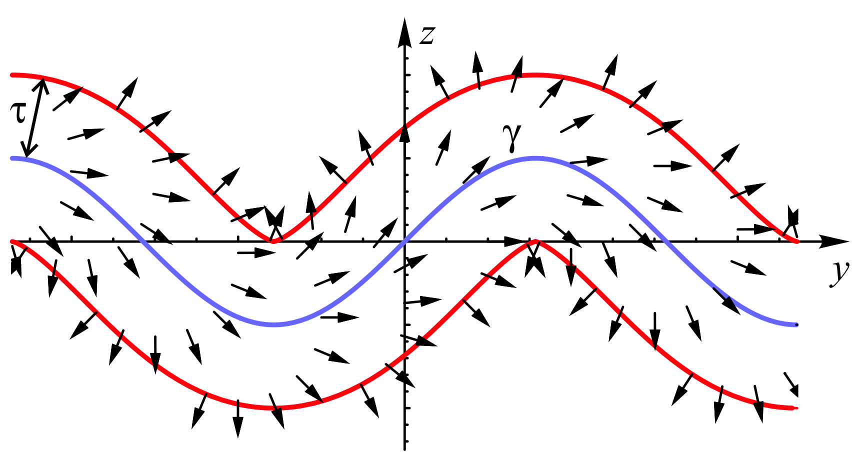
Let us now turn to interpretation of implicit equation (12) for function ω. Let the function F has been fixed. This define a curve $\gamma : F(y, z) = 0$ on plane Oyz.



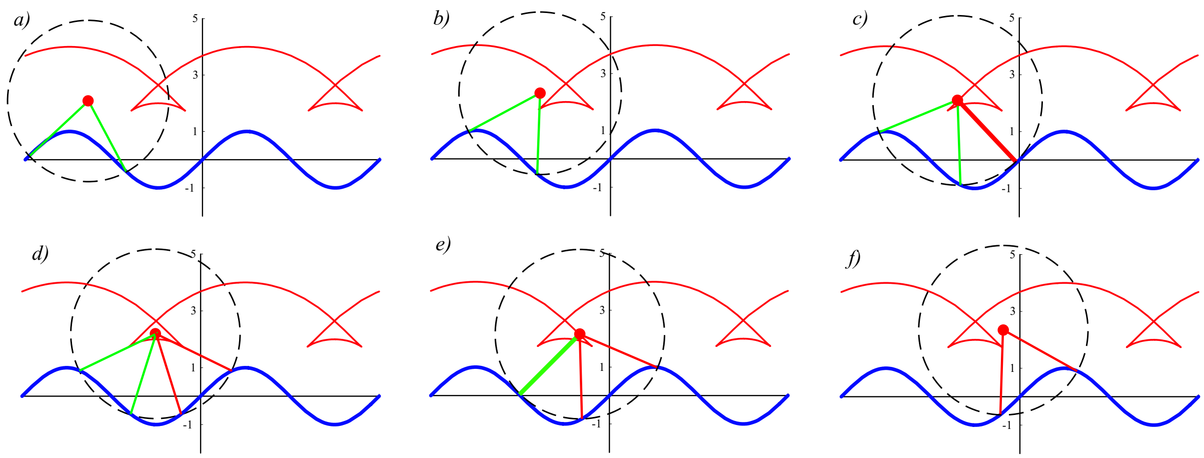
In order to find angle ω at arbitrary point $M = (y, z)$ it is required to draw a line segment AM of length $\tau = 1/h$ such that $A \in \gamma$. The direction of AM defines angle ω as it is shown in the figure.

The main properties of the obtained vector field are the following

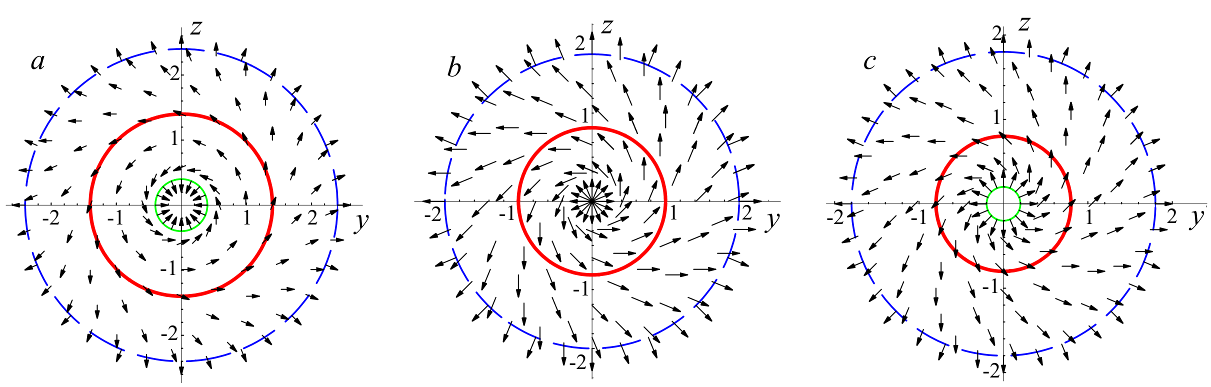
- The solution is determined only inside the stripe of width 2τ with curve $\gamma : \ F(y, z) = 0$ as a centerline.



- The vector field is directed orthogonally to the boundaries of the stripe over the boundaries.
- If $\tau > \min_{\mathbf{x}\in\gamma} R(\mathbf{x})$, where $R(\mathbf{x})$ is a curvature radius of γ at point $\mathbf{x}$, then the boundary of the stripe has singularities of a "dovetail" type. In this case the vector field can not be defined continuously over all borders of the dovetail.



One can show a vector field without singularities. Let us choose, for example, $\gamma : y^2 + z^2 = R^2$. The figure shows vector fields obtained for different relations between τ and R. In order to show



$R > \tau$ $R = \tau$ $R < \tau$

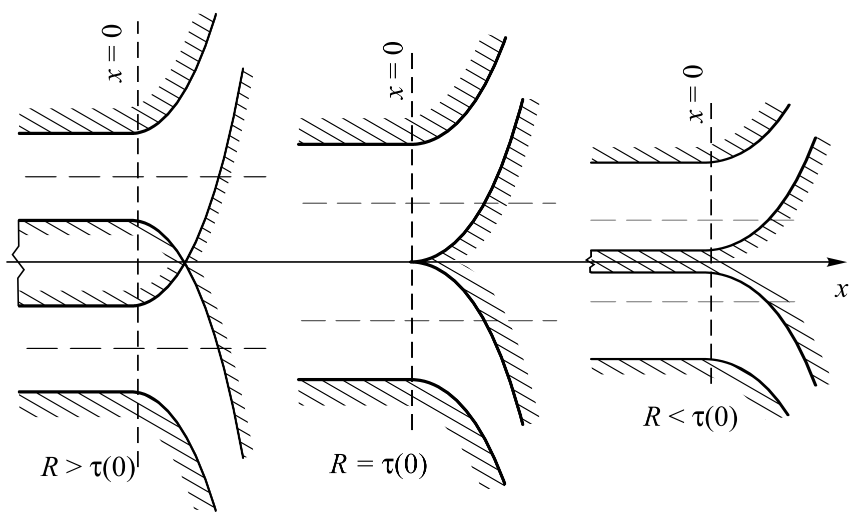
the corresponding 3D-motion of plasma let us choose a particular stationary solution of invariant subsystem (11):

$$
\begin{aligned}
U &= \frac{H_0^2}{\cosh x}, \quad V = H_0^2 \tanh x, \quad \tau = \cosh x, \\
H &= \frac{H_0}{\cosh x}, \quad N = H_0 \tanh x, \quad \rho = H_0^{-2}, \quad S = S_0.
\end{aligned} \tag{17}
$$

Streamlines and magnetic field lines here coincide and are given by formula

$$
l(x) = \cosh x - 1. \tag{18}
$$

With the vector field in Oyz plane from previous example one can build solution, which describe a flow of ideal plasma in axisymmetric canal. The axial section of the canal is given in figures below. We assume that uniform flow for $x < 0$ changes at $x = 0$ to the

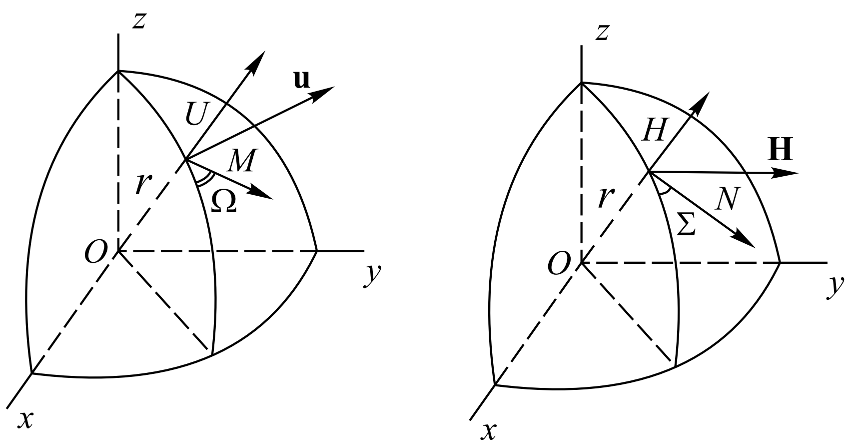


flow, described by the solution (17). Depending on the relation between $\tau(0)$ and R different pictures of motion are possible.

Solution with spherical waves

We observe the following representation of solution in spherical coordinate system:

$$\begin{aligned}
&U = U(t,r), \quad M = M(t,r), \quad H = H(t,r), \quad N = N(t,r),\\
&\Omega = \omega(t,r,\theta,\varphi), \quad \Sigma = \sigma(t,r) + \omega(t,r,\theta,\varphi),\\
&p = p(t,r), \quad \rho = \rho(t,r).
\end{aligned} \tag{19}$$



Theorem 3. *Irreducible solution to the ideal magnetohydrodynamics equations (1) of the form (19) exists only when* $\sigma \equiv 0$. *At that the velocity and the magnetic field vectors at any particle are coplanar to its radius-vector. The invariant functions are determined from the invariant system of equation*

$$\begin{aligned}
&\widetilde{D}\,M_1 + \frac{2}{r}UM_1 - \frac{H_0}{r^4\rho\cos\tau}N_{1r} = 0, && \widetilde{D} = \partial_t + U\partial_r,\\
&\widetilde{D}\,N_1 + N_1U_r - \frac{H_0}{\cos\tau}M_{1r} - M_1N_1\tan\tau = 0, &&\\
&\widetilde{D}\,p + A(p,\rho)\left(U_r + \frac{2}{r}U - M_1\tan\tau\right) = 0, && H = \frac{H_0}{r^2\cos\tau}\\
&\widetilde{D}\,U + \frac{1}{\rho}p_r + \frac{N_1N_{1r}}{r^2\rho} - rM_1^2 = 0, && H_0\tau_r = N_1\cos\tau,
\end{aligned} \tag{20}$$

$$\widetilde{D}\,\rho + \rho\left(U_r + \frac{2}{r}U - M_1\tan\tau\right) = 0\widetilde{D}\,\tau = M_1,$$

$$M_1 = \frac{M}{r},\quad H = \frac{H_0}{r^2\cos\tau},\quad N_1 = r\,N.$$

Here H_0 is an arbitrary constant, and $\tau \in (\pi/2, \pi/2)$ is some function of t and r. The non-invariant function ω is defined by the following implicit equation

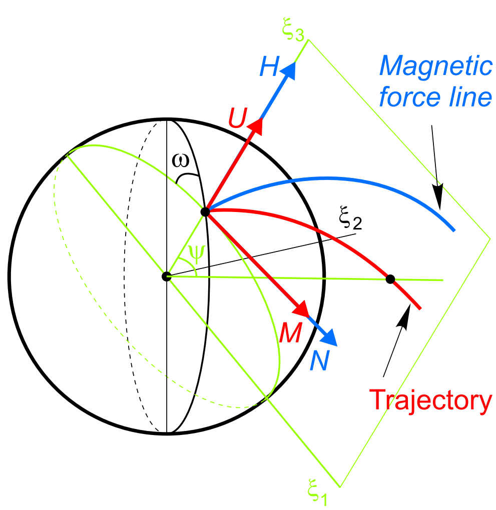
$$\eta = f(\zeta). \tag{21}$$

with arbitrary smooth function f;

$$\begin{aligned}\eta &= \sin\theta\cos\omega\cos\tau - \cos\theta\sin\tau,\\ \zeta &= \varphi + \arctan\frac{\sin\omega\cos\tau}{\cos\theta\cos\omega\cos\tau + \sin\theta\sin\tau}.\end{aligned} \tag{22}$$

One can show that particles' trajectories and magnetic force lines are flat curves. All particles, which start from the same sphere $r = \text{const}$ move along similar curves. The location and orientation of trajectories in the 3D space depend on the particle's initial position and its initial velocity vector.

Trajectories and magnetic field lines are given by solutions of the following Cauchy problem.



$$\frac{dr}{dt} = U(t,r),\ r(t_0) = r_*.$$

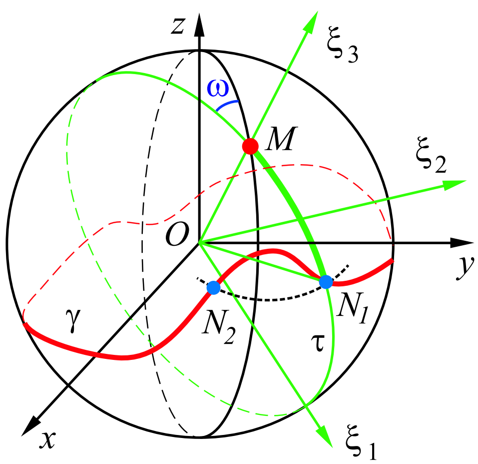
Particle's trajectory in the polar coordinate system is determined by

$$r = r(t),\ \psi = \tau(t, r(t)) - \tau(t_0, r_*).$$

The magnetic force line at the time $t = t_0$ is

$$\psi = \tau(t_0, r) - \tau(t_0, r_*).$$

Description of the plasma motion as a whole depends on the value of function $\omega(\tau, \theta, \varphi)$ on some initial sphere. For given point M in the unit sphere, defined by its spherical coordinates (θ, φ) we introduce auxiliary Cartesian frame of reference $O\xi_1\xi_2\xi_3$ according to the figure. Here we pretend to know angle ω at M.

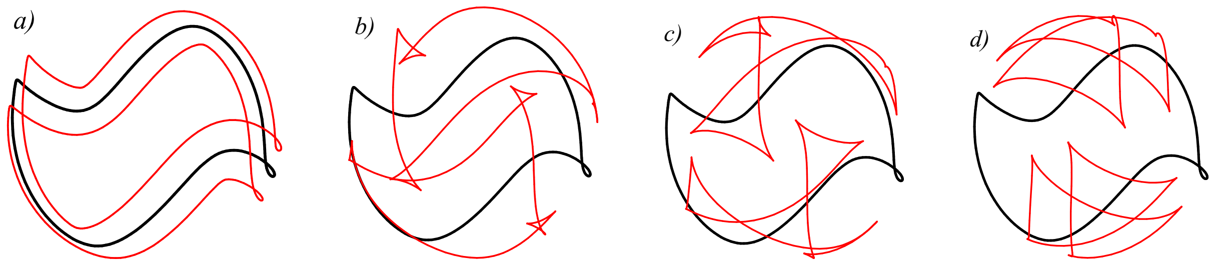


Let $\angle MON_1 = \pi/2 - \tau$. One can show that coordinates of point N_1 in Oxy frame of reference are given by $(x, y, z) = (\cos\zeta, \sin\zeta, \eta)$ with η and ζ from (22). This observation allows us giving the following algorithm of finding ω at arbitrary point on a sphere for given dependence (21).

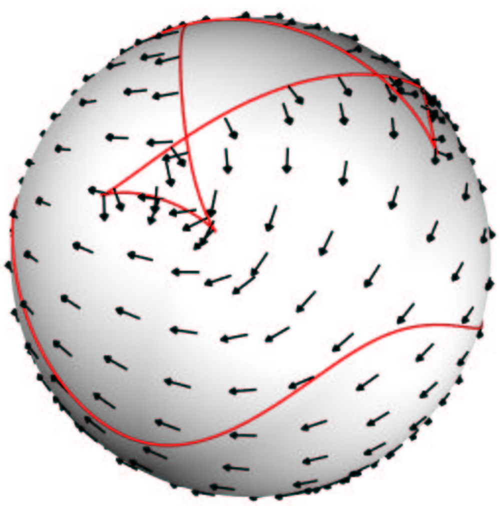
Let a curve γ on the sphere $|\mathbf{x}| = R$ be determined by the relation $z = f(\varphi)$, where (r, φ, z) are cylindrical coordinates, and f is the same as in (21). For given point M let us draw a circle S^1 on the sphere S^2 with the center M and the geodesic radius $\pi/2 - \tau$. Note, that the angle τ is the same for each points of the sphere.

Theorem 4. *We denote by N_i, $i = 1, \ldots, k$ points of intersection of the curve γ with the circle S^1. The angles between the meridian, which pass through M, and the geodesic curves, which follow from M to each N_i define all possible values of ω, which satisfy the relation* (21).

Like in planar case, tangential vector field on a sphere is defined only inside of a stripe of width $\pi - 2\tau$ with the midline γ. For small or negative τ there could be singularities of a dovetail type on the boundaries of the stripe.



In the next figure we give an example of a vector field, generated in accordance with the presented algorithm. Curve $\gamma : \theta = \pi/2 + 1/4 \sin 3\varphi$ and its northern equidistant $\gamma^+(\delta)$ are shown. One can see the multiple-valuedness of the vector field inside the dovetail. The dovetail appears only when value of $\pi/2 - \tau$ is big enough. The following theorem shows how big τ can be.



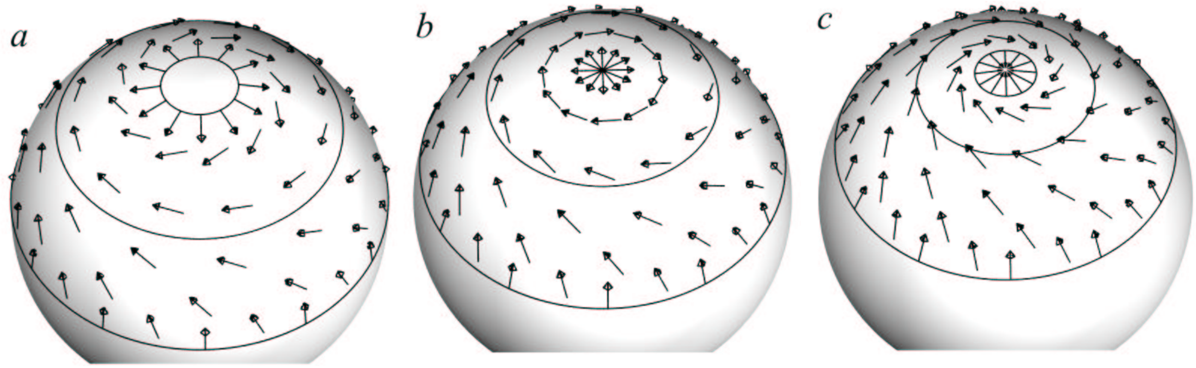
Theorem 5. *Let γ be a smooth curve on the sphere $|\mathbf{x}| = R$. The equidistants $\gamma^\pm(\delta)$ are smooth curves if and only if the following inequality holds*

$$\tan \delta < \min_{\mathbf{x}\in\gamma} R/k_g(\mathbf{x}). \tag{23}$$

Here k_g is a geodesic curvature of the curve γ.

Next figure gives an example of vector field without singularities.

The middle curve γ is the parallel $\theta = \theta_* < \pi/2$. The vector field is determined in the stripe between the equidistants $\gamma^\pm : \theta = \theta_* \mp \delta$. Three possibilities are shown: (a) $\theta_* > \delta$; (b) $\theta_* = \delta$; and (c) $\theta_* < \delta$.



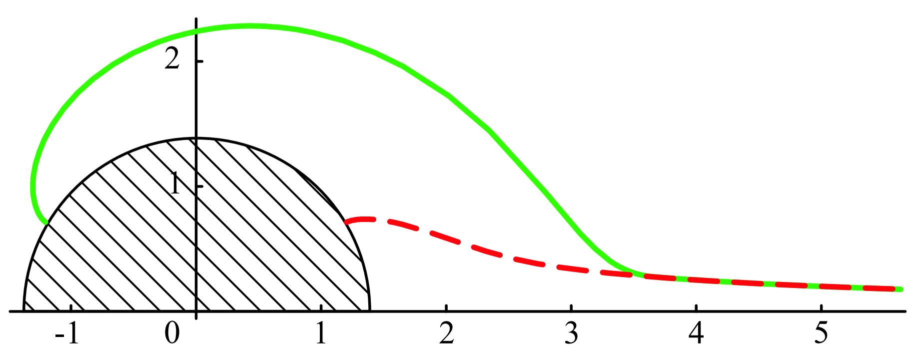
Let us choose a particular stationary solution of invariant subsystem (20) in the form

$$M_1 = \frac{H_0^2 \tau'}{r^2 \cos\tau}, \quad N_1 = \frac{H_0 \tau'}{\cos\tau}, \quad U = \frac{H_0^2}{r^2 \cos\tau}, \quad \rho = \frac{1}{H_0^2}. \quad (24)$$

Here function $\tau(r)$ satisfies the following equation

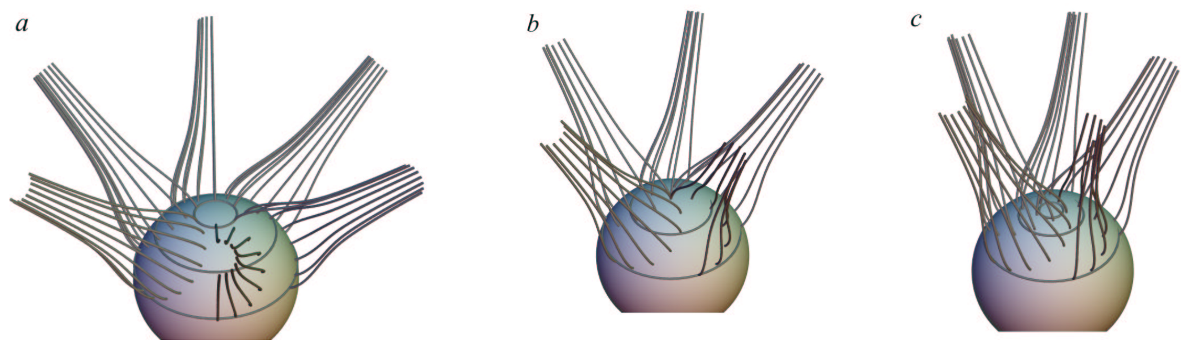
$$\tau'^2 = r^2 \cos^2\tau - r^{-2}. \quad (25)$$

For this solution streamlines and magnetic field lines coincide and are shown in the figure below. Two different curves correspond to different initial data of equation (25). Shaded region is a spherical source of plasma.

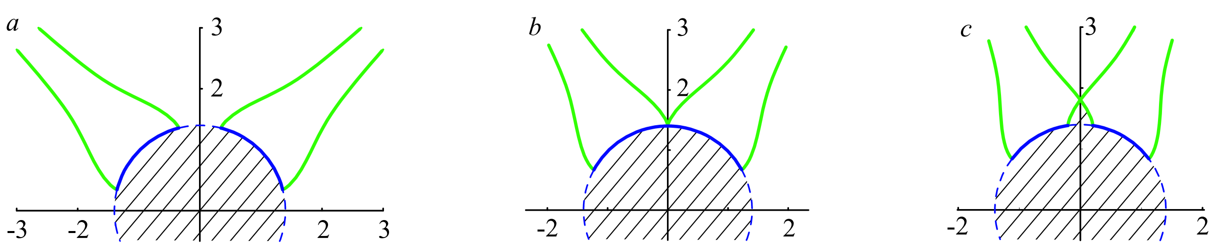


Now one can construct 3D-picture of the flow. The following figure depicts typical magnetic field lines of the stationary flow.

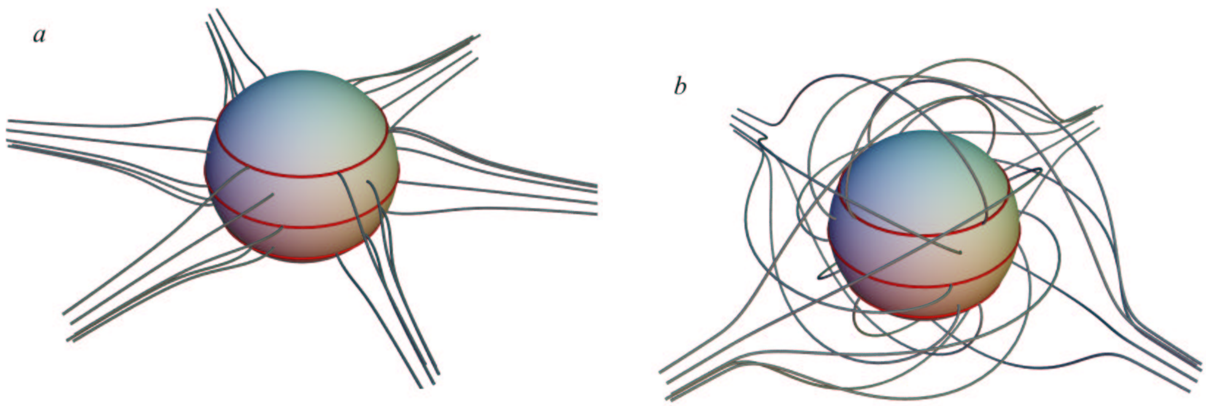
Here the tangential vector field on a sphere and magnetic field lines are taken from the previous figures. Domains of the latter

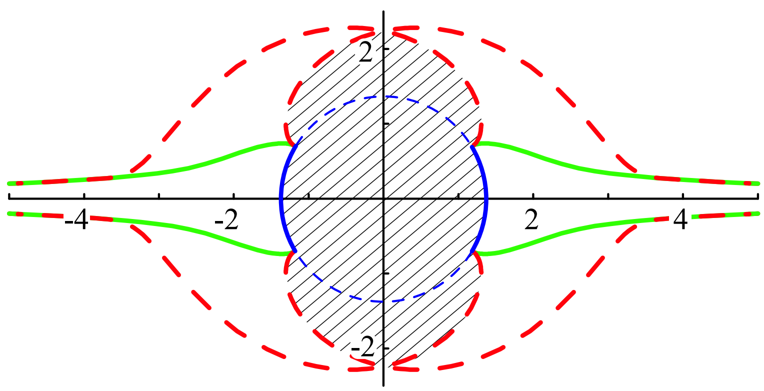


flows are axially symmetric. Below we provide figures of axial sections of the domains of the flows. The shaded region is the spherical source of plasma. The flow is determined only inside the area bounded by the limiting magnetic field lines (solid lines in the diagrams).

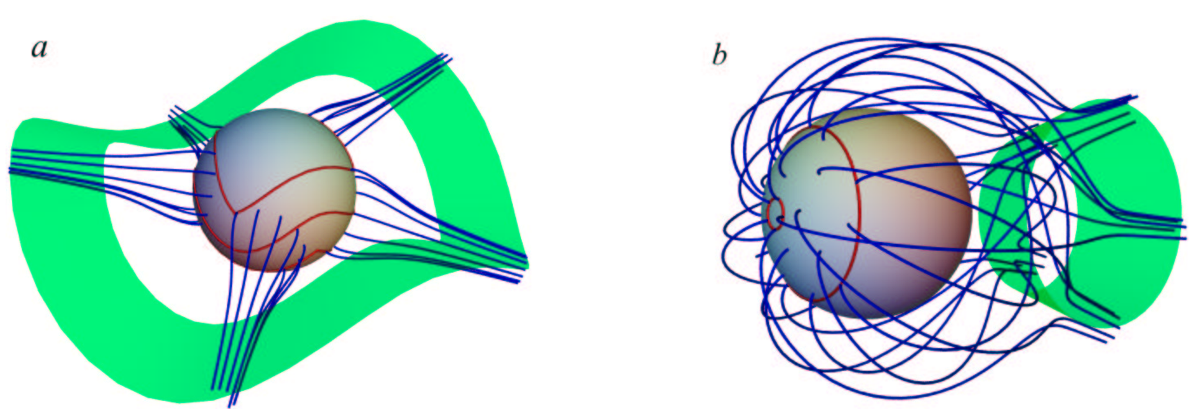


Another examples of magnetic field lines of the flow from the spherical source is shown below. The curve γ as the equator $\theta = \pi/2$. The magnetic field line is (a) short curve; (b) long curve depicted in upper figure. The sphere is a source of plasma. Axial section of domains of the flows in previous figure. The solutions are determined outside the shaded region in the area between the limiting magnetic curves. Solid curves are boundaries of the flow depicted in figure (a), the dashed ones bound the flow in figure (b). The solid part of the boundary of the shaded region is a source of plasma.





In the next pictures the curve γ is (a) $\theta = \pi/2 + 1/4 \sin 3\varphi$; (b) $\theta = 7/8\pi$. At the infinite distance from the spherical source all particles approach to the surface, which is partially shown in the diagrams.



References

[1] Ovsyannikov L.V. *Group Analysis of Differential Equations* New York: Academic Press, 1982

[2] Golovin S.V. Singular vortex in magnetohydrodynamics, *J. Phys. A: Math. Gen.* **38** (2005), 4501-4516

[3] Golovin S.V. Invariant solutions of the singular vortex in magnetohydrodynamics, *J. Phys. A: Math. Gen.* **38** (2005) 8169-8184

[4] Golovin S.V. Generalization of the one-dimensional ideal plasma flow with spherical waves. *J. Phys. A: Math. Gen.* **39** 7579-7595 (2006)

[5] Ovsyannikov L.V. Singular vortex, *J. Appl. Mech. Tech. Phys.* **36** (1995), 360-366

[6] Chupakhin A.P. Invariant submodels of the singular vortex, *J. Appl. Math. Mech.* **67** (2003), 351-364