COLORING GRAPHS
WITH FORBIDDEN PATHS

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COLORING IS NPC

3-COLORING IS NPC

BEST KNOWN APPROX: (THORUP & KAWARABAYASHI)

AN n-VERTEX 3-COLORABLE GRAPH CAN BE \( \frac{4}{11} \)-COLORED IN POLYNOMIAL TIME
COLORING GRAPHS WITH CERTAIN
INDUCED S.G.'S EXCLUDED

THM (KAMINSKI, LOZIN)

∀k ≥ 3, g ≥ 3 THE k-COLORING
PROBLEM IS NPC FOR GRAPHS OF
GIRTH ≥ g

COR ∀k ≥ 3, k-COLORING IS
NPC FOR H-FREE GRAPHS IF
H HAS A CYCLE
THM (HOLYER)

\textit{K-EDGE-COLORING} is \textit{NPC} for \textit{K}\geq3

COR \textit{K-COLORING} \textit{CLAW-FREE}

\textit{GRAPHS} is \textit{NPC} FOR \textit{K}\geq3

\begin{center}
\includegraphics[width=2cm]{claw.png}
\end{center}

\textit{CLAW}
REMAINS OPEN:

- \( H \) IS THE DISJOINT UNION OF PATHS
- \( G \) IS \( H \)-FREE
- \( k \geq 3 \)

? \( k \)-COLOR \( G \)

FOCUS ON: \( H \) IS A PATH

\( P_t \) \( t \)-VERTEX PATH
<table>
<thead>
<tr>
<th>t ≤ 5</th>
<th>k = 3</th>
<th>k = 4</th>
<th>k = 5</th>
<th>k &gt; 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
</tr>
<tr>
<td>t = 6</td>
<td>P</td>
<td>?</td>
<td>NPC</td>
<td>NPC</td>
</tr>
<tr>
<td>t = 7</td>
<td>?</td>
<td>NPC</td>
<td>NPC</td>
<td>NPC</td>
</tr>
<tr>
<td>t &gt; 8</td>
<td>NPC</td>
<td>NPC</td>
<td>NPC</td>
<td>NPC</td>
</tr>
</tbody>
</table>

From: Three complexity results on coloring \( P_k \)-free graphs (Broersma, Fomin, Golovach, Paulusma)

Improved complexity results on \( k \)-coloring \( P_t \)-free graphs (Huang)
Our Results:

1. A poly-time alg with
   Input: \( p_7 \)-free graph \( G \)
   Output: a 3-coloring of \( G \), or a determination that none exists

3-coloring \( p_7 \)-free graphs is polynomial

2. A poly-time alg with
   Input: \( \{p_6, c_5\} \)-free graph \( G \)
   Output: a 4-coloring of \( G \), or a determination that none exists

4-coloring \( \{p_6, c_5\} \)-free graphs is polynomial
EVERY $\{P_8, C_3, C_4\}$-FREE GRAPH IS 3-COLORABLE

EVERY $\{P_8, C_3, C_5\}$-FREE GRAPH IS 3-COLORABLE

A MINIMAL NON-3-COLORABLE $\{P_8, C_4, C_5\}$-FREE GRAPH HAS AT MOST 12 VERTICES.
CONJECTURE (SEYMOUR)

\[ \forall \mathcal{F} \text{ a minimal non-3-colorable } \mathcal{F}_e \text{-free graph has } \leq f(\mathcal{F}) \text{ vertices} \]

would imply a poly-time algorithm to test if a \( \mathcal{F}_e \)-free graph is 3-colorable
LIST COLORING

G \text{ GRAPHS}

L = \{L(v)\}_{v \in V(G)} \text{ LISTS OF COLORS}

(G, L) \text{ IS COLORABLE IF } \exists \text{ COLORING c OF } V(G), \text{ s.t. } \forall v \in V(G) c(v) \in L(v)

IDEA: REDUCE COLORING G TO \{(G_1, L_1), \ldots, (G_k, L_k)\} \text{ s.t. } G \text{ IS COLORABLE IFF } (G_i, L_i) \text{ IS COLORABLE FOR SOME } i
IMPORTANT TOOL:

**THM (EDWARDS)**

- $G$ GRAPH
- $L = \{ L(v) \}_{v \in V(G)}$ LISTS
- $\forall v \, |L(v)| \leq 2$

**THERE IS A POLY-TIME ALG TO TEST IF** $(G, L)$ **IS COLORABLE**

**PROOF:** REDUCE TO 2-SAT
IMPORTANT TOOL:

THM (BFGP)

• G: GRAPH
• \( L = \{ L(v) \}_{v \in V(G)} \): lists
• \( \forall v \ |L(v)| \leq 2 \)
• \( S_1, \ldots, S_k \subseteq V(G) \).

THERE IS A POLY-TIME ALG TO TEST IF \((G, L)\) IS COLORABLE S.T. EACH OF \(S_1, \ldots, S_k\) IS MONOCHROMATIC

PROOF: REDUCE TO 2-SAT
ON THM 1

IN THE PERFECT WORLD

• FIND A DOMINATING SET $S$ WITH $|S| \leq 100$

• TRY EACH OF $3^{100}$ 3-COLORINGS OF $S$

• USE EDWARD'S THM TO CHECK IF THEY EXTEND TO A 3-COLORING OF $G$
G is reducible if \( \exists G' \) s.t.
- \( |V(G')| \leq |V(G)| \)
- \( G' \) is \( P_7 \)-free
- \( G' \) can be computed in poly-time starting from \( G \)
- Given a 3-coloring of \( G' \), can compute a 3-coloring of \( G' \)
- \( G \) is 3-colorable iff \( G' \) is

Example: If \( \exists u \in V(G) \) s.t. \( N(u) \cap N(v) \), then \( G \) is reducible

\[ G' = G \setminus u \]
TWO PARTS TO THE ALG

1. $G$ \(\Delta\)-FREE
   USE INFO ABOUT ADJACENCY

2. $G$ HAS A $\Delta$ \(xyz\)
   TRY ALL POSSIBLE COLORINGS OF \(\{x, y, z\}\)
   USE INFO ABOUT FORBIDDEN COLORS

WMA $G$ IS NOT REDUCIBLE
Δ-FREE CASE

- If G is bip, output a bipartition
- WMA G contains C₅ or C₇

|L(v)| ≤ 2

Pre-WLDR

Dominated vertices

Need to deal with
Lemma: If $\exists v \in V(G)$ s.t. $G \mid N(v)$ is connected, then $G$ is reducible.
A TRIPED (A, B, C)

\{ x_1, ... , x_m \}

\( x_1, x_2, x_3 \) is \( \overrightarrow{A} \).

\( \forall i \geq 4, \ x_i \in A \) has a nbr in 

\( B \cap \{ x_1, ... , x_{i-3} \} \) & 

\( B \cap \{ x_1, ... , x_{i-3} \} \)

SAME FOR \( x_i \in B, x_i \in C \)

COLOR \( \{ x_1, x_2, x_3 \} \)

\( \rightarrow \) THE COLORING OF (A, B, C) IS FORCED
\( b \)

\( \Delta \ a \ c \)

- \((A,B,C) \) MAX 'L TRIPOD
  - \( a \in A \), \( b \in B \), \( c \in C \)

- IF \( A \) OR \( B \) OR \( C \) NOT STABLE, OUTPUT NO

- \( \forall x \in V(G) \setminus (A \cup B \cup C), \) \( \checkmark \)
  - HAS NBRS IN AT MOST ONE OF \( A,B,C \)
\[|L(v)| = 1 \quad |L(v)| \leq 2\]

"Dominated vertices or connected neighborhoods"
ON THM 2

4-COLOURING \{P_6, C_5\}-FREE GRAPHS

\forall x \in X \quad \text{EITHER}
- \quad N(x) \cap V(C) \text{ contains big }
- \quad x \text{ starts a P}_4 \text{ in } C+X
\text{PRE-COLOR C}

- \textbf{\underline{x is BIG \implies |L(x)| = 1}}
- \textbf{\underline{x is SMALL \implies}}
  \textbf{\underline{x is not mixed on a component of 4+2}}
  \textbf{\underline{\&}}
  \textbf{\underline{|L(x)| = 2}}
IF SOME COMPONENT D OF Y+Z ONLY ATTACHES AT BIG VERTICES OF X \Rightarrow "CLIQUE UMTSET"
Lemma: If $s, t$ are in the same anticomponent of $S$, then $L(s) = L(t)$.
- Grow structures

- No "Connected Star Cutsets"
\exists \text{ PATH FROM } u \text{ TO } \{a, b, c, d\} \text{ AVOIDING } N(v).