ON THE CONSTRUCTION OF ISOSPECTRAL MANIFOLDS

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ABSTRACT. We derive a criterion for closed Riemannian manifolds to be isospectral. We apply it to show that there are isospectral metrics on $S^2 \times S^3$. Our work relies heavily on recent work of Dorothee Schueth, see [Sch].

INTRODUCTION

Let M be a closed Riemannian manifold. Let Δ be the Laplace operator of M. Then the spectrum of M consists of the sequence

$$0 = \lambda_0 \le \lambda_1 \le \lambda_2 \le \dots$$

of eigenvalues of Δ, where we count each eigenvalue as often as its multiplicity requires. We say that closed Riemannian manifolds M and M' are *isospectral* if the corresponding sequences of eigenvalues coincide.

There is a general method for establishing isospectrality, in its original form due to Toshikazu Sunada [Su], which concerns subcovers of a given Riemannian manifold. Examples arising from this method have the same local geometry, they can only be distinguished by global invariants.

The Criteria of Gordon and Schueth. The first pair of manifolds — with boundary though — which are isospectral but not locally isometric is due to Zoltan Szabo [Sz]. Motivated by his work, Carolyn Gordon constructed the first closed examples and came up with the following general criterion, compare [Go1, Go2], see also Theorem 1.6 below.

THEOREM A. *Let M and M' be closed Riemannian manifolds. Suppose that a torus T acts freely and isometrically on M and M'. Suppose furthermore that the following hold:*

1. *The orbits of the action are totally geodesic.*
2. *For any closed subgroup S of T of codimension at most one, the quotient manifolds M/S and M'/S are isospectral.*

Then M and M' are isospectral.

Date: September 5, 2000.
1991 *Mathematics Subject Classification.* 58J53.
Key words and phrases. Laplace operator, spectrum.
Partially supported by SFB256 (U Bonn).

Note the difference between Gordon's and Sunada's method. Whereas we assume isospectrality of quotients in the first, we establish isospectrality of quotients in the latter. Gordon's criterion has turned out to be a very powerful tool in the construction of isospectral manifolds; for a more elaborate historical account I refer to [Sch].

Let T be a torus and $P \to N$ be a principal bundle with structure group T, where N is a closed and connected Riemannian manifold. Fix a left invariant metric g_T on T. Then for any principal connection ω on P, there is a unique Riemannian metric g_ω on P such that the projection $P \to N$ is a Riemannian submersion with totally geodesic fibers isometric to T and horizontal distribution equal to $\ker \omega$, see Theorem 9.59 in [Be].

In her recent work, Dorothee Schueth used Theorem A to derive the following very useful more special criterion, see Theorem 1.6 in [Sch].

THEOREM B. *Let ω and ω' be principal connections on P. Suppose that for each ζ in the dual of the Lie algebra $\mathfrak{t}$ of T there is an automorphism $\Phi_\zeta : P \to P$ such that*

1. *Φ_ζ factors over an isometry of N.*
2. *$\zeta \circ \omega' \circ \Phi_{\zeta *} = \zeta \circ \omega$.*

Then (P, g_ω) and $(P, g_{\omega'})$ are isospectral.

In her proof, Schueth showed that for any closed subgroup S of T of codimension at most one, the quotient manifolds $(P, g_\omega)/S$ and $(P, g_{\omega'})/S$ are isometric. In one of her applications, Schueth obtained pairs of isospectral metrics on $S^2 \times T^2$.

One of the aims of this paper is to extend the criteria of Gordon and Schueth and to explain why tori play such a prominent role in their criteria.

Results. Let N be a closed and connected Riemannian manifold and $P \to N$ be a principal bundle with structure group G, where G is a compact Lie group. Let F be a closed Riemannian manifold and $\mu : G \to \mathrm{Iso}(F)$ be a representation. Set $M := P \times_\mu F$.

Given a principal connection ω on P, there is precisely one Riemannian metric g_ω on M such that the natural projection $\pi : M \to N$ is a Riemannian submersion with totally geodesic fibers isometric to F and horizontal distribution induced by ω, see Theorem 9.59 in [Be]. Extending Theorem B, we obtain the following criterion for manifolds to be isospectral.

THEOREM C. *Let ω and ω' be principal connections on P. Suppose that for each irreducible unitary representation ρ of G there is an automorphism $\Phi_\rho : P \to P$ such that*

1. *Φ_ρ factors over an isometry of N.*
2. *$\rho_* \circ \omega' \circ \Phi_{\rho *} = \rho_* \circ \omega$.*

Then (M, g_ω) and $(M, g_{\omega'})$ are isospectral.

Our proof of Theorem C is quite different from the proof of Theorem B in [Sch]. We use a correspondence between the Laplacian of M and the Laplacians of certain vector bundles.

In this work we discuss just one application of Theorem C, others are possible. In our application, we rely on results from Chapter 2 in [Sch].

THEOREM D. *Suppose that F has dimension ≥ 3 and constant scalar curvature and that T^2 acts effectively on F. Then $M = S^2 \times F$ admits isospectral Riemannian metrics for which the projection $M \to S^2$ is a Riemannian submersion with totally geodesic fibers isometric to F.*

In particular, there is a pair of isospectral metrics on $S^2 \times S^3$. So far, this is the lowest dimensional known simply connected closed Riemannian manifold which admits a pair of isospectral metrics.

I should point out that independently of this work, Gordon extended the work from [Sch] in a different direction. Building on Gordon's extension, Schueth also obtains — among others — isospectral Riemannian metrics on $S^2 \times S^3$.

Remarks. In Theorems B and C, it would be overkill to assume that there is an automorphism $\Phi : P \to P$ which factors over an isometry $\phi : N \to N$ such that $\omega' \circ \Phi_* = \omega$. This would imply that (P, g_ω) and $(P, g_{\omega'})$ or, respectively, (M, g_ω) and $(M, g_{\omega'})$ are isometric, which is not what we are after. One of the highlights in the work of Schueth is the construction of non–trivial examples where Theorem B applies, see Chapter 2 of [Sch] and Section 2 below.

Theorem C explains why tori are crucial in the constructions of isospectral manifolds of Gordon and Schueth: If G has an irreducible unitary representation ρ with injective differential, then by Assumption 2 of Theorem C, $\omega' \circ \Phi_{\rho *} = \omega$, and then Φ_ρ is an isometry between (M, g_ω) and $(M, g_{\omega'})$.

Actually Assumption 2 in Theorem C can be relaxed somewhat: Only the irreducible unitary representations ρ of G which occur in the natural representation of G on $L^2(F)$ need to be considered.

Acknowledgment. I would like to thank Dorothee Schueth for helpful remarks and discussions.

1. PROOF OF THEOREM C

Let M and N be closed Riemannian manifolds and $\pi : M \to N$ be a Riemannian submersion. We denote the fiber over $q \in N$ by F_q.

Let $c : [0,1] \to N$ be a piecewise smooth curve from $q_0 = c(0)$ to $q_1 = c(1)$. Then the horizontal displacement $h_c : F_{q_o} \to F_{q_1}$ is a diffeomorphism. We recall the following well known fact, see Theorem 9.56 in [Be].

1.1. LEMMA. *The fibers of π are totally geodesic if and only if, for each piecewise smooth curve $c : [0,1] \to N$, h_c is an isometry.*

From now on we assume that N is connected and that the fibers of π are totally geodesic. Then all fibers of π are isometric. We fix an origin $o \in N$ and set $F := F_o$. The induced metric on F is denoted g_F.

We denote by $H \subset \mathrm{Iso}(F)$ the holonomy group of π at o; that is, H consists of all isometries h_c, where $c : [0,1] \to N$ is a piecewise smooth curve with $c(0) = c(1) = o$.

We fix a closed subgroup $G \subset \mathrm{Iso}(F)$ containing H and let $P \to N$ be the principal bundle with structure group G and fibers

$$P_q := \{h_c \circ g \mid g \in G\}\,, \quad q \in N\,,$$

where $c : [0,1] \to N$ is some piecewise smooth curve from o to q. Since $H \subset G$, P_q does not depend on the choice of c. Composition on the right defines the right action of G on P. Horizontal displacement defines a principal connection ω on P: Let $c : [0,1] \to N$ be piecewise smooth and $k \in G_{c(0)}$. Let $h_t : F_{c(0)} \to F_{c(t)}$ be the horizontal displacement along $c|[0,t]$. Then $h_t \circ k$, $0 \le t \le 1$, is the horizontal lift of c to P starting in k.

Let $\mu : G \to \mathrm{Iso}(F)$ be the inclusion. Then the canonical map

$$P \times_\mu F \to M\,, \quad [k,p] \mapsto k(p)\,,$$

is a diffeomorphism. Moreover, the horizontal distribution induced by ω coincides with the original horizontal distribution of π. Hence we recover the Riemannian metric on M from the Riemannian metrics on F and the base N.

The Laplace operator of F is denoted Δ_F. Since F is closed, $L^2(F)$ is the Hilbert space sum of the eigenspaces of Δ_F. We fix a Hilbert space decomposition

$$L^2(F) = \hat{\oplus}_{\alpha \in A} V_\alpha$$

into pairwise orthogonal and G–invariant subspaces such that, for each α, the induced representation $\rho_\alpha : G \to U(V_\alpha)$ is irreducible. It follows that each V_α is contained in an eigenspace of Δ_F. In particular, V_α is a finite dimensional Euclidean space. We set

$$W_\alpha := P \times_{\rho_\alpha} V_\alpha\,.$$

Since V_α is Euclidean and ρ_α is unitary, W_α carries a canonical Hermitian metric. The connection ω on P induces a Hermitian covariant derivative ∇^α on W_α. We denote the corresponding connection Laplacian by Δ_α. In terms of a local orthonormal frame $Y_1, \dots, Y_n$ on N we have

$$\Delta_\alpha = -\sum (\nabla^\alpha_{Y_j} \nabla^\alpha_{Y_j} - \nabla^\alpha_{\nabla^N_{Y_j} Y_j}) \, ,$$

where ∇^N denotes the Levi–Civita connection of N.

We recall that V_α consists of functions $f : F \to \mathbb{C}$. Hence we obtain a linear map from sections of W_α to functions on M,

$$\big(\sigma : q \mapsto [k(q), v(q)]\big) \mapsto \big(f : p \mapsto v(q)(k(q)^{-1}(p))\big) \, , \quad \text{where } q = \pi(p) \, .$$

Therefore, sections of W_α are in canonical one–to–one correspondence with functions $f : M \to \mathbb{C}$ such that $f \circ h_c \in V_\alpha$ for all piecewise smooth curves $c : [0,1] \to N$ with $c(0) = o$. In particular, via this correspondence

$$L^2(M) = \hat{\oplus}_\alpha L^2(W_\alpha)$$

is a decomposition into pairwise perpendicular closed subspaces.

Let $(U_1, \dots, U_k, X_1, \dots, X_n)$ be a local orthonormal frame of M, where $(U_1, \dots, U_k)$ is a local orthonormal vertical frame and $(X_1, \dots, X_n)$ is the horizontal lift of a local orthonormal frame of N. Then we have

$$\Delta = -\sum (U_i^2 - \nabla_{U_i} U_i) - \sum (X_j^2 - \nabla_{X_j} X_j) \, . \tag{1.2}$$

Since the fibers are totally geodesic, the vertical part

$$\Delta_{\mathcal{V}} := -\sum (U_i^2 - \nabla_{U_i} U_i) \tag{1.3}$$

corresponds to the Laplace operator of the fibers. We show now that the horizontal part

$$\Delta_{\mathcal{H}} := -\sum (X_j^2 - \nabla_{X_j} X_j) \tag{1.4}$$

correponds to Δ_α via the above correspondence between sections of W_α and functions on M.

1.5. LEMMA. *If a smooth section σ of W_α corresponds to a smooth function f on M, then $\Delta_\alpha \sigma$ corresponds to $\Delta_{\mathcal{H}} f$.*

Proof. Let $q_0 \in N$ and $p \in F_{q_0}$ be a point over q_0. Choose $\varepsilon > 0$ smaller than the injectivity radius of N at q_0. Choose a smooth curve $c : [0, 1/2] \to N$ from o to q_0. For $q \in B_\varepsilon(q_0)$ let $c_q : [0,1] \to N$ be the piecewise smooth curve with $c|[0,1/2]) = c$ and such that $c|[1/2, 1]$ is the unique shortest geodesic from q_0 to q. Let $\tilde{c}_q$ be the horizonzal lift of c_q with $\tilde{c}_q(1/2) = p$.

Let $v_1, \dots, v_s$ be an orthonormal basis of V_α. Then for $q \in B_\varepsilon(q_0)$, the vectors $\sigma_i(q) := [h_{c_q}, v_i]$, $1 \le i \le s$, are an orthonormal basis of the fiber of W_α over q. Moreover, the local frame $(\sigma_1, \dots, \sigma_s)$ is smooth and parallel along the curves c_q. Hence locally, σ can be expressed as a linear combination

$$\sigma(q) = \sum \varphi_i(q) \, \sigma_i(q)$$

with smooth coefficient functions $\varphi_1, \dots, \varphi_s$ and

$$(\Delta_\alpha \sigma)(q_0) = \sum (\Delta_N \varphi_i)(q_0)\, \sigma_i(q_0)\,,$$

where Δ_N denotes the Laplace operator of N.

By the definition of c_q, $p_0 = h_{c_q}^{-1}(\tilde{c}_q(1)) \in F$ is independent of $q \in B_\varepsilon(q_0)$. Hence the function f corresponding to σ satisfies

$$f(\tilde{c}_q(1)) = \sum \varphi_i(q)\, v_i(p_0)\,.$$

Therefore,

$$(\Delta_{\mathcal{H}} f)(p) = \sum (\Delta_N \varphi_i)(q_0)\, v_i(p_0)\,.$$

Hence $\Delta_\alpha \sigma$ correponds to $\Delta_{\mathcal{H}} f$. □

For each α, there is an eigenvalue λ_α of Δ_F such that V_α is contained in the corresponding eigenspace of Δ_F. Therefore,

$$\Delta f = \lambda_\alpha f + \Delta_{\mathcal{H}} f$$

for any smooth function f such that $f \circ h_c \in V_\alpha$ for all piecewise smooth curves $c : [0,1] \to N$ with $c(0) = o$. Now list the eigenvalues of Δ_α in increasing order and include any eigenvalue as often as its multiplicity,

$$\lambda_{\alpha,0} \le \lambda_{\alpha,1} \le \lambda_{\alpha,2} \le \dots$$

Then by the above, the spectrum of Δ is given by

$$\lambda_\alpha + \lambda_{\alpha,k}\,, \quad \alpha \in A\,,\ k \ge 0\,,$$

where each eigenvalue occurs as often as its multiplicity demands.

Our aim is a criterion for isospectrality of Riemannian manifolds. Let $\pi : M \to N$ and $\pi' : M' \to N'$ be Riemannian submersions as above with totally geodesic fibers and N and N' connected. Fix decompositions

$$L^2(F) = \hat{\oplus}_{\alpha \in A} V_\alpha \quad \text{and} \quad L^2(F') = \hat{\oplus}_{\alpha \in A} V'_\alpha$$

as above (and with the same set A of indices). Our first criterion is still of a rather general and non–effective nature. It is a variation of the general criterion of Gordon in Theorem A.

1.6. THEOREM. *Suppose that for each $\alpha \in A$,*

1. $\lambda_\alpha = \lambda'_\alpha$ *and*
2. Δ_α *and* Δ'_α *are isospectral.*

Then M and M' are isospectral. □

We are now ready for the proof of Theorem C. As above, we fix a Hilbert space decomposition $L^2(F) = \hat{\oplus}_{\alpha \in A} V_\alpha$ into pairwise orthogonal subspaces such that, for each α, the representation $\rho_\alpha : G \to U(V_\alpha)$ is irreducible. Then there exist G–equivariant bundle isomorphisms $\Phi_\alpha : P \to P$ satisfying Assumptions 1 and 2 of Theorem C with $\rho = \rho_\alpha$. For each α, we have a Hermitian vector bundle

$$W_\alpha := P \times_{\rho_\alpha} V_\alpha$$

over N. The map

$$\Psi_\alpha : W_\alpha \to W_\alpha \,, \quad \Psi_\alpha([k,v]) = [\Phi_\alpha(k), v] \,,$$

is well defined, hence a vector bundle isomorphism which preserves the metrics of the fibers. Now by Assumption 1, Ψ_α factors over an isometry of the base manifolds, hence Ψ_α induces a unitary transformation $L^2(W_\alpha) \to L^2(W_\alpha)$.

We denote the covariant derivatives on W_α induced by ω and ω' by ∇^α and ∇'^α, respectively. Now ∇^α and ∇'^α only depend on $\rho_{\alpha*} \circ \omega$ and $\rho_{\alpha*} \circ \omega'$, respectively. Hence by Assumption 2, $\Psi_\alpha^*(\nabla'^\alpha) = \nabla^\alpha$. It follows that Ψ_α intertwines the Laplacians Δ_α and Δ'_α. Hence by Theorem 1.6, (M, g_ω) and $(M, g_{\omega'})$ are isospectral.

1.7. REMARK. The main point in the proof of Theorem C is the decomposition of the Laplacian into vertical and horizontal part, $\Delta = \Delta_{\mathcal{V}} + \Delta_{\mathcal{H}}$, as in (1.2). There is a similar formula in the case where the metric in the vertical direction is rescaled by a positive function ψ on N,

$$g_\psi := \psi^2 g_{\mathcal{V}} + g_{\mathcal{H}} \,,$$

where $g_{\mathcal{V}}$ and $g_{\mathcal{H}}$ denote the restrictions of the original metric on M to the vertical and horizontal distribution. The corresponding Laplacian Δ_ψ is given by

$$\Delta_\psi = \frac{1}{\psi^2} \Delta_{\mathcal{V}} + \Delta_{\mathcal{H}} - kZ \,,$$

where $k = \dim F$ and Z is the horizontal lift of $\operatorname{grad} \psi$.

Note that for the rescaled metric, the fibers are totally umbilic submanifolds with second fundamental form $S(U,V) = -\langle U, V \rangle Z$ and the horizontal displacements are homotheties. Vice versa, suppose the horizontal displacements of a Riemannian submersion with compact fibers are homotheties. Then rescaling vertically so that the fibers are of constant volume we obtain a Riemannian submersion with totally geodesic fibers.

Theorem C extends to this situation: The data are connections ω and ω' as in Theorem C and functions ψ and ψ' on N. Rescaling the metrics g_ω and $g_{\omega'}$ in the vertical directions by ψ and ψ', respectively, we obtain metrics $g_{\psi,\omega}$ and $g_{\psi',\omega'}$ on M. Then $(M, g_{\psi,\omega})$ and $(M, g_{\psi',\omega'})$ are isospectral if Assumptions 1 and 2 of Theorem C hold and, in addition, $\psi' \circ \phi_\rho = \psi$ for all ρ. Compare Theorem 4.3 in [Sch] for a special version (where we subsume the rescaling of the metric g_N on the base there into g_N here).

2. PROOF OF THEOREM D

We use ideas and results from Chapter 2 of [Sch]. Let $N = S^2 = \{q \in \mathbb{R}^3 \mid |q| = 1\}$ and $P = S^2 \times T^2$, where T^2 acts on P by right multiplication on the factor T^2. Let F be a compact Riemannian manifold of dimension ≥ 3 and constant scalar curvature and $\mu : T^2 \times F \to F$ be an effective isometric action. In particular, $M := P \times_\mu F$ is diffeomorphic to $S^2 \times F$.

Consider the following two pairs of symmetric matrices,

$$c_1 = c_1' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad c_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \quad c_2' = \begin{pmatrix} 0 & & \sqrt{2} \\ 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \end{pmatrix} .$$

The remarkable property of these matrices is that for any pair $\zeta = (\zeta_1, \zeta_2)$ of real numbers, the matrices $\zeta_1 c_1 + \zeta_2 c_2$ and $\zeta_1 c_1' + \zeta_2 c_2'$ have the same eigenvalues, that is, there is a matrix $A_\zeta \in SO(3)$ such that

$$A_\zeta(\zeta_1 c_1 + \zeta_2 c_2)A_\zeta^{-1} = \zeta_1 c_1' + \zeta_2 c_2' ,$$

but A_ζ cannot be chosen independently of ζ, see Proposition 2.4 in [Sch]. This latter fact actually follows from our results below.

Define 1–forms $\lambda = (\lambda_1, \lambda_2)$ and $\lambda' = (\lambda_1', \lambda_2')$ on S^2 with values in the Lie algebra $\mathfrak{t}^2 \cong \mathbb{R}^2$ of T^2 by

$$\text{(2.1)} \qquad \lambda_i(q)(X) = \langle c_i(q) \times q, X\rangle \quad \text{and} \quad \lambda_i'(q)(X) = \langle c_i'(q) \times q, X\rangle ,$$

where $\langle . , . \rangle$ denotes the inner product of $\mathbb{R}^3$. Note that the dependence on the foot point $q \in S^2$ is quadratic. Let $\zeta = (\zeta_1, \zeta_2)$ be a pair of real numbers. Then A_ζ as above is an isometry of S^2. Considering ζ as an element of the dual of $\mathfrak{t}^2$, we have

$$\zeta \circ \lambda' \circ A_{\zeta *} = \zeta \circ \lambda$$

by the above property of A_ζ.

Define connection forms ω and ω' on P with values in $\mathfrak{t}^2$ by

$$\omega((X, V)) = \lambda(X) + V \quad \text{and} \quad \omega'((X, V)) = \lambda'(X) + V ,$$

where we write vector fields on P in the form (X, V) with X tangent to N and V tangent to T^2. Then for any element ζ in the dual of $\mathfrak{t}^2$ we have

$$\zeta \circ \omega' \circ \Phi_{\zeta *} = \zeta \circ \omega ,$$

where $\Phi_\zeta = (A_\zeta, \mathrm{id})$. It follows that the connection forms ω and ω' satisfy the assumptions of Theorem C. Hence (M, g_ω) and $(M, g_{\omega'})$ are isospectral.

It remains to show that (M, g_ω) and $(M, g_{\omega'})$ are not isometric. To that end we follow [Sch] and study the loci of maximal scalar curvature. We start with some more general considerations.

Let M and N be Riemannian manifolds and $\pi : M \to N$ be a Riemannian submersion with totally geodesic fibers. Let s_M, s_N and $s_{\mathcal{V}}$ be the scalar curvature of M, N

and fibers of π, respectively. Then in $p \in M$,

$$s_M(p) = s_N(\pi(p)) + s_{\mathcal{V}}(p) - \frac{1}{4}\sum_{i,j} \left\| [X_i, X_j]_{\mathcal{V}}(p) \right\|^2 , \tag{2.2}$$

where $X_1, \dots, X_n$ is a local orthonormal frame about p of the horizontal distribution $\mathcal{H}$ and where the index $\mathcal{V}$ indicates the vertical component, see Corollary 9.37 in [Be].

Now we consider the case where $P = N \times T^2$ and $M = P \times_\mu F$. We also assume that ω is of the form $\omega((X, V)) = \lambda(X) + V$, where λ is a 1–form on N with values in $\mathfrak{t}^2$.

For a left invariant vector field V on T^2, let V^* be the vector field on F defined by

$$V^*(p) = \partial_t(e^{tV}(p))|_{t=0} .$$

Then the horizontal distribution on M induced by ω is given by the following rule:

$$(X, W) \text{ is horizontal} \Longleftrightarrow W(q, p) = \{-\lambda_q(X(q, p))\}^*(p) \text{ for all } (q, p).$$

In short: (X, W) is horizontal if and only if $W = -\lambda(X)^*$.

In order to apply (2.2), we need to compute the vertical part of the Lie bracket of horizontal lifts of vector fields X and Y on N, $[(X, -\lambda(X)^*), (Y, -\lambda(Y)^*)]_{\mathcal{V}}$. Then X, Y and λ only depend on $q \in N$. Since T^2 is abelian and $\lambda(X)$ and $\lambda(Y)$ are left invariant vector fields on T^2, we have $[\lambda(X)^*, \lambda(Y)^*] = 0$. By the definition of the horizontal distribution, the vertical part of $([X, Y], 0)$ is $\lambda([X, Y])^*$. Therefore,

$$\begin{aligned}[(X, -\lambda(X)^*), (Y, -\lambda(Y)^*)]_{\mathcal{V}} &= (0, \{-X(\lambda(Y)) + Y(\lambda(X)) + \lambda([X, Y])\}^*) \\ &= (0, \{-d\lambda(X, Y)\}^*) .\end{aligned}$$

We conclude: If (q, p) is a point in M, if $(X_1, \dots, X_n)$ is a local orthonormal frame about q in N and $((X_1, -\lambda(X_1)^*), \dots, (X_n, -\lambda(X_n)^*))$ is its horizontal lift to M, then

$$\sum_{i,j} \left\| [(X_i, -\lambda(X_i)^*), (X_j, -\lambda(X_j)^*)]_{\mathcal{V}}(q, p) \right\|^2 = \sum_{i,j} \left\| \{(d\lambda(q))(X_i(q), X_j(q))\}^*(p) \right\|^2 .$$

We now return to the case in question, where $N = S^2$. We let F^* be the set of points in F with a twodimensional T^2–orbit. That is $p \in F^*$ if and only if $\{V^*(p) \mid V \in \mathfrak{t}^2\}$ is twodimensional. It is immediate from the slice theorem that $F \setminus F^*$ has codimension at least two in F. Now $N = S^2$, hence up to order, there is only one summand $\left\| \{(d\lambda)_q(X_i(q), X_j(q))\}^*(p) \right\|^2$. For $p \in F^*$ it vanishes precisely in the points (q, p) where $d\lambda(q) = 0$. Now for λ, λ' as in (2.1),

$$\{q \in S^2 \mid d\lambda(q) = 0\} = \{q \in S^2 \mid q_1 = -q_3 \text{ or } q = (\pm 1/\sqrt{2}, 0, \pm 1/\sqrt{2})\}$$

and

$$\{q \in S^2 \mid d\lambda'(q) = 0\} = \{(0, \pm 1, 0)\} ,$$

see Proposition 2.10 in [Sch]. It follows from (2.2) and the fact that s_N and $s_{\mathcal{V}}$ are constant that in the first case, the locus where the scalar curvature attains its maximum has components of codimension one, in the second case the locus has codimension two. Hence (M, g_ω) and $(M, g_{\omega'})$ are not isometric.

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