

## ALGEBRAIC APPROACHES: QUASI-EXACTLY-SOLVABLE PROBLEMS

(Lie-algebraic theory of polynomial solutions of differential and finite-difference linear equations)

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Lie-algebraic theory of polynomial solutions of differential and finite-difference linear equations is presented. It is based on an observation that any known polynomial solution can be identified with an element of finite-dimensional representation space of a Lie algebra differential (finite-difference) operators. There are no counter-examples so far. Three types of linear operators are studied: (i) differential  $L(x, \partial_x)$ , (ii) finite-difference  $L(x, D_\delta)$  on uniform lattice, where  $D_\delta f(x) = \frac{f(x+\delta) - f(x)}{\delta}$  and (iii) finite-difference  $L(x, \mathcal{D}_q)$  on exponential lattice, where  $\mathcal{D}_q f(x) = \frac{f(qx) - f(x)}{x(q-1)}$ , discrete or Jackson derivative. Operators admitting polynomial eigenfunctions are naturally divided in two classes: having a finite number of polynomial eigenfunctions (we call these operators *quasi-exactly-solvable*) and having infinitely-many polynomial eigenfunctions (we call these operators *exactly-solvable*). Thus, the theory of polynomial solutions is related to the representation theory of Lie algebras of differential (finite-difference operators).

A complete classification of one-dimensional differential and finite-difference linear equations admitting polynomial solutions is given in terms of the algebra  $sl(2)$  realized in action on the line (quasi-exactly-solvable case) and its Borel sub-algebra  $b_2 \subset sl(2)$  (exactly-solvable case). In particular, it is shown that for the 2nd order differential equations: the Heun equation is the only equation which can possess finite number of polynomial solutions (i), while the hypergeometrical equation is the most general possessing the infinitely-many polynomial solutions (ii). In turn, the Hahn polynomials are the most general polynomial eigenfunctions emerged from the three-point difference operators on uniform lattice.

It is demonstrated though a complete classification of multi-dimensional differential and finite-difference linear equations admitting polynomial solutions is absent the present approach provides wealth of non-trivial cases related to the algebras  $gl(n)$ ,  $gl(n) \oplus gl(2) \dots \oplus gl(2)$  etc realized by differential (finite-difference) operators and having finite-dimensional representations. Two-dimensional case is studied in details. It is shown that so-called Krall-Sheffer classification is a very particular degenerate case of Lie-algebraic approach.