Complex variables for separation of Hamilton-Jacobi equation on real pseudo-Riemannian manifolds

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Symmetries and Overdetermined Systems
of Partial Differential Equations

Classical SoV: Definitions

Definition

A solution $W(q^\mu, c_\nu)$ of the Hamilton-Jacobi equation

$$H\left(p_\mu = \frac{\partial W}{\partial q^\mu}, q^\mu\right) = E$$

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- is additively separated in the coordinates q^μ if

$$W = W_1(q^1, c_\nu) + \cdots + W_n(q^n, c_\nu)$$

Classical SoV: Levi-Civita criterion

Theorem (Levi-Civita 1904)

Given a symplectic manifold M and an Hamiltonian $H : M \to \mathbb{R}$, the Hamilton-Jacobi equation for H admits a complete integral, additively separated in the Darboux coordinates (q^μ, p_μ) if and only if

$$\partial^\mu H \partial^\nu H \partial_{\mu\nu} H + \partial_\mu H \partial_\nu H \partial^{\mu\nu} H - \partial^\mu H \partial_\nu H \partial^\nu_\mu H - \partial_\mu H \partial^\nu H \partial^\mu_\nu H = 0$$

where $\partial_\mu = \dfrac{\partial}{\partial q^\mu}$, $\partial^\mu = \dfrac{\partial}{\partial p_\mu}$ *and* $\mu \neq \nu$ *are not summed.*

Classical SoV: Stäckel-Eisenhart theory

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$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu + V(q^\mu) = G + V.$$

As a consequence H is separable only if its geodesic part G is separable in the same coordinates.

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- The separable coordinates are orthogonal:

$$G = \frac{1}{2}\left(g^1 p_1^2 + \cdots + g^n p_n^2 \right).$$

Classical SoV: Stäckel-Eisenhart theory

Theorem (Stäckel 1891)

A geodesic Hamiltonian, written in orthogonal separable coordinates, assume a Stäckel form: the g^{μ} form a row of the inverse of a Stäckel matrix

$$S = \begin{pmatrix} \Phi_{11}(q^1) & \cdots & \Phi_{1n}(q^1) \\ \vdots & & \vdots \\ \Phi_{n1}(q^n) & \cdots & \Phi_{nn}(q^n) \end{pmatrix}.$$

Classical SoV: Stäckel-Eisenhart theory

Theorem (Eisenhart 1934)

A natural Hamiltonian is orthogonally separable if and only if it admits n independent first integrals in involution of the form

$$\begin{aligned} H = H_{(1)} &= \frac{1}{2}\left(g^1 p_1^2 + \cdots + g^n p_n^2\right) + V; \\ H_{(2)} &= \frac{1}{2}\left(\lambda^1_{(2)} g^1 p_1^2 + \cdots + \lambda^n_{(2)} g^n p_n^2\right) + V_{(2)}; \\ &\vdots \\ H_{(n)} &= \frac{1}{2}\left(\lambda^1_{(n)} g^1 p_1^2 + \cdots + \lambda^n_{(n)} g^n p_n^2\right) + V_{(n)}. \end{aligned}$$

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Eisenhart equations

$$g^\nu \frac{\partial \lambda^\nu_{(\sigma)}}{\partial q^\mu} = (\lambda^\mu_{(\sigma)} - \lambda^\nu_{(\sigma)}) \frac{\partial g^\nu}{\partial q^\mu}$$

Classical SoV: Killing-Stäckel algebras and separable webs

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A 2-tensor **K** is a Killing 2-tensor if

$$\{K^{\mu\nu}p_\mu p_\nu, G\} = 0 \qquad \Longleftrightarrow \qquad \nabla_{(\sigma}K_{\mu\nu)} = 0.$$

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Killing tensors allow to formulate the Eisenhart theorem in an intrinsic way.

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Theorem (Eisenhart, Kalnins & Miller, Benenti...)

A natural Hamiltonian $H = G + V$ is orthogonally separable if and only if a characteristic Killing tensor $\mathbf{K}$ *exist (or equivalently a Killing-Stäckel algebra $\mathcal{K}$) such that*

$$\mathrm{d}(\mathbf{K}\mathrm{d}V) = 0.$$

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Separable webs and coordinates

The integration of the n distributions orthogonal to each eigenvector of a characteristic Killing tensor gives the associated separable web. A set of adapted coordinates can be found integrating the corresponding closed eigenforms.

$$\mathbf{K}\mathrm{d}z^\mu = \lambda^\mu \mathbf{g}\mathrm{d}z^\mu$$

Example: parabolic coordinates in $\mathbb{M}_2$

Metric tensor

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Killing tensor

$$K^{\mu\nu} = \begin{pmatrix} 1+2t & x+t \\ x+t & 1+2x \end{pmatrix}$$

Pseudo-Cartesian coordinates:

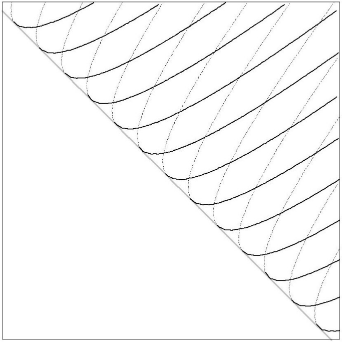
$$q^1 = x, \quad q^2 = t.$$

Looking for eigenforms:

$$\mathbf{K}\mathrm{d}z^\mu = \lambda^\mu \mathbf{g}\mathrm{d}z^\mu$$

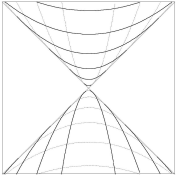
Separation coordinates:

$$\begin{aligned} z^1 &= x - t + \sqrt{1+2(x+t)} \\ z^2 &= x - t - \sqrt{1+2(x+t)} \end{aligned}$$

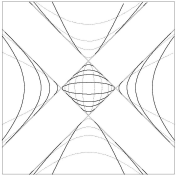


Example: other separable coordinates in $\mathbb{M}_2$

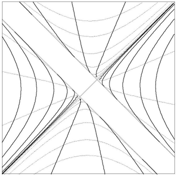
$$\begin{pmatrix} 2x & t \\ t & 0 \end{pmatrix}$$



$$\begin{pmatrix} x^2 & xt \\ xt & t^2-1 \end{pmatrix}$$



$$\begin{pmatrix} x^2-1 & xt-1 \\ xt-1 & t^2-1 \end{pmatrix}$$



Characteristic Killing tensor, standard definition

Notations

$(Q, \mathbf{g})$ is a pseudo-Riemannian n-dimensional manifold with coordinates q^μ.
Its cotangent bundle T^*Q has canonical coordinates (q^μ, p_μ).
The Hamiltonian is geodesic: $H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu$.

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The Killing tensor $\mathbf{K}$ is characteristic if:

- has real pointwise distinct eigenvalues;
- has eigenvectors all independent and orthogonally integrable (and hence n functionally independent functions z^μ exist).

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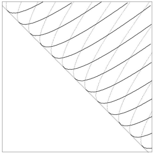

Characteristic Killing tensor, extended definition

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Complex variables, conjugate momenta and associated vector fields

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$$\frac{\partial}{\partial z^\mu} = J^\mu_\nu \frac{\partial}{\partial q^\nu}, \qquad \frac{\partial}{\partial P_\mu} = \frac{\partial z^\mu}{\partial q^\nu}\frac{\partial}{\partial p_\nu}.$$

Complex variables and associated vector fields

Lemma

- $\left[\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}\right] = \left[\frac{\partial}{\partial P_\mu}, \frac{\partial}{\partial P_\nu}\right] = \left[\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial P_\nu}\right] = 0;$
- $\frac{\partial}{\partial \bar{z}^\mu} = \overline{\frac{\partial}{\partial z^\mu}}$ *and* $\frac{\partial}{\partial \overline{P}_\mu} = \overline{\frac{\partial}{\partial P_\mu}};$
- $\overline{\frac{\partial}{\partial z^\mu}(W)} = \frac{\partial}{\partial \bar{z}^\mu}(\overline{W})$ *and* $\overline{\frac{\partial}{\partial P_\mu}(W)} = \frac{\partial}{\partial \overline{P}_\mu}(\overline{W}).$

Lemma

The "change of variables" $(q^\mu, p_\mu) \mapsto (z^\mu, P_\mu)$ *is canonical:*

$$\{z^\mu, P_\nu\} = \frac{\partial z^\mu}{\partial q^\sigma}\frac{\partial P_\nu}{\partial p_\sigma} - \frac{\partial P_\nu}{\partial q^\sigma}\frac{\partial z^\mu}{\partial p_\sigma} = \delta^\mu_\nu$$
$$\{z^\mu, z^\nu\} = \{P_\mu, P_\nu\} = 0$$

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- The n functions z^μ, eventually all complex, are too many to be considered as independent coordinates on the n-dimensional real manifold.
- Nevertheless, the variables z^μ always define only n real coordinates on Q through their real and imaginary parts.

Real and imaginary parts

Locally, the variables z^μ can be divided in two sets: z^a are the real ones, while z^α and $z^{\bar\alpha}$ are the complex conjugate pairs.

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Wirtinger formal derivatives

If $z^\alpha = x^\alpha + iy^\alpha$ then:

$$\begin{aligned}\frac{\partial}{\partial z^\alpha} &= \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} - i\frac{\partial}{\partial y^\alpha}\right);\\ \frac{\partial}{\partial z^{\bar\alpha}} &= \frac{1}{2}\left(\frac{\partial}{\partial x^\alpha} + i\frac{\partial}{\partial y^\alpha}\right).\end{aligned}$$

Poisson bracket and Hamilton equations

Definition

The Poisson bracket in (z^μ, P_μ) variables of two functions $F_1,\ F_2 : T^*Q \longrightarrow \mathbb{C}$ can be defined as:

$$\{F_1, F_2\} = \frac{\partial F_1}{\partial z^\mu}\frac{\partial F_2}{\partial P_\mu} - \frac{\partial F_2}{\partial z^\mu}\frac{\partial F_1}{\partial P_\mu}.$$

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Definition

The Hamilton equations for an Hamiltonian $H : T^*Q \longrightarrow \mathbb{C}$ are:

$$\begin{cases} \dot{z}^\mu & = & \dfrac{\partial H}{\partial P_\mu} \\ \dot{P}_\mu & = & -\dfrac{\partial H}{\partial z^\mu} \end{cases}$$

Extended SoV: Hamilton-Jacobi equation

Problem

When the Hamilton-Jacobi equation

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Observation

$$\left|\frac{\partial^2 W}{\partial z^\mu \partial c_\nu}\right| = \left|J^\sigma_\mu\right| \left|\frac{\partial^2 W}{\partial q^\sigma \partial c_\nu}\right| \qquad (c_\nu \in \mathbb{R}).$$

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Theorem

The Hamilton-Jacobi equation admits a separated complete integral if and only if the Levi-Civita criterion is satisfied:

$$\partial^\mu H \partial^\nu H \partial_{\mu\nu} H + \partial_\mu H \partial_\nu H \partial^{\mu\nu} H - \partial^\mu H \partial_\nu H \partial^\nu_\mu H - \partial_\mu H \partial^\nu H \partial^\mu_\nu H = 0$$

where $\partial_\mu = \dfrac{\partial}{\partial z^\mu}$ *and* $\partial^\mu = \dfrac{\partial}{\partial P_\mu}$.

Extended SoV: Levi-Civita theorem (proof)

Set $w_\mu = \frac{\partial W}{\partial z^\mu}$, separated solutions of Hamilton-Jacobi equation coincide with the solutions of the n systems of PDE:

$$\frac{\partial w_\mu}{\partial z^\nu} = 0; \quad \frac{\partial w_\mu}{\partial z^\mu} = R_\mu(w_\nu, z^\nu) \equiv -\frac{\frac{\partial H}{\partial z^\mu}}{\frac{\partial H}{\partial P_\mu}}.$$

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These systems can be rewritten as

$$\left\{\begin{aligned} \frac{\partial w_a}{\partial z^a} &= R_a \\ \frac{\partial w_a}{\partial z^b} &= 0 \\ \frac{\partial w_a}{\partial x^\alpha} &= 0 \\ \frac{\partial w_a}{\partial y^\alpha} &= 0 \end{aligned}\right. \qquad \left\{\begin{aligned} \frac{\partial w_\alpha}{\partial x^\alpha} &= R_\alpha \\ \frac{\partial w_\alpha}{\partial y^\alpha} &= iR_\alpha \\ \frac{\partial w_\alpha}{\partial x^\beta} &= 0 \\ \frac{\partial w_\alpha}{\partial y^\beta} &= 0 \\ \frac{\partial w_\alpha}{\partial z^a} &= 0 \end{aligned}\right. \qquad \left\{\begin{aligned} \frac{\partial w_{\bar\alpha}}{\partial x^\alpha} &= R_{\bar\alpha} \\ \frac{\partial w_{\bar\alpha}}{\partial y^\alpha} &= -iR_{\bar\alpha} \\ \frac{\partial w_{\bar\alpha}}{\partial x^\beta} &= 0 \\ \frac{\partial w_{\bar\alpha}}{\partial y^\beta} &= 0 \\ \frac{\partial w_{\bar\alpha}}{\partial z^a} &= 0 \end{aligned}\right.$$

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The integrability conditions give the Levi-Civita criterion.

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Proposition

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If H is real then $R_{\bar\alpha} = \overline{R}_\alpha$. Therefore

$$\begin{aligned}
\frac{\partial}{\partial x^\alpha}(w_\alpha - \overline{w}_{\bar\alpha}) &= R_\alpha - \overline{R}_{\bar\alpha} = 0; \\
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But $w_\alpha - \overline{w}_{\bar\alpha}$ has the same imaginary part of $w_\alpha + w_{\bar\alpha}$, it is constant and can be chosen to be zero.
Finally, W_α is always added to $W_{\bar\alpha}$ in W, and $\dfrac{\mathrm{d}W_\mu}{\mathrm{d}z^\mu} = w_\mu$.

Extended SoV: Eisenhart theorem

Proposition

All the tensors of the Killing-Stäckel algebra associated to the variables (z^μ, P_μ) are simultaneously diagonalized:

$$\begin{aligned} G = G_{(1)} &= \frac{1}{2}\Big(g^1 P_1^2 + \cdots + g^n P_n^2\Big); \\ G_{(2)} &= \frac{1}{2}\Big(\lambda^1_{(2)} g^1 P_1^2 + \cdots + \lambda^n_{(2)} g^n P_n^2\Big); \\ &\vdots \\ G_{(n)} &= \frac{1}{2}\Big(\lambda^1_{(n)} g^1 P_1^2 + \cdots + \lambda^n_{(n)} g^n P_n^2\Big). \end{aligned}$$

Extended SoV: Eisenhart theorem

Observation

The Hamilton-Jacobi equations for the geodesic Hamiltonians $G_{(\sigma)}$ can be written in matrix form:

$$\underbrace{\begin{pmatrix} g^1 & \cdots & g^n \\ \lambda^1_{(2)} g^1 & \cdots & \lambda^n_{(2)} g^n \\ \vdots & & \vdots \\ \lambda^1_{(n)} g^1 & \cdots & \lambda^n_{(n)} g^n \end{pmatrix}}_{\mathbf{S}^{-1}} \begin{pmatrix} (\partial_1 W)^2 \\ (\partial_2 W)^2 \\ \vdots \\ (\partial_n W)^2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

Extended SoV: Eisenhart theorem

Proposition

Any function $G_{(\sigma)} = \frac{1}{2}\sum_\mu \lambda^\mu_{(\sigma)} g^\mu {P_\mu}^2$ is a first integral for the geodesic Hamiltonian $G = \frac{1}{2}\sum_\nu g^\nu {P_\nu}^2$ if and only if the Eisenhart equations are satisfied:

$$g^\nu \frac{\partial \lambda^\nu_{(\sigma)}}{\partial z^\mu} = (\lambda^\mu_{(\sigma)} - \lambda^\nu_{(\sigma)}) \frac{\partial g^\nu}{\partial z^\mu}.$$

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Theorem

The Eisenhart equations are satisfied if and only if $\frac{\partial}{\partial z^\mu} S^\sigma_\nu = 0$ $(\mu \neq \nu)$ *hold for the components of the matrix* $\mathbf{S}$.

In other words, if and only if $\mathbf{S}$ *is a Stäckel matrix, hence the coefficients* g^μ *are in Stäckel form.*

Example: parabolic coordinates in $\mathbb{M}_2$

Metric tensor

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Killing tensor

$$K^{\mu\nu} = \begin{pmatrix} 1+2t & x+t \\ x+t & 1+2x \end{pmatrix}$$

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Separation coordinates

$$z^1 = x - t + \sqrt{1+2(x+t)}, \qquad z^2 = x - t - \sqrt{1+2(x+t)}.$$

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Hamiltonians

$$G_{(1)} = 4\,\frac{{P_1}^2 - {P_2}^2}{z^1 - z^2}, \quad G_{(2)} = 4\,\frac{z^1 {P_2}^2 - z^2 {P_1}^2}{z^1 - z^2}.$$

Example: parabolic coordinates in $\mathbb{M}_2$

Hamilton-Jacobi equations

$$\begin{cases} \left(\dfrac{\partial W}{\partial z^1}\right)^2 - \left(\dfrac{\partial W}{\partial z^2}\right)^2 = \dfrac{c_1}{4}(z^1 - z^2) \\ z^1\left(\dfrac{\partial W}{\partial z^2}\right)^2 - z^2\left(\dfrac{\partial W}{\partial z^1}\right)^2 = \dfrac{c_2}{4}(z^1 - z^2) \end{cases}$$

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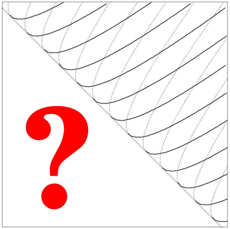
Then, the separated complete integral is the real function:

$$W = \frac{1}{3c_1}\sqrt{(z^1 c_1 - c_2)^3} + \frac{1}{3c_1}\sqrt{(z^2 c_1 - c_2)^3}.$$

Example: parabolic coordinates in $\mathbb{M}_2$

Problem

What happens where the eigenvalues are complex?



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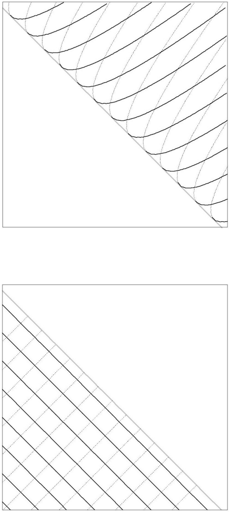
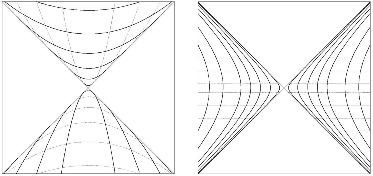


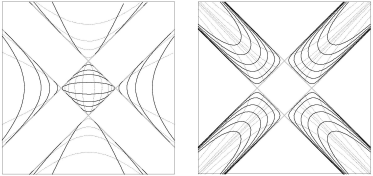
Drawing $\mathrm{Re}(z^1)$ and $\mathrm{Im}(z^1)$:

Example: other separable coordinates in $\mathbb{M}_2$

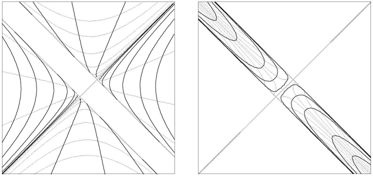
$$\begin{pmatrix} 2x & t \\ t & 0 \end{pmatrix}$$



$$\begin{pmatrix} x^2 & xt \\ xt & t^2-1 \end{pmatrix}$$



$$\begin{pmatrix} x^2-1 & xt-1 \\ xt-1 & t^2-1 \end{pmatrix}$$



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- The definition of a characteristic Killing tensor is extended.
- Levi-Civita criterion and Eisenhart theorem can be reformulated in the extended framework.
- Starting from a real Hamiltonian a real complete integral of the Hamilton-Jacobi equation is found.
- The complete integral of the Hamilton-Jacobi equation is defined both where the eigenvalues are real and where they are complex, assumed that they are distinct.