Entropy Stable FEM Discretization of First-Order Systems of Conservation Laws

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We consider the discontinuous Galerkin (DG) finite element discretization [RH73, CLS89, CHS90, Coc99] of first-order systems of conservation laws such as those occurring in hydrodynamics. Stability of the discontinuous Galerkin method depends critically on the choice of the numerical flux and any other stabilization added in element interiors or on element interfaces. In this lecture, we give a self-contained review of global and elementwise local entropy analysis for the DG method and how the analysis is used to design stable numerical fluxes. Consequently, using these discretizations, global and elementwise local entropy inequalities are formally obtained. For example, if U denotes the convex entropy, then a formal space-time global entropy inequality for a spatial domain Ω is obtained

$$\int_\Omega U(T_0 + n\Delta t)\, dx \leq \int_\Omega U(T_0)\, dx \quad + \text{boundary conditions}, \quad n = 0, 1, 2, ...$$

for a prescribed time slab increment Δt.

We then expand the discussion to include a first-order system of conservation laws with solenoidal constraint such as those occurring in compressible magnetohydrodynamics (MHD). The objective is to again show how entropy analysis plays an invaluable role in designing numerical fluxes and stabilization terms. Unlike standard hydrodynamics, the entropy analysis for MHD reveals the subtle role of the solenoidal constraint in obtaining global and elementwise local entropy stability through

- strong or weak (penalty) satisfaction of $\text{div} B = 0$ in element interiors
- strong or weak (penalty) satisfaction of $[B \cdot n] = 0$ on element interfaces.

Although one could satisfy $[B \cdot n] = 0$ strongly (see for example Brezzi-Douglas-Marini (1985)), this would add considerable complexity to the discontinuous Galerkin method and would destroy the simple block diagonal mass matrix structure in the DG method. Alternatively, one can use the entropy analysis to design a $[B \cdot n] = 0$ penalty term with magnitude dictated by the entropy analysis so that global and elementwise local entropy inequalities are obtained and the simple mass matrix structure retained. Finally, we consider the generalization of these procedures to other systems of conservations laws with constraints. Throughout the lecture, numerical results of interest are given to verify the analysis.

References

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