

# A Geometric-Optics Proof of a Theorem on Boundary Control Given a Convex Function

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## Abstract

We consider a hyperbolic equation of the form

$$u_{tt} - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + \text{lower order terms} = 0$$

in a cylindrical region  $\Omega \times (0, T)$  in space-time. In the important paper “Inverse/Observability Estimates for Second-Order Hyperbolic Equations with Variable Coefficients” by Lasiecka, Triggiani and Yao [5], those authors use Carleman estimates to show that the equation can be controlled from a subset of the boundary  $\partial\Omega \times (0, T)$  if there is a positive function  $v$  on  $\Omega$  which is strictly convex with respect to the metric defined by the coefficients of the equation, if that convex function has non-positive outward normal derivative on the uncontrolled part of the boundary. The time needed for control is a function of the maximum value of  $v$  on  $\Omega$  and a lower bound on its convexity. In the present paper we will show that control in the same time is established by a simpler geometric optics argument—in fact it comes down to a short calculus computation on the value of  $v$  along a bicharacteristic of the equation.

## References

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