# Finite Element Exterior Calculus and Its Applications

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# Outline

1. Introduction and motivating examples
2. Roots and ingredients of FEEC
3. Applications related to the Hodge Laplacian
4. Application to elasticity via BGG

# Main reference

DNA, R. Falk, R. Winther: Finite element exterior calculus, homological techniques, and applications, *Acta Numerica* 15 (2006), pp. 1–155.

# Introduction

- A great strength of finite element methods is that they often admit a mathematical convergence theory, allowing validation and comparison of methods.
- Approximability, consistency, and stability $\Longrightarrow$ convergence
- Stability, like its continuous analogue, well-posedness, can be extremely subtle

  Well-posedness + approximability + consistency $\not\Longrightarrow$ stability
- Exterior calculus, Hodge theory, de Rham cohomology, . . . , were developed to get at well-posedness.
  FEEC adapts these tools to the discrete level to get at stability.

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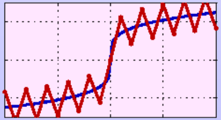

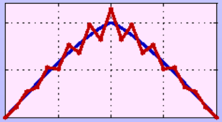

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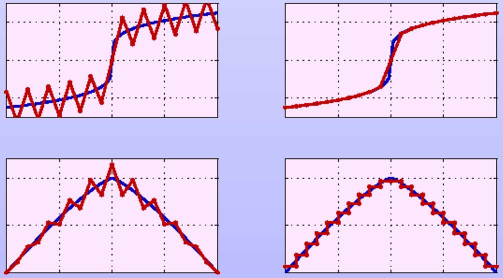

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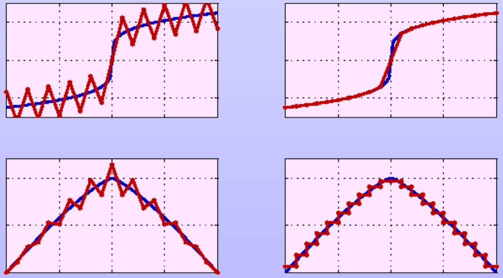


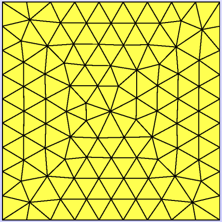
In higher dimensions, the solution is not obvious!

2D: Raviart–Thomas '76; Brezzi–Douglas–Marini '85; 3D: Nedelec '86

# Ex. 2a: Maxwell eigenvalue problem, unstructured mesh

$$\int_\Omega \operatorname{curl} u \cdot \operatorname{curl} v = \lambda \int_\Omega u \cdot v \quad \forall v$$

$$\lambda = m^2 + n^2 = 0, 1, 1, 2, 4, 4, 5, 5, 8, \ldots$$



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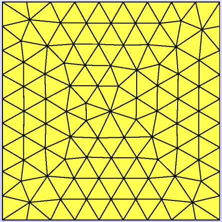

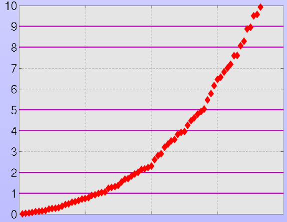


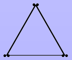

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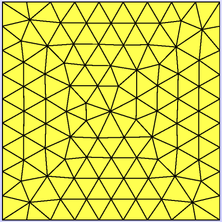

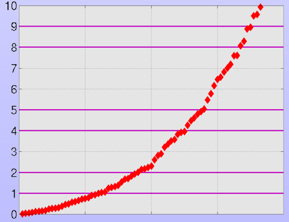


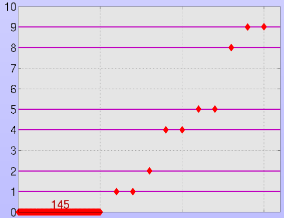


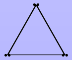

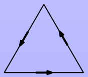

# Ex. 2b: Maxwell eigenvalue problem, regular mesh

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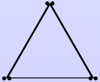

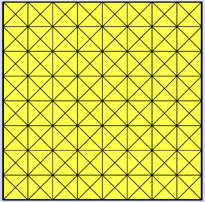

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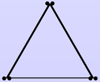

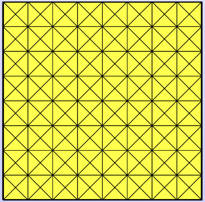

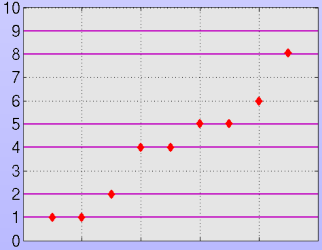


| 254 | 574 | 1022 | 1598 |
|---|---|---|---|
| 1.0043 | 1.0019 | 1.0011 | 1.0007 |
| 1.0043 | 1.0019 | 1.0011 | 1.0007 |
| 2.0171 | 2.0076 | 2.0043 | 2.0027 |
| 4.0680 | 4.0304 | 4.0171 | 4.0110 |
| 4.0680 | 4.0304 | 4.0171 | 4.0110 |
| 5.1063 | 5.0475 | 5.0267 | 5.0171 |
| 5.1063 | 5.0475 | 5.0267 | 5.0171 |
| 5.9229 | 5.9658 | 5.9807 | 5.9877 |
| 8.2713 | 8.1215 | 8.0685 | 8.0438 |

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# De Rham's Theorem and ...

$\Omega \subset \mathbb{R}^n$, $H\Lambda^k(\Omega) = \{\, \omega \in L^2\Lambda^k(\Omega) \mid d\omega \in L^2\Lambda^{k+1}(\Omega) \,\}$

$$0 \to H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^n(\Omega) \to 0$$

If $\Omega$ is furnished with a simplicial decomposition $\mathcal{T}$, a many-to-one correspondence $\Lambda^k(\Omega) \to C_k^*(\mathcal{T})$ (space of $k$-cochains) is given by

$$\omega \mapsto \left(c \mapsto \int_c \omega\right) \qquad (\star)$$

By Stokes theorem it's a cochain map, so induces a map from de Rham to simplicial cohomology.

$$\begin{array}{ccc} \Lambda^k(\Omega) & \xrightarrow{d} & \Lambda^{k+1}(\Omega) \\ \downarrow & & \downarrow \\ C_k^*(\mathcal{T}) & \xrightarrow{\partial^*} & C_{k+1}^*(\mathcal{T}) \end{array}$$

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- Whitney '57 constructed a cochain map $C_k^*(\mathcal{T}) \to H\Lambda^k(\Omega)$ which is a one-sided inverse to $(\star)$. Its range consists of certain piecewise linear $k$-forms ($\mathcal{P}_1^-\Lambda^k(\mathcal{T})$ in my notation). In this way the simplicial cochain complex is identified with a subcomplex of the de Rham complex:

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- Bossavit '88 observed that these spaces of Whitney forms coincided with the lowest order cases of mixed finite elements developed by Raviart–Thomas '76 and Nedelec '80 for 1-forms and 2-forms in 2D and 3D.

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# Finite Element de Rham subcomplexes

This is the fundamental structure of FEEC.

- A finite element subcomplex of the de Rham complex

$$0 \longrightarrow H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^n(\Omega) \longrightarrow 0$$

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The $\Lambda^k(\mathcal{T})$ are finite element spaces in the sense that they can be *assembed* from the following data on each simplex:

- finite dimensional space of polynomials forms on the simplex, and
- a decomposition of its dual space into subspaces associated to the subsimplices (degrees of freedom)

# Construction of FE differential forms

The key to the construction is the Koszul differential $\kappa : \Lambda^k \to \Lambda^{k-1}$:

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$\kappa$ is a contracting chain homotopy

- $\mathcal{H}_r\Lambda^k = d\mathcal{H}_{r+1}\Lambda^{k-1} \oplus \kappa\mathcal{H}_{r-1}\Lambda^{k+1}$

Using the Koszul differential, we define a special space of polynomial differential $k$-forms between $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_{r-1}\Lambda^k$:

$$\mathcal{P}_r\Lambda^k := \mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{H}_{r-1}\Lambda^{k+1} + d\mathcal{H}_{r+1}\Lambda^{k-1}$$

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Note that $\mathcal{P}_r^-\Lambda^0 = \mathcal{P}_r\Lambda^0$ and $\mathcal{P}_r^-\Lambda^n = \mathcal{P}_{r-1}\Lambda^n$

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*God made $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$,*
*all the rest is the work of man.*

# $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$

Using the Koszul differential, we define a special space of polynomial differential $k$-forms between $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_{r-1}\Lambda^k$:

$$\mathcal{P}_r^-\Lambda^k := \mathcal{P}_{r-1}\Lambda^k + \kappa\mathcal{H}_{r-1}\Lambda^{k+1} + d\mathcal{H}_{r+1}\Lambda^{k-1}$$

Note that $\mathcal{P}_r^-\Lambda^0 = \mathcal{P}_r\Lambda^0$ and $\mathcal{P}_r^-\Lambda^n = \mathcal{P}_{r-1}\Lambda^n$

*God made $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$,*
*all the rest is the work of man.*

Proven with representation theory...

# Degrees of freedom

To obtain *finite element* differential forms—not just pw polynomials—we need degrees of freedom, i.e., a decomposition of the dual spaces $(\mathcal{P}_r\Lambda^k(T))^*$ and $(\mathcal{P}_r^-\Lambda^k(T))^*$ ($T$ a simplex), into subspaces associated to subsimplices $f$ of $T$.

DOF for $\mathcal{P}_r\Lambda^k(T)$: to a subsimplex $f$ of dimension $d$ we associate

$$\omega \mapsto \int_f \operatorname{Tr}_f \omega \wedge \eta, \quad \eta \in \mathcal{P}_{r+k-d}^- \Lambda^{d-k}(f)$$

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Hiptmair

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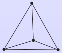
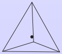
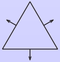
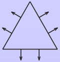
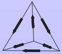
Hiptmair

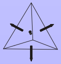
Given a triangulation $\mathcal{T}$, we can then define $\mathcal{P}_r\Lambda^k(\mathcal{T})$, $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$. They are subspaces of $H\Lambda^k(\Omega)$.

# Finite element differential forms/Mixed FEM

- $\mathcal{P}_r^-\Lambda^0(\mathcal{T}) = \mathcal{P}_r\Lambda^0(\mathcal{T}) \subset H^1$ Lagrange elts 
- $\mathcal{P}_r^-\Lambda^n(\mathcal{T}) = \mathcal{P}_{r-1}\Lambda^n(\mathcal{T}) \subset L^2$ discontinuous elts 
- $n = 2$: $\mathcal{P}_r^-\Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Raviart–Thomas elts 

  $\mathcal{P}_r\,\Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Brezzi–Douglas–Marini elts 
- $n = 3$: $\mathcal{P}_r^-\Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Nedelec 1st kind edge elts 

  $\mathcal{P}_r\,\Lambda^1(\mathcal{T}) \subset H(\mathrm{curl})$ Nedelec 2nd kind edge elts 

  $\mathcal{P}_r^-\Lambda^2(\mathcal{T}) \subset H(\mathrm{div})$ Nedelec 1st kind face elts 

  $\mathcal{P}_r\,\Lambda^2(\mathcal{T}) \subset H(\mathrm{div})$ Nedelec 2nd kind face elts 

# Finite element de Rham subcomplexes

- For every $r \geq 1$, the $\mathcal{P}_r^-\Lambda^k$ spaces give a FE de Rham subcomplex:

$$0 \to \mathcal{P}_r^-\Lambda^0(\mathcal{T}) \xrightarrow{d} \mathcal{P}_r^-\Lambda^1(\mathcal{T}) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_r^-\Lambda^n(\mathcal{T}) \to 0$$

For $r = 1$ this is Whitney's complex.

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- The projections $\Pi^k : \Lambda^k(\Omega) \to \mathcal{P}_r^- \Lambda^k(\mathcal{T})$ defined through the DOF form a cochain projection. (They are not defined on all of $H\Lambda^k(\Omega)$ but modified cochain projections can be defined which are bounded on $H\Lambda^k$.)

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- There are many ways to form the spaces $\mathcal{P}_r \Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^- \Lambda^k(\mathcal{T})$ into a discrete de Rham subcomplex with a cochain projection:
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- In every case the cochain projection induces an isomorphism on cohomology.

# Outline

# Mixed Hodge Laplacian

$\Omega \subset \mathbb{R}^n$, $0 \le k \le n$, $f \in L^2\Lambda^k(\Omega)$

$$\sigma \in H\Lambda^{k-1}(\Omega), \quad u \in H\Lambda^k(\Omega):$$

$$\begin{aligned} \langle \sigma, \tau \rangle - \langle d\tau, u \rangle &= 0 && \forall \tau \in H\Lambda^{k-1}(\Omega) \\ \langle d\sigma, v \rangle + \langle du, dv \rangle &= \langle f, v \rangle && \forall v \in H\Lambda^k(\Omega) \end{aligned}$$

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- $k = 0$: ordinary Laplacian
- $k = n$: mixed Laplacian
- $k = 1, n = 3$: $\sigma = -\operatorname{div} u, \quad \operatorname{grad}\sigma + \operatorname{curl}\operatorname{curl} u = f$
- $k = 2, n = 3$: $\sigma = \operatorname{curl} u, \quad \operatorname{curl}\sigma - \operatorname{grad}\operatorname{div} u = f$

IMA

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For special $f$ these reduce to

- $\operatorname{div} u = f$, $\operatorname{curl} u = 0$
- $\operatorname{curl}\operatorname{curl} u = f$, $\operatorname{div} u = 0$

# Well-posedness of the Hodge Laplacian

To obtain well-posedness we must handle the harmonic forms

$$\mathfrak{h}^k := \{\, u \in H\Lambda^k \mid du = 0,\ \langle d\tau, u\rangle = 0\ \forall \tau \in H\Lambda^{k-1} \,\}$$

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Need to control $\|\sigma\|_{H\Lambda} + \|u\|_{H\Lambda} + \|p\|$ by a bounded choice of $\tau$, $v$, and $q$.

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$v = u$ controls $\|du\|$, *How to control* $\|u\|$ *??*

IMA

# Well-posedness from the Hodge decomposition

$$H\Lambda^{k-1} \xrightarrow{d} H\Lambda^k \xrightarrow{d} H\Lambda^{k+1}$$

$$u \in H\Lambda^k = dH\Lambda^{k-1} \oplus (dH\Lambda^{k-1})^\perp$$

Since $dH\Lambda^{k-1} \oplus \mathfrak{h}^k = \mathcal{N}(d)$, $(dH\Lambda^{k-1})^\perp = \mathfrak{h}^k \oplus \mathcal{N}(d)^\perp$. Thus

$$H\Lambda^k = dH\Lambda^{k-1} \oplus \mathfrak{h}^k \oplus \mathcal{N}(d)^\perp$$ Hodge decomposition

$$u = d\tau + q + z, \quad \tau \in H\Lambda^{k-1}, q \in \mathfrak{h}^k, z \in \mathcal{N}(d)^\perp$$

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$$u = d\tau + q + z, \quad \tau \in H\Lambda^{k-1}, q \in \mathfrak{h}^k, z \in \mathcal{N}(d)^\perp$$

Easy to bound $\|d\tau\|$ and $\|q\|$. To bound $\|z\|$ we use Poincaré's inequality $\|z\| \le c\|dz\|$ for $z \in \mathcal{N}(d)^\perp$, and the fact that $dz = du$, which is already under control.

# Stability of the FE for the mixed Hodge Laplacian

Analogous reasoning using the finite element de Rham complex, establishes stability of the finite element. In place of the Poincaré inequality we use the Poincaré inequality on the continuous level and the boundedness of the cochain projections. A full convergence theory follows for four different families of mixed finite elements!

$$\mathcal{P}_r^- \Lambda^{k-1}(\mathcal{T}) \times \mathcal{P}_r^- \Lambda^k(\mathcal{T})$$

$$\mathcal{P}_r \Lambda^{k-1}(\mathcal{T}) \times \mathcal{P}_r^- \Lambda^k(\mathcal{T})$$

$$\mathcal{P}_{r+1}^- \Lambda^{k-1}(\mathcal{T}) \times \mathcal{P}_r \Lambda^k(\mathcal{T})$$

$$\mathcal{P}_{r+1} \Lambda^{k-1}(\mathcal{T}) \times \mathcal{P}_r \Lambda^k(\mathcal{T})$$

# There are lots of other applications of FEEC

- Maxwell's equations and related EM problems
- Mixed eigenvalue problems
- Preconditioning and multigrid
- Stable mixed FEM for elasticity

# Outline

# The equations of elasticity

$\Omega \subset \mathbb{R}^3$, $f : \Omega \to \mathbb{R}^3$ imposed load.
Find stress $\sigma : \Omega \to \mathbb{R}^{3\times 3}_{\text{sym}}$, displacement $u : \Omega \to \mathbb{R}^3$ such that

$$A\sigma = \epsilon(u), \quad \operatorname{div} \sigma = f$$

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Finding stable finite elements for this first order system is a long open, very challenging, and very important problem.

# The elasticity complex

For the equations of elasticity, the relevant elliptic complex is

$$0 \to C^\infty(\Omega, \mathbb{R}^3) \xrightarrow{\epsilon} C^\infty(\Omega, \mathbb{R}^{3\times3}_{\mathrm{sym}}) \xrightarrow{J} C^\infty(\Omega, \mathbb{R}^{3\times3}_{\mathrm{sym}}) \xrightarrow{\mathrm{div}} C^\infty(\Omega, \mathbb{R}^3) \to 0$$

$J\tau = \mathrm{curl}(\mathrm{curl}\,\tau)^T$, second order

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$$\underset{\text{displacement}}{\uparrow} \qquad \underset{\text{strain}}{\uparrow} \qquad \underset{\text{stress}}{\uparrow} \qquad \underset{\text{load}}{\uparrow}$$

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With weakly imposed symmetry the relevant sequence is

$$0 \to C^\infty(\mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{\mathrm{skw}}) \xrightarrow{(\mathrm{grad}, -I)} C^\infty(\mathbb{R}^{3\times 3}) \xrightarrow{J} C^\infty(\mathbb{R}^{3\times 3}) \xrightarrow{\begin{pmatrix}\mathrm{div}\\ \mathrm{skw}\end{pmatrix}} C^\infty(\mathbb{R}^3 \times \mathbb{R}^{3\times 3}_{\mathrm{skw}}) \to 0$$

where $J$ is extended by zero to skew matrices.

IMA

# Bernstein–Gelfand–Gelfand construction, I

$\mathbb{V} = \mathbb{R}^n$, $\mathbb{K} = \mathbb{V} \wedge \mathbb{V}$, $\mathbb{W} = \mathbb{K} \times \mathbb{V}$.

1. Start with the de Rham sequence with values in $\mathbb{W}$:

$$0 \to \Lambda^0(\Omega; \mathbb{W}) \xrightarrow{\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}} \Lambda^1(\Omega; \mathbb{W}) \xrightarrow{\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}} \cdots \xrightarrow{\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}} \Lambda^n(\Omega; \mathbb{W}) \to 0$$

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$$\Phi = \begin{pmatrix} I & K \\ 0 & I \end{pmatrix}, \qquad \Phi^{-1} = \begin{pmatrix} I & -K \\ 0 & I \end{pmatrix}$$

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# BGG II

5. Note that $\mathcal{A} = \begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}$, where $S = dK - Kd : \Lambda^k(\mathbb{V}) \to \Lambda^{k+1}(\mathbb{K})$ is given by

$$(S\omega)_x(v_1, \ldots, v_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} v_j \wedge \omega_x(v_1, \ldots, \hat{v}_j, \ldots, v_{k+1}).$$

Properties: $S$ is algebraic. For $k = n - 2$, $S$ is an isomorphism. $dS = -Sd$.

# BGG II

5. Note that $\mathcal{A} = \begin{pmatrix} d & -S \\ 0 & d \end{pmatrix}$, where $S = dK - Kd : \Lambda^k(\mathbb{V}) \to \Lambda^{k+1}(\mathbb{K})$ is given by

$$(S\omega)_x(v_1, \ldots, v_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} v_j \wedge \omega_x(v_1, \ldots, \hat{v}_j, \ldots, v_{k+1}).$$

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6. Define subspaces $\Gamma^k \subset \Lambda^k(\mathbb{W})$ satisfying $\mathcal{A}(\Gamma^k) \subset \Gamma^{k+1}$ and projections

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$$\pi^{n-2} = \begin{pmatrix} I & 0 \\ S^{-1}d & 0 \end{pmatrix} : \Lambda^{n-2}(\mathbb{W}) \to \Gamma^{n-2}, \quad \pi^{n-1} = \begin{pmatrix} 0 & 0 \\ dS^{-1} & I \end{pmatrix} : \Lambda^{n-1}(\mathbb{W}) \to \Gamma^{n-1}.$$

IMA

# BGG III

7. The following diagram with vertical projections commutes ($dS = -Sd$)

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Therefore, the subcomplex on the bottom row is exact if the top is.

8. This subcomplex may be identified with the elasticity complex.

# Mixed finite elements for elasticity

We can mimic the BGG construction on the discrete level.

We begin by picking two *different* finite element de Rham sequences

$$\cdots \to \Lambda^{n-3}(\mathcal{T}) \longrightarrow \Lambda^{n-2}(\mathcal{T}) \longrightarrow \Lambda^{n-1}(\mathcal{T}) \longrightarrow \Lambda^{n}(\mathcal{T}) \longrightarrow 0$$

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for $k = n-2$, $S_{\mathcal{T}}$ is onto.
If this holds, we finally conclude that the spaces $\tilde{\Lambda}^{n-1}(\mathcal{T};\mathbb{V})$ for $\sigma$, $\tilde{\Lambda}^{n}(\mathcal{T};\mathbb{V})$ for $u$, and $\Lambda^{n}(\mathcal{T};\mathbb{K})$ for $p$ gives a stable discretization for elasticity.

# Stable elasticity elements

There are many pairs of finite element de Rham complexes satisfying the compability condition. The simplest is:

$$\cdots \to \mathcal{P}_{r+1}^{-}\Lambda^{n-3} \to \mathcal{P}_{r+1}^{-}\Lambda^{n-2} \to \mathcal{P}_{r+1}^{-}\Lambda^{n-1} \to \mathcal{P}_{r}\Lambda^{n} \to 0$$

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They satisfy the compatibility condition because $\mathcal{P}^-_{r+2}\Lambda^{n-2}$ includes face DOFs, needed for surjectivity of $S_{\mathcal{T}}$ onto $\mathcal{P}^-_{r+1}\Lambda^{n-1}$.

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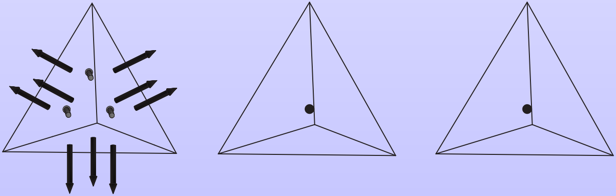
They satisfy the compatibility condition because $\mathcal{P}_{r+2}^{-}\Lambda^{n-2}$ includes face DOFs, needed for surjectivity of $S_{\mathcal{T}}$ onto $\mathcal{P}_{r+1}^{-}\Lambda^{n-1}$. This choice leads to the following stable elements for elasticity:

| | |
|---|---|
| stress | $\mathcal{P}_{r+1}\Lambda^{n-1}(\mathcal{T};\mathbb{V})$ |
| displacement | $\mathcal{P}_{r}\Lambda^{n}(\mathcal{T};\mathbb{V})$ |
| multiplier | $\mathcal{P}_{r}\Lambda^{n}(\mathcal{T};\mathbb{K})$ |

# Simplest case

$r = 0$



*Far* simpler than the elements than any previously devised stable mixed elasticity elements.

# Conclusions

- FEEC provides a very natural framework for the design and understanding of subtle stability issues that arise in the discretization of a wide variety of PDE systems.
- FEEC brings to bear tools from geometry, topology, and algebra to develop discretizations which are compatible with the geometric, topological, and algebraic structure of the PDE system, and so obtain stability.
- FEEC has been used to unify, clarify, and refine many known finite element methods.
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