Counting independent sets in hypergraphs and its applications

József Balogh
U. of Illinois at U.C.

September 2014
Transference theorems

Theorem (Conlon–Gowers [2009+], Schacht [2009+])

extremal result $\mathcal{R}$

$\mathcal{R} + \implies$ random analogue of $\mathcal{R}$.

supersaturation

Dr D. Conlon  Sir W.T. Gowers  Dr M. Schacht
Szemerédi’s theorem

Theorem (Szemerédi [1975])

For every $k \geq 3$, the largest subset of $\{1, \ldots, n\}$ with no $k$-term AP has $o(n)$ elements.

Endre Szemerédi
Random analogue of Szemerédi’s theorem

**Theorem (Kohayakawa–Łuczak–Rödl [1996])**

For every $\delta > 0$, there exists a $C$ such that if $p(n) \geq Cn^{-1/2}$, then a.a.s.: the $p$-random subset $[n]_p$ satisfies:

Every $A \subseteq [n]_p$ with $|A| \geq \delta|[n]_p|$ contains a 3-term AP.

Y. Kohayakawa  
T. Łuczak  
V. Rödl
Theorem (Conlon–Gowers [2009+], Schacht [2009+])

extremal result $\mathcal{R}$

$\Rightarrow$ random analogue of $\mathcal{R}$

supersaturation
Transference theorems — corollary

**Theorem (Conlon–Gowers [2009+], Schacht [2009+])**

extremal result $\mathcal{R}$

$$+ \quad \implies \quad \text{random analogue of } \mathcal{R}$$

supersaturation

**Corollary (Random analogue of Szemerédi’s theorem)**

For every $k \geq 3$ and $\delta > 0$, if $p(n) \geq C(k, \delta) \cdot n^{-\frac{1}{k-1}}$, then a.a.s. $[n]_p$ satisfies that every $A \subseteq [n]_p$ with $|A| \geq \delta|[n]_p|$ contains a $k$-term AP.
Transference theorems — corollary

Theorem (Turán [1941])

For every \( k \geq 3 \),

\[
\text{ex}(n, K_k) = e(T_{k-1}(n)) = \left(1 - \frac{1}{k-1} + o(1)\right) \binom{n}{2}.
\]
Transference theorems — corollary

**Theorem (Turán [1941])**

For every $k \geq 3$,

$$\text{ex}(n, K_k) = e(T_{k-1}(n)) = \left(1 - \frac{1}{k - 1} + o(1)\right) \binom{n}{2}.$$  

Motivated by: Haxell, Kohayakawa, Łuczak, Rodl.

by many others later (or earlier): Babai, Gerke, Simonovits, Spencer, Steger, Szabó, Sudakov, Vu,...
Transference theorems — corollary

**Theorem (Turán [1941])**

For every $k \geq 3$,

$$\text{ex}(n, K_k) = e(T_{k-1}(n)) = \left( 1 - \frac{1}{k-1} + o(1) \right) \binom{n}{2}.$$  

Motivated by: Haxell, Kohayakawa, Łuczak, Rodl.
by many others later (or earlier): Babai, Gerke, Simonovits, Spencer, Steger, Szabó, Sudakov, Vu, . . .

**Theorem (Conlon–Gowers [2009+], Schacht [2009+])**

For $p = p(n) \gg n^{-\frac{2}{k+1}}$ a.a.s.:

$$\text{ex}(G(n, p), K_k) = \left( 1 - \frac{1}{k-1} + o(1) \right) e(G(n, p)).$$
Transference theorems — corollary

**Theorem (Turán [1941])**

For every $k \geq 3$,

$$\text{ex}(n, K_k) = e(T_{k-1}(n)) = \left(1 - \frac{1}{k-1} + o(1)\right) \binom{n}{2}.$$  

Motivated by: Haxell, Kohayakawa, Łuczak, Rodl.  
by many others later (or earlier): Babai, Gerke, Simonovits, Spencer, Steger, Szabó, Sudakov, Vu, . . .

**Theorem (Conlon–Gowers [2009+], Schacht [2009+])**

For $p = p(n) \gg n^{-\frac{2}{k+1}}$ a.a.s.:

$$\text{ex}(G(n, p), K_k) = \left(1 - \frac{1}{k-1} + o(1)\right) \cdot e(G(n, p)).$$

This is usually referred to as the random analogue of Turán’s theorem.
Certain hypergraphs have only few independent sets.
Counting Independent sets in Hypergraphs

Balogh–Morris–Samotij, Saxton–Thomason [2012+]

Certain hypergraphs have only few independent sets.

Corollary (Random analogue of Szemerédi’s theorem)

For every $k \geq 3$ and $\delta > 0$, if $p(n) \geq C(k, \delta) \cdot n^{-\frac{1}{k-1}}$, then a.a.s. $[n]_p$ satisfies that every $A \subseteq [n]_p$ with $|A| \geq \delta |[n]_p|$ contains a $k$-term AP.

Corollary (Random analogue of Turán’s theorem)

For $p(n) \gg n^{-\frac{1}{k+1}}$ a.a.s.: $\text{ex}(G(n, p), K_k) = (1 - \frac{1}{k-1} + o(1)) \cdot e(G(n, p))$. 
Balogh–Morris–Samotij, Saxton–Thomason [2012+]  

Certain hypergraphs have only few independent sets.

Corollary (Random analogue of Szemerédi’s theorem)

For every $k \geq 3$ and $\delta > 0$, if $p(n) \geq C(k, \delta) \cdot n^{-\frac{1}{k-1}}$, then a.a.s. $[n]_p$ satisfies that every $A \subseteq [n]_p$ with $|A| \geq \delta|[n]_p|$ contains a $k$-term AP.

Corollary (Random analogue of Turán’s theorem)

For $p = p(n) \gg n^{-\frac{2}{k+1}}$ a.a.s.:

$$\text{ex}(G(n, p), K_k) = \left(1 - \frac{1}{k-1} + o(1)\right) \cdot e(G(n, p)).$$
Counting Independent sets in Hypergraphs

Balogh–Morris–Samotij, Saxton–Thomason [2012+]

Certain hypergraphs have only few independent sets.
Counting Independent sets in Hypergraphs

Balogh–Morris–Samotij, Saxton–Thomason [2012+]

Certain hypergraphs have only few independent sets.

Corollary (Counting analogue of Szemerédi’s theorem)

For every $k \geq 3$ and $\delta > 0$, if $m \geq C(k, \delta)n^{1-\frac{1}{k-1}}$, then

$$\#m\text{-subsets of } [n] \text{ with no } k\text{-term AP } \leq \binom{\delta n}{m}.$$
Counting Independent sets in Hypergraphs

Balogh–Morris–Samotij, Saxton–Thomason [2012+]

Certain hypergraphs have only few independent sets.

Corollary (Counting analogue of Szemerédi’s theorem)

For every $k \geq 3$ and $\delta > 0$, if $m \geq C(k, \delta) n^{1 - \frac{1}{k-1}}$, then

$$\#m\text{-subsets of } [n] \text{ with no } k\text{-term AP} \leq \binom{\delta n}{m}.$$

Theorem (Erdős–Kleitman–Rothschild [1976])

There are at most $2^{(1+o(1)) \cdot \text{ex}(n,K_k)} K_k$-free graphs on $n$ vertices.
Counting Independent sets in Hypergraphs

Balogh–Morris–Samotij, Saxton–Thomason [2012+]

Certain hypergraphs have only few independent sets.

Corollary (Counting analogue of Szemerédi’s theorem)

For every $k \geq 3$ and $\delta > 0$, if $m \geq C(k, \delta)n^{1-\frac{1}{k-1}}$, then

$$\# m\text{-subsets of } [n] \text{ with no } k\text{-term AP} \leq \binom{\delta n}{m}.$$ 

Theorem (Erdős–Kleitman–Rothschild [1976])

There are at most $2^{(1+o(1)) \cdot \text{ex}(n,K_k)} K_k$-free graphs on $n$ vertices.

Corollary (Sparse analogue of Erdős–Kleitman–Rothschild’s theorem)

A.a. $K_k$-free graph with $m$ edges can be made $(k - 1)$-partite by removing at most $o(m)$ edges when $m \gg n^{2-\frac{2}{k}}$. 
Sharp sparse analogue of Erdős–Kleitman–Rothschild’s theorem

Almost every $K_k$-free $n$ vertex graph with $m > m(n)$ edges is $(k - 1)$-partite. $m(n)$ is best possible.

Sharp sparse analogue of Erdős–Kleitman–Rothschild’s theorem

Balogh–Morris–Samotij–Warnke [2013+]

For every $r \geq 3$, there exists a $d_r = d_r(n) = \Theta(n)$ such that the following holds for every $\varepsilon > 0$. Define

$$\theta_r = \frac{r - 1}{2r} \cdot \left[ r \cdot \left( \frac{2r + 2}{r + 2} \right)^{\frac{1}{r-1}} \right]^{\frac{2}{r+2}}$$

and

$$m_r = m_r(n) = \theta_r n^{2-\frac{2}{r+2}} (\log n)^{\frac{1}{(r+1)^2}}.$$ 

If $F_{n,m}$ is the uniformly chosen random element of $\mathcal{F}_{n,m}(K_{r+1})$, then

$$\lim_{n \to \infty} \Pr[F_{n,m} \text{ is } r\text{-partite}] = \begin{cases} 
1, & \text{if } m \leq (1 - \varepsilon) d_r, \\
0, & \text{if } (1 + \varepsilon) d_r \leq m \leq (1 - \varepsilon) m_r, \\
1, & \text{if } m \geq (1 + \varepsilon) m_r.
\end{cases}$$
New applications of the “Counting Method”:

**Sperner (1928)**

The size of the largest antichain in the Boolean lattice over 
\([n] = \{1, \ldots, n\}\) is \(\left(\begin{array}{c} n \\ \lfloor n/2 \rfloor \end{array}\right)\).
New applications of the “Counting Method”:

**Sperner (1928)**

The size of the largest antichain in the Boolean lattice over \([n] = \{1, \ldots, n\}\) is \(\binom{n}{\lfloor n/2 \rfloor}\).

**Problem [Kohayakawa–Kreuter–Osthus (2000)]**

With probability \(p\) keep elements of the Boolean lattice over \([n]\).

For what \(p = p(n)\) will have the largest antichain size \((1 + o(1))p \cdot \binom{n}{\lfloor n/2 \rfloor}\)?
New applications of the “Counting Method”:

<table>
<thead>
<tr>
<th>Sperner (1928)</th>
</tr>
</thead>
<tbody>
<tr>
<td>The size of the largest antichain in the Boolean lattice over $[n] = {1, \ldots, n}$ is $\binom{n}{\lfloor n/2 \rfloor}$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Problem [Kohayakawa–Kreuter–Osthus (2000)]</th>
</tr>
</thead>
<tbody>
<tr>
<td>With probability $p$ keep elements of the Boolean lattice over $[n]$. For what $p = p(n)$ will have the largest antichain size $(1 + o(1))p \cdot \binom{n}{\lfloor n/2 \rfloor}$?</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Osthus (2000)</th>
</tr>
</thead>
<tbody>
<tr>
<td>For $p \gg (\log n)/n$: YES. For $p = O(1/n)$: NO.</td>
</tr>
</tbody>
</table>
New applications of the “Counting Method”:

**Sperner (1928)**

The size of the largest antichain in the Boolean lattice over $[n] = \{1, \ldots, n\}$ is $\binom{n}{\lfloor n/2 \rfloor}$.

**Problem [Kohayakawa–Kreuter–Osthus (2000)]**

With probability $p$ keep elements of the Boolean lattice over $[n]$. For what $p = p(n)$ will have the largest antichain size $(1 + o(1))p \cdot \binom{n}{\lfloor n/2 \rfloor}$?


For $p \gg (\log n)/n$: YES.

For $p = O(1/n)$: NO.

**Collares Neto–Morris, Balogh–Mycroft–Treglown [2014+]**

For $p \gg 1/n$: YES.
New applications of the “Counting Method”:


For

\[
\frac{\log n}{n^t} \ll p \ll \frac{1}{n^{t-1}}
\]

the size of the largest antichain is

\[
(t + o(1)) p \cdot \binom{n}{\lfloor n/2 \rfloor}.
\]

**Balogh–Mycroft–Treglown [2014]**

For

\[
\frac{1}{n^t} \ll p \ll \frac{1}{n^{t-1}}
\]

the size of the largest antichain is

\[
(t + o(1)) p \cdot \binom{n}{\lfloor n/2 \rfloor}.
\]
New applications of the “Counting Method”:

Balogh–Mycroft–Treglown [2014]

For

\[ \frac{1}{n^t} \ll p \ll \frac{1}{n^{t-1}} \]

the size of the largest antichain is \((t + o(1)) p \cdot \left(\begin{array}{c} n \\ \lfloor n/2 \rfloor \end{array}\right)\).
Question

How many integers from \( \{1, \ldots, n\} \) can we select without creating a solution of

\[
x + y = z
\]
The Cameron–Erdős problem

Question
How many integers from \( \{1, \ldots, n\} \) can we select without creating a solution of
\[ x + y = z? \]

Observation
- Set of odds is sum-free.
The Cameron–Erdős problem

**Question**

How many integers from \( \{1, \ldots, n\} \) can we select without creating a solution of

\[
x + y = z
\]

**Observation**

- Set of odds is sum-free.
- \( \{n/2 + 1, n/2 + 2, \ldots, n\} \) is sum-free.
- \( \{n/2, n/2 + 1, \ldots, n - 1\} \) is sum-free.
The Cameron–Erdős problem

Question
How many integers from \(\{1, \ldots, n\}\) can we select without creating a solution of
\[x + y = z?\]

Observation
- Set of odds is sum-free.
- \(\{n/2 + 1, n/2 + 2, \ldots, n\}\) is sum-free.
- \(\{n/2, n/2 + 1, \ldots, n - 1\}\) is sum-free.

Cameron – Erdős Conjecture (1990)
The number of sum-free subsets of \([n]\) is \(O(2^{n/2})\).
The Cameron–Erdős problem

Question
How many integers from \( \{1, \ldots, n\} \) can we select without creating a solution of

\[
x + y = z?
\]

Observation
- Set of odds is sum-free.
- \( \{n/2 + 1, n/2 + 2, \ldots, n\} \) is sum-free.
- \( \{n/2, n/2 + 1, \ldots, n − 1\} \) is sum-free.

Cameron – Erdős Conjecture (1990)
The number of sum-free subsets of \([n]\) is \(O(2^{n/2})\).

Remark
The number of sum-free subsets of \([n]\) is more than \(2 \times 2^{n/2}\).
Any subset of \( \{n/2, n/2 + 1, \ldots, n − 1\} \) is sum-free, etc...
The Cameron–Erdős problem

Cameron – Erdős Conjecture (1990)
The number of sum-free subsets of \([n]\) is \(O(2^{n/2})\).

There are constants \(c_e\) and \(c_o\) s.t. the number of sum-free subsets of \([n]\) is

\[(1 + o(1))c_e2^{n/2}, \quad (1 + o(1))c_o2^{n/2}\]

depending on the parity of \(n\).
The Cameron–Erdős problem

Cameron – Erdős Conjecture (1999)

There is $c > 0$ that the number of maximal sum-free subsets of $[n]$ is

$$O(2^{n/2-cn}).$$

There are at least $2^{n/4}$ maximal sum-free subsets of $[n]$. 
Cameron–Erdős Conjecture (1999)

There is $c > 0$ that the number of maximal sum-free subsets of $[n]$ is

$$O(2^{n/2 - cn}).$$

There are at least $2^{n/4}$ maximal sum-free subsets of $[n]$.

Łuczak and Schoen (2001)

The number of maximal sum-free subsets of $[n]$ is at most $O(2^{n/2 - 2^{-28}n})$. 

The Cameron–Erdős problem

Cameron – Erdős Conjecture (1999)
There is $c > 0$ that the number of **maximal** sum-free subsets of $[n]$ is

$$O(2^{n/2 - cn}).$$

There are at least $2^{n/4}$ **maximal** sum-free subsets of $[n]$.

Łuczak and Schoen (2001)
The number of **maximal** sum-free subsets of $[n]$ is at most $O(2^{n/2 - 2^{-28}n})$.

Wolfowitz (2009)
The number of **maximal** sum-free subsets of $[n]$ is at most $2^{3n/8 - o(n)}$. 
### The Cameron–Erdős problem

#### Cameron – Erdős Conjecture (1999)

There is $c > 0$ such that the number of **maximal** sum-free subsets of $[n]$ is $O(2^{n/2 - cn})$.

There are at least $2^{n/4}$ **maximal** sum-free subsets of $[n]$.

#### Łuczak and Schoen (2001)

The number of **maximal** sum-free subsets of $[n]$ is at most $O(2^{n/2 - 2^{-28} n})$.

#### Wolfowitz (2009)

The number of **maximal** sum-free subsets of $[n]$ is at most $2^{3n/8 - o(n)}$.


The number of **maximal** sum-free subsets of $[n]$ is $2^{n/4 + o(n)}$. 
The Cameron–Erdős problem

Cameron – Erdős Conjecture (1999)
There is \( c > 0 \) that the number of maximal sum-free subsets of \([n]\) is

\[
O\left(2^{n/2 - cn}\right).
\]

There are at least \( 2^{n/4} \) maximal sum-free subsets of \([n]\).

Łuczak and Schoen (2001)
The number of maximal sum-free subsets of \([n]\) is at most \( O\left(2^{n/2 - 2^{-28} n}\right)\).

Wolfowitz (2009)
The number of maximal sum-free subsets of \([n]\) is at most \(2^{3n/8 - o(n)}\).

The number of maximal sum-free subsets of \([n]\) is \( O\left(2^{n/4}\right)\).
New applications of the “Counting Method”:

**Definition**
- **Permutation** $\pi = \pi(n)$ is a bijective map from $[n]$ to $[n]$. 
New applications of the “Counting Method”:

**Definition**

- **Permutation** $\pi = \pi(n)$ is a bijective map from $[n]$ to $[n]$.
- Permutations $\rho, \pi$ are **intersecting** if there is an $i$ that $\rho(i) = \pi(i)$.

---


(i) The number of intersecting families of permutations is $2(1+o(1))(n−1)!$.

(ii) Almost every intersecting permutation family is trivially intersecting.
New applications of the “Counting Method”:

**Definition**

- **Permutation** \( \pi = \pi(n) \) is a bijective map from \([n]\) to \([n]\).
- Permutations \( \rho, \pi \) are **intersecting** if there is an \( i \) that \( \rho(i) = \pi(i) \).
- \( \Pi \) is an **intersecting family** of permutations if for every \( \rho, \pi \in \Pi \), \( \rho, \pi \) are intersecting.


(i) The number of intersecting families of permutations is \( 2^{(1+o(1))(n-1)!} \).

(ii) Almost every intersecting permutation family is trivially intersecting.
New applications of the “Counting Method”:

### Definition

- **Permutation** $\pi = \pi(n)$ is a bijective map from $[n]$ to $[n]$.
- Permutations $\rho, \pi$ are **intersecting** if there is an $i$ that $\rho(i) = \pi(i)$.
- $\Pi$ is an **intersecting family** of permutations if for every $\rho, \pi \in \Pi$, $\rho, \pi$ are intersecting.
- $\Pi(i, j) := \{\pi : \pi(i) = j\}$ is a **trivially intersecting family**; of size $(n - 1)!$. 


(i) The number of intersecting families of permutations is $2(1+o(1))(n-1)!$.

(ii) Almost every intersecting permutation family is trivially intersecting.
New applications of the “Counting Method”:

### Definition

- **Permutation** $\pi = \pi(n)$ is a bijective map from $[n]$ to $[n]$.
- Permutations $\rho, \pi$ are **intersecting** if there is an $i$ that $\rho(i) = \pi(i)$.
- $\Pi$ is an **intersecting family** of permutations if for every $\rho, \pi \in \Pi$, $\rho, \pi$ are intersecting.
- $\Pi(i, j) := \{\pi : \pi(i) = j\}$ is a **trivially intersecting family**; of size $(n - 1)!$.
- The number of intersecting families is at least $(1 - o(1)) \cdot n^2 \cdot 2^{(n-1)!}$. 

---


1. The number of intersecting families of permutations is $2(1 + o(1))(n - 1)!$.
2. Almost every intersecting permutation family is trivially intersecting.
New applications of the “Counting Method”:

**Definition**

- **Permutation** $\pi = \pi(n)$ is a bijective map from $[n]$ to $[n]$.
- Permutations $\rho, \pi$ are **intersecting** if there is an $i$ that $\rho(i) = \pi(i)$.
- $\Pi$ is an **intersecting family** of permutations if for every $\rho, \pi \in \Pi$, $\rho, \pi$ are intersecting.
- $\Pi(i,j) := \{\pi : \pi(i) = j\}$ is a **trivially intersecting family**; of size $(n - 1)!$.
- The number of intersecting families is at least $(1 - o(1)) \cdot n^2 \cdot 2^{(n-1)!}$.

**Balogh–Das–Delcourt–Liu–Sharifzadeh [2014++]**

(i) The number of intersecting families of permutations is

$$2^{(1+o(1))(n-1)!}.$$
New applications of the “Counting Method”:

**Definition**

- **Permutation** $\pi = \pi(n)$ is a bijective map from $[n]$ to $[n]$.
- Permutations $\rho, \pi$ are **intersecting** if there is an $i$ that $\rho(i) = \pi(i)$.
- $\Pi$ is an **intersecting family** of permutations if for every $\rho, \pi \in \Pi$, $\rho, \pi$ are intersecting.
- $\Pi(i, j) := \{\pi : \pi(i) = j\}$ is a **trivially intersecting family**; of size $(n-1)!$.
- The number of intersecting families is at least $(1 - o(1)) \cdot n^2 \cdot 2^{(n-1)!}$.

**Balogh–Das–Delcourt–Liu–Sharifzadeh [2014++]**

(i) The number of intersecting families of permutations is

$$2^{(1+o(1))(n-1)!}.$$ 

(ii) Almost every intersecting permutation family is trivially intersecting.
The number of intersecting families of permutations is
\[ 2^{(1+o(1))(n-1)!}. \]

Proof follows Alon–Balogh–Morris–Samotij [2014]:

- Form graph: \( V := \text{permutations}, E := \text{non-intersecting pairs}. \)
- Apply Alon–Chung Expander-Mixing Lemma:
  \[ |G[S]| \geq D^2 N |S|^2 + \lambda^2 N |S| (N-|S|). \]
- Ellis:
  \[ \lambda = (1-e+o(1))(n-1)!, \]
  \[ N = n!, D = (1+o(1)) N, |S| = (1+o(1))(n-1)! \]
  \( G[S] \) spans many edges \( \rightarrow \) \( G \) does not have 'many' independent sets.
Permutations:


The number of intersecting families of permutations is

\[ 2^{(1+o(1))(n-1)!}. \]

- Proof follows Alon–Balogh–Morris–Samotij [2014]:
- Form graph: \( V := \) permutations, \( E := \) non-intersecting pairs.

The number of intersecting families of permutations is

\[ 2^{(1+o(1))(n-1)!}. \]

- Proof follows Alon–Balogh–Morris–Samotij [2014]:
- Form graph: \( V := \text{permutations}, \ E := \text{non-intersecting pairs}. \)
- Apply Alon–Chung Expander-Mixing Lemma:
The number of intersecting families of permutations is

\[ 2^{(1+o(1))(n-1)!} \, . \]

- Proof follows Alon–Balogh–Morris–Samotij [2014]:
- Form graph: \( V \) := permutations, \( E \) := non-intersecting pairs.
- Apply Alon–Chung Expander-Mixing Lemma:
  Let \( G \) be a \( D \)-regular graph on \( N \) vertices, and let \( \lambda \) be its smallest eigenvalue. Then for all \( S \subseteq V(G) \),

\[
e(G[S]) \geq \frac{D}{2N} |S|^2 + \frac{\lambda}{2N} |S| (N - |S|) .
\]
The number of intersecting families of permutations is

\[ 2^{(1 + o(1))(n-1)!}. \]

Proof follows Alon–Balogh–Morris–Samotij [2014]:

Form graph: \( V := \) permutations, \( E := \) non-intersecting pairs.

Apply Alon–Chung Expander-Mixing Lemma:
Let \( G \) be a \( D \)-regular graph on \( N \) vertices, and let \( \lambda \) be its smallest eigenvalue. Then for all \( S \subseteq V(G) \),

\[
e(G[S]) \geq \frac{D}{2N} |S|^2 + \frac{\lambda}{2N} |S| (N - |S|).\]

Ellis: \( \lambda = \left(-\frac{1}{e} + o(1)\right)(n-1)!, \)
\( N = n!, \) \( D = \left(\frac{1}{e} + o(1)\right)N, \) \( |S| = (1 + o(1))(n-1)! \)
The number of intersecting families of permutations is

\[ 2^{(1+o(1))(n-1)!}. \]

Proof follows Alon–Balogh–Morris–Samotij [2014]:
Form graph: \( V := \text{permutations}, \ E := \text{non-intersecting pairs}. \)
Apply Alon–Chung Expander-Mixing Lemma:
Let \( G \) be a \( D \)-regular graph on \( N \) vertices, and let \( \lambda \) be its smallest eigenvalue. Then for all \( S \subseteq V(G) \),

\[ e(G[S]) \geq \frac{D}{2N} |S|^2 + \frac{\lambda}{2N} |S| (N - |S|). \]

Ellis: \( \lambda = \left(-\frac{1}{e} + o(1)\right)(n-1)!, \)
\( N = n!, \) \( D = \left(\frac{1}{e} + o(1)\right)N, \) \( |S| = (1 + o(1))(n-1)! \)
\( G[S] \) spans many edges \( \rightarrow \) \( G \) does not have ‘many’ independent sets.
Almost every intersecting permutation family is trivially intersecting.
Almost every intersecting permutation family is trivially intersecting.
Permutations:


Almost every intersecting permutation family is trivially intersecting.

- Count maximal intersecting families. Let \( \Pi \) be such family.
Almost every intersecting permutation family is trivially intersecting.

- **Count maximal** intersecting families. Let \( \Pi \) be such family.
- \( \mathcal{I}(\Gamma) := \{ \pi \in S_n : \pi \cap \rho \neq \emptyset, \forall \rho \in \Gamma \} \).
Almost every intersecting permutation family is trivially intersecting.

- Count maximal intersecting families. Let $\Pi$ be such family.

- $\mathcal{I}(\Gamma) := \{ \pi \in S_n : \pi \cap \rho \neq \emptyset, \ \forall \rho \in \Gamma \}$.

- $\Gamma \subset \Pi$ is a generating set of $\Pi$ if $\mathcal{I}(\Gamma) = \Pi$. 

Almost every intersecting permutation family is trivially intersecting.

- Count **maximal** intersecting families. Let $\Pi$ be such family.
- $\mathcal{I}(\Gamma) := \{\pi \in S_n : \pi \cap \rho \neq \emptyset, \forall \rho \in \Gamma\}$.
- $\Gamma \subset \Pi$ is a **generating set** of $\Pi$ if $\mathcal{I}(\Gamma) = \Pi$.
- Every $\Pi$ has DIFFERENT **minimal generating sets**.
Almost every intersecting permutation family is trivially intersecting.

- Count maximal intersecting families. Let $\Pi$ be such family.
  \[ I(\Gamma) := \{ \pi \in S_n : \pi \cap \rho \neq \emptyset, \forall \rho \in \Gamma \}. \]
- $\Gamma \subset \Pi$ is a generating set of $\Pi$ if $I(\Gamma) = \Pi$.
- Every $\Pi$ has DIFFERENT minimal generating sets.
- Count minimal generating sets!
Permutations:

Almost every intersecting permutation family is trivially intersecting.

- Count **maximal** intersecting families. Let \( \Pi \) be such family.
- \( \mathcal{I}(\Gamma) := \{ \pi \in S_n : \pi \cap \rho \neq \emptyset, \forall \rho \in \Gamma \} \).
- \( \Gamma \subset \Pi \) is a **generating set** of \( \Pi \) if \( \mathcal{I}(\Gamma) = \Pi \).
- Every \( \Pi \) has DIFFERENT **minimal generating sets**.
- Count minimal generating sets!
- \( \forall \rho_i \in \Gamma \) there is a \( \pi_i \not\in \Pi \) that \( (\rho_i, \pi_i) \) is not an intersecting pair, but \( \forall \rho_j \in \Gamma \) with \( i \neq j \), \( (\rho_j, \pi_i) \) is an intersecting pair.
Almost every intersecting permutation family is trivially intersecting.

- Count maximal intersecting families. Let $\Pi$ be such family.
  \[
  \mathcal{I}(\Gamma) := \{\pi \in S_n : \pi \cap \rho \neq \emptyset, \forall \rho \in \Gamma\}.
  \]
- $\Gamma \subseteq \Pi$ is a **generating set** of $\Pi$ if $\mathcal{I}(\Gamma) = \Pi$.
- Every $\Pi$ has **DIFFERENT minimal generating sets**.
- Count minimal generating sets!
- $\forall \rho_i \in \Gamma$ there is a $\pi_i \not\in \Pi$ that $(\rho_i, \pi_i)$ is not an intersecting pair, but $\forall \rho_j \in \Gamma$ with $i \neq j$, $(\rho_j, \pi_i)$ is an intersecting pair.
- $\rho \to \{(i, \rho(i) : i \in [n]\}$ maps an $n$-uniform hypergraph.
Almost every intersecting permutation family is trivially intersecting.

- Count **maximal** intersecting families. Let \( \Pi \) be such family.
- \( \mathcal{I}(\Gamma) := \{ \pi \in S_n : \pi \cap \rho \neq \emptyset, \ \forall \rho \in \Gamma \} \).
- \( \Gamma \subset \Pi \) is a **generating set** of \( \Pi \) if \( \mathcal{I}(\Gamma) = \Pi \).
- Every \( \Pi \) has **DIFFERENT minimal generating sets**.
- Count minimal generating sets!
- \( \forall \rho_i \in \Gamma \) there is a \( \pi_i \notin \Pi \) that \( (\rho_i, \pi_i) \) is not an intersecting pair, but \( \forall \rho_j \in \Gamma \) with \( i \neq j \), \( (\rho_j, \pi_i) \) is an intersecting pair.
- \( \rho \rightarrow \{ (i, \rho(i) : i \in [n] \} \) maps an \( n \)-uniform hypergraph.
- **Bollobás** set-pair inequality: \( |\Gamma| \leq \binom{2n}{n} \).
Permutations:


Almost every intersecting permutation family is trivially intersecting.

- Count **maximal** intersecting families. Let $\Pi$ be such family.
- $\mathcal{I}(\Gamma) := \{ \pi \in S_n : \pi \cap \rho \neq \emptyset, \forall \rho \in \Gamma \}$.
- $\Gamma \subset \Pi$ is a **generating set** of $\Pi$ if $\mathcal{I}(\Gamma) = \Pi$.
- Every $\Pi$ has **DIFFERENT minimal generating sets**.
- Count minimal generating sets!
- $\forall \rho_i \in \Gamma$ there is a $\pi_i \notin \Pi$ that $(\rho_i, \pi_i)$ is not an intersecting pair, but $\forall \rho_j \in \Gamma$ with $i \neq j$, $(\rho_j, \pi_i)$ is an intersecting pair.
- $\rho \rightarrow \{(i, \rho(i)) : i \in [n]\}$ maps an $n$-uniform hypergraph.
- **Bollobás** set-pair inequality: $|\Gamma| \leq \binom{2n}{n}$.
- **Ellis (2011)**: Largest non-trivial intersecting permutation family has size at most $(1 - \frac{1}{e} + o(1))(n - 1)!$. 
Almost every intersecting permutation family is trivially intersecting.

- Count maximal intersecting families. Let \( \Pi \) be such family.
- \( \mathcal{I}(\Gamma) := \{ \pi \in S_n : \pi \cap \rho \neq \emptyset, \ \forall \rho \in \Gamma \} \).
- \( \Gamma \subset \Pi \) is a \textbf{generating set} of \( \Pi \) if \( \mathcal{I}(\Gamma) = \Pi \).
- Every \( \Pi \) has \textbf{DIFFERENT minimal generating sets}.
- Count minimal generating sets!
- \( \forall \rho_i \in \Gamma \) there is a \( \pi_i \notin \Pi \) that \( (\rho_i, \pi_i) \) is not an intersecting pair, but \( \forall \rho_j \in \Gamma \) with \( i \neq j \), \( (\rho_j, \pi_i) \) is an intersecting pair.
- \( \rho \rightarrow \{(i, \rho(i)) : i \in [n]\} \) maps an \( n \)-uniform hypergraph.
- \textbf{Bollobás} set-pair inequality: \( |\Gamma| \leq \binom{2n}{n} \).
- \textbf{Ellis (2011)}: Largest non-trivial intersecting permutation family has size at most \( (1 - \frac{1}{e} + o(1))(n - 1)! \).
- \( \left( \binom{n!}{\binom{2n}{n}} \right) \)
Almost every intersecting permutation family is trivially intersecting.

- Count **maximal** intersecting families. Let \( \Pi \) be such family.
- \( \mathcal{I}(\Gamma) := \{ \pi \in S_n : \pi \cap \rho \neq \emptyset, \forall \rho \in \Gamma \} \).
- \( \Gamma \subset \Pi \) is a **generating set** of \( \Pi \) if \( \mathcal{I}(\Gamma) = \Pi \).
- Every \( \Pi \) has DIFFERENT **minimal generating sets**.
- Count minimal generating sets!
- \( \forall \rho_i \in \Gamma \) there is a \( \pi_i \notin \Pi \) that \( (\rho_i, \pi_i) \) is not an intersecting pair, but \( \forall \rho_j \in \Gamma \) with \( i \neq j \), \( (\rho_j, \pi_i) \) is an intersecting pair.
- \( \rho \to \{(i, \rho(i) : i \in [n]\} \) maps an \( n \)-uniform hypergraph.
- **Bollobás** set-pair inequality: \( |\Gamma| \leq \binom{2n}{n} \).
- **Ellis (2011):** Largest non-trivial intersecting permutation family has size at most \( (1 - \frac{1}{e} + o(1))(n - 1)! \).
- \( \left( \frac{n!}{\binom{2n}{n}} \right) \cdot 2(1 - 1/e + o(1))(n-1)! \ll 2^n \log n \cdot 2^{2n} \cdot 2(1 - 1/e + o(1))(n-1)! \ll 2^{(n-1)!} \).
New applications of the “Counting Method”:

**Theorem (Erdős–Kleitman–Rothschild [1976])**

Almost all triangle-free graphs are bipartite.

Remark

Most bipartite graphs are not maximal; there are only "few" complete bipartite graphs.

**Question (Erdős [1996])**

What is the number of maximal triangle-free graphs on \( n \) vertices?

\[
2^{n^2/8} \cdot 2^{n^2/4 - c n^2/2}.
\]
New applications of the “Counting Method”:

**Theorem (Erdős–Kleitman–Rothschild [1976])**
Almost all triangle-free graphs are bipartite.

**Remark**
Most bipartite graphs are not maximal; there are only “few” complete bipartite graphs.
New applications of the “Counting Method”:

**Theorem (Erdős–Kleitman–Rothschild [1976])**
Almost all triangle-free graphs are bipartite.

**Remark**
Most bipartite graphs are not maximal; there are only “few” complete bipartite graphs.

**Question (Erdős [1996])**
What is the number of **maximal** triangle-free graphs on $n$ vertices?
New applications of the “Counting Method”:

Theorem (Erdős–Kleitman–Rothschild [1976])
Almost all triangle-free graphs are bipartite.

Remark
Most bipartite graphs are not maximal; there are only “few” complete bipartite graphs.

Question (Erdős [1996])
What is the number of maximal triangle-free graphs on $n$ vertices?

$2^{n^{3/2+o(1)}}$
New applications of the “Counting Method”:

**Theorem (Erdős–Kleitman–Rothschild [1976])**
Almost all triangle-free graphs are bipartite.

**Remark**
Most bipartite graphs are not maximal; there are only “few” complete bipartite graphs.

**Question (Erdős [1996])**
What is the number of maximal triangle-free graphs on \( n \) vertices?

\[
2^{n^{3/2+o(1)}} \quad 2^{o(n^2)}
\]
New applications of the “Counting Method”:

**Theorem (Erdős–Kleitman–Rothschild [1976])**
Almost all triangle-free graphs are bipartite.

**Remark**
Most bipartite graphs are not maximal; there are only “few” complete bipartite graphs.

**Question (Erdős [1996])**
What is the number of **maximal** triangle-free graphs on $n$ vertices?

\[
2^{n^3/2+o(1)} \quad 2^{o(n^2)} \quad 2^{n^2/8}
\]
New applications of the “Counting Method”:

**Theorem (Erdős–Kleitman–Rothschild [1976])**

Almost all triangle-free graphs are bipartite.

**Remark**

Most bipartite graphs are not maximal; there are only “few” complete bipartite graphs.

**Question (Erdős [1996])**

What is the number of maximal triangle-free graphs on $n$ vertices?

\[ 2^{n^3/2+o(1)} \quad 2^{o(n^2)} \quad 2^{n^2/8} \quad 2^{(1/4−c)n^2} \]
New applications of the “Counting Method”:

**Theorem (Erdős–Kleitman–Rothschild [1976])**

Almost all triangle-free graphs are bipartite.

**Remark**

Most bipartite graphs are not maximal; there are only “few” complete bipartite graphs.

**Question (Erdős [1996])**

What is the number of maximal triangle-free graphs on $n$ vertices?

$2^{n^3/2+o(1)}$, $2^{o(n^2)}$, $2^{n^2/8}$, $2^{(1/4-c)n^2}$, $2^{n^2/4}$. 
New applications of the “Counting Method”:

Folklore

There are at least $2^{n^2/8}$ maximal triangle-free graphs on $n$ vertices.
New applications of the “Counting Method”:

Folklore
There are at least $2^{n^2/8}$ maximal triangle-free graphs on $n$ vertices.

- Let $X := \{u_1v_1, \ldots, u_{n/4}v_{n/4}\}$ be a matching;
New applications of the “Counting Method”:

Folklore

There are at least $2^{n^2/8}$ maximal triangle-free graphs on $n$ vertices.

- Let $X := \{u_1v_1, \ldots, u_{n/4}v_{n/4}\}$ be a matching;
- $Y$ be an independent set of size $n/2$. 
New applications of the “Counting Method”:

Folklore

There are at least $2^{n^2/8}$ maximal triangle-free graphs on $n$ vertices.

- Let $X := \{u_1v_1, \ldots, u_{n/4}v_{n/4}\}$ be a matching;
- $Y$ be an independent set of size $n/2$.
- For every $i$: partition $Y := A_i \cup B_i$. 
New applications of the “Counting Method”:

Folklore

There are at least $2^{n^2/8}$ maximal triangle-free graphs on $n$ vertices.

- Let $X := \{u_1v_1, \ldots, u_{n/4}v_{n/4}\}$ be a matching;
- $Y$ be an independent set of size $n/2$.
- For every $i$: partition $Y := A_i \cup B_i$.
- Add all edges between $u_i$ and $A_i$; add all edges between $v_i$ and $B_i$. 

Balogh–Petříčková [2014+]

There are at most $2^{n^2/8 + o(n^2)}$ maximal triangle-free graphs on $n$ vertices.
New applications of the “Counting Method”:

**Folklore**

There are at least $2^{n^2/8}$ maximal triangle-free graphs on $n$ vertices.

- Let $X := \{u_1 v_1, \ldots, u_{n/4} v_{n/4}\}$ be a matching;
- $Y$ be an independent set of size $n/2$.
- For every $i$: partition $Y := A_i \cup B_i$.
- Add all edges between $u_i$ and $A_i$; add all edges between $v_i$ and $B_i$.
- Most of these graphs will be maximal triangle-free.
New applications of the “Counting Method”:

Folklore

There are at least $2^{n^2/8}$ maximal triangle-free graphs on $n$ vertices.

- Let $X := \{u_1v_1, \ldots, u_{n/4}v_{n/4}\}$ be a matching;
- $Y$ be an independent set of size $n/2$.
- For every $i$: partition $Y := A_i \cup B_i$.
- Add all edges between $u_i$ and $A_i$; add all edges between $v_i$ and $B_i$.
- Most of these graphs will be maximal triangle-free.
- Number of graphs: $(2^{n/2})^{n/4} = 2^{n^2/8}$.
New applications of the “Counting Method”:

**Folklore**

There are at least $2^{n^2/8}$ maximal triangle-free graphs on $n$ vertices.

- Let $X := \{u_1v_1, \ldots, u_{n/4}v_{n/4}\}$ be a matching;
- $Y$ be an independent set of size $n/2$.
- For every $i$: partition $Y := A_i \cup B_i$.
- Add all edges between $u_i$ and $A_i$; add all edges between $v_i$ and $B_i$.
- Most of these graphs will be maximal triangle-free.
- Number of graphs: $(2^{n/2})^{n/4} = 2^{n^2/8}$.

**Balogh–Petříčková [2014+]**

There are at most $2^{n^2/8 + o(n^2)}$ maximal triangle-free graphs on $n$ vertices.
New applications of the “Counting Method”:


Almost every maximal triangle-free graph has the above structure.
The number of triangle-free graphs:
Regularity Lemma approach

Theorem (Erdős–Kleitman–Rothschild [1976])

The number of triangle-free graphs is $2^{n^2/4+o(n^2)}$.

- Apply Szemerédi Regularity Lemma for a $G_n$ triangle-free graph.
The number of triangle-free graphs: 
Regularity Lemma approach

**Theorem (Erdős–Kleitman–Rothschild [1976])**

The number of triangle-free graphs is $2^{n^2/4+o(n^2)}$.

- Apply Szemerédi Regularity Lemma for a $G_n$ triangle-free graph.
- Obtain cluster graph $R_t$. 
The number of triangle-free graphs: 
Regularity Lemma approach

**Theorem (Erdős–Kleitman–Rothschild [1976])**

The number of triangle-free graphs is $2^{n^2/4+o(n^2)}$.

- Apply Szemerédi Regularity Lemma for a $G_n$ triangle-free graph.
- Obtain cluster graph $R_t$.
- Clean $G_n$: remove edges inside clusters, between sparse pairs, and irregular pairs.
The number of triangle-free graphs:
Regularity Lemma approach

**Theorem (Erdős–Kleitman–Rothschild [1976])**

The number of triangle-free graphs is $2^{n^2/4+o(n^2)}$.

- Apply Szemerédi Regularity Lemma for a $G_n$ triangle-free graph.
- Obtain cluster graph $R_t$.
- Clean $G_n$: remove edges inside clusters, between sparse pairs, and irregular pairs.
- $C_n :=$ blow up $R_t$ to $n$ vertices.
The number of triangle-free graphs:  
Regularity Lemma approach

**Theorem (Erdős–Kleitman–Rothschild [1976])**

The number of triangle-free graphs is $2^{n^2/4+o(n^2)}$.

- Apply Szemerédi Regularity Lemma for a $G_n$ triangle-free graph.
- Obtain cluster graph $R_t$.
- Clean $G_n$: remove edges inside clusters, between sparse pairs, and irregular pairs.
- $C_n :=$ blow up $R_t$ to $n$ vertices.
- $C_n$ contains all but $o(n^2)$ edges of $G_n$. [Approximate Container]
The number of triangle-free graphs: 
Regularization Lemma approach

Theorem (Erdős–Kleitman–Rothschild [1976])

The number of triangle-free graphs is $2^{n^2/4+o(n^2)}$.

- Apply Szemerédi Regularity Lemma for a $G_n$ triangle-free graph.
- Obtain cluster graph $R_t$.
- Clean $G_n$: remove edges inside clusters, between sparse pairs, and irregular pairs.
- $C_n :=$ blow up $R_t$ to $n$ vertices.
- $C_n$ contains all but $o(n^2)$ edges of $G_n$. [Approximate Container]
- $C_n$ is triangle-free, hence $e(C_n) \leq n^2/4$. 
The number of triangle-free graphs: 
Regularity Lemma approach

**Theorem (Erdős–Kleitman–Rothschild [1976])**

The number of triangle-free graphs is $2^{\frac{n^2}{4} + o(n^2)}$.

- Apply Szemerédi Regularity Lemma for a $G_n$ triangle-free graph.
- Obtain cluster graph $R_t$.
- Clean $G_n$: remove edges inside clusters, between sparse pairs, and irregular pairs.
- $C_n :=$ blow up $R_t$ to $n$ vertices.
- $C_n$ contains all but $o(n^2)$ edges of $G_n$. [Approximate Container]
- $C_n$ is triangle-free, hence $e(C_n) \leq n^2/4$.
- Number of choices for $C_n$ is $O(1) \cdot n^n$. 

The number of triangle-free graphs: 
Regularity Lemma approach

**Theorem (Erdős–Kleitman–Rothschild [1976])**

The number of triangle-free graphs is $2^{n^2/4+o(n^2)}$.

- Apply Szemerédi Regularity Lemma for a $G_n$ triangle-free graph.
- Obtain cluster graph $R_t$.
- Clean $G_n$: remove edges inside clusters, between sparse pairs, and irregular pairs.
- $C_n :=$ blow up $R_t$ to $n$ vertices.
- $C_n$ contains all but $o(n^2)$ edges of $G_n$. [Approximate Container]
- $C_n$ is triangle-free, hence $e(C_n) \leq n^2/4$.
- Number of choices for $C_n$ is $O(1) \cdot n^n$.
- Number of choices for $G_n$ is

$$O(1) \cdot n^n.$$
The number of triangle-free graphs: 
Regularity Lemma approach

**Theorem (Erdős–Kleitman–Rothschild [1976])**

The number of triangle-free graphs is $2^{n^2/4 + o(n^2)}$.

- Apply Szemerédi Regularity Lemma for a $G_n$ triangle-free graph.
- Obtain cluster graph $R_t$.
- Clean $G_n$: remove edges inside clusters, between sparse pairs, and irregular pairs.
- $C_n :=$ blow up $R_t$ to $n$ vertices.
- $C_n$ contains all but $o(n^2)$ edges of $G_n$. [Approximate Container]
- $C_n$ is triangle-free, hence $e(C_n) \leq n^2/4$.
- Number of choices for $C_n$ is $O(1) \cdot n^n$.
- Number of choices for $G_n$ is

$$O(1) \cdot n^n \cdot 2^{n^2/4}.$$
The number of triangle-free graphs: Regularity Lemma approach

**Theorem (Erdős–Kleitman–Rothschild [1976])**

The number of triangle-free graphs is $2^{n^2/4+o(n^2)}$.

- Apply Szemerédi Regularity Lemma for a $G_n$ triangle-free graph.
- Obtain cluster graph $R_t$.
- Clean $G_n$: remove edges inside clusters, between sparse pairs, and irregular pairs.
- $C_n :=$ blow up $R_t$ to $n$ vertices.
- $C_n$ contains all but $o(n^2)$ edges of $G_n$. [Approximate Container]
- $C_n$ is triangle-free, hence $e(C_n) \leq n^2/4$.
- Number of choices for $C_n$ is $O(1) \cdot n^n$.
- Number of choices for $G_n$ is

$$O(1) \cdot n^n \cdot 2^{n^2/4} \cdot \left( \frac{n^2}{o(n^2)} \right) = 2^{n^2/4+o(n^2)}.$$
The number of triangle-free graphs: ‘New approach’

Theorem (Erdős–Kleitman–Rothschild [1976])

The number of triangle-free graphs is \( 2^{n^2/4+o(n^2)} \).
The number of triangle-free graphs: ‘New approach’

**Theorem (Erdős–Kleitman–Rothschild [1976])**

The number of triangle-free graphs is \(2^{n^2/4+o(n^2)}\).

**Balogh–Morris–Samotij, Saxton–Thomason [2012+]**

There is a \(t < 2^{O(\log n \cdot n^{3/2})}\) and a set \(\{G_1, \ldots, G_t\}\) of graphs, each containing at most \(o(n^3)\) triangles, such that for every triangle-free graph \(H\) there is an \(i \in [t]\) such that \(H \subseteq G_i\).
The number of triangle-free graphs: ‘New approach’

**Theorem (Erdős–Kleitman–Rothschild [1976])**

The number of triangle-free graphs is $2^{n^2/4+o(n^2)}$.

**Balogh–Morris–Samotij, Saxton–Thomason [2012+]**

There is a $t < 2^{O(\log n \cdot n^{3/2})}$ and a set $\{G_1, \ldots, G_t\}$ of graphs, each containing at most $o(n^3)$ triangles, such that for every triangle-free graph $H$ there is an $i \in [t]$ such that $H \subseteq G_i$.

- For each $F_n$ triangle-free graph there is an $i$ that $F_n \subset G_i$. 
**The number of triangle-free graphs: ‘New approach’**

<table>
<thead>
<tr>
<th>Theorem (Erdős–Kleitman–Rothschild [1976])</th>
</tr>
</thead>
<tbody>
<tr>
<td>The number of triangle-free graphs is $2^{n^2/4+o(n^2)}$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Balogh–Morris–Samotij, Saxton–Thomason [2012+]</th>
</tr>
</thead>
<tbody>
<tr>
<td>There is a $t &lt; 2^{O(\log n \cdot n^{3/2})}$ and a set ${G_1, \ldots, G_t}$ of graphs, each containing at most $o(n^3)$ triangles, such that for every triangle-free graph $H$ there is an $i \in [t]$ such that $H \subseteq G_i$.</td>
</tr>
</tbody>
</table>

- For each $F_n$ triangle-free graph there is an $i$ that $F_n \subset G_i$.
- $e(G_i) \leq n^2/4 + o(n^2)$. |
The number of triangle-free graphs: ‘New approach’

**Theorem (Erdős–Kleitman–Rothschild [1976])**

The number of triangle-free graphs is $2^{n^2/4+o(n^2)}$.

**Balogh–Morris–Samotij, Saxton–Thomason [2012+]**

There is a $t < 2^{O(\log n \cdot n^{3/2})}$ and a set $\{G_1, \ldots, G_t\}$ of graphs, each containing at most $o(n^3)$ triangles, such that for every triangle-free graph $H$ there is an $i \in [t]$ such that $H \subseteq G_i$.

- For each $F_n$ triangle-free graph there is an $i$ that $F_n \subseteq G_i$.
- $e(G_i) \leq n^2/4 + o(n^2)$.
- Number of choices for $F_n$ is
The number of triangle-free graphs: ‘New approach’

**Theorem (Erdős–Kleitman–Rothschild [1976])**
The number of triangle-free graphs is \(2^{n^2/4+o(n^2)}\).

**Balogh–Morris–Samotij, Saxton–Thomason [2012+]**
There is a \(t < 2^{O(\log n \cdot n^{3/2})}\) and a set \(\{G_1, \ldots, G_t\}\) of graphs, each containing at most \(o(n^3)\) triangles, such that for every triangle-free graph \(H\) there is an \(i \in [t]\) such that \(H \subseteq G_i\).

- For each \(F_n\) triangle-free graph there is an \(i\) that \(F_n \subset G_i\).
- \(e(G_i) \leq n^2/4 + o(n^2)\).
- Number of choices for \(F_n\) is \(t\).
## The number of triangle-free graphs: ‘New approach’

### Theorem (Erdős–Kleitman–Rothschild [1976])

The number of triangle-free graphs is $2^{n^2/4+o(n^2)}$.

### Balogh–Morris–Samotij, Saxton–Thomason [2012+]

There is a $t < 2^{O(\log n \cdot n^{3/2})}$ and a set $\{G_1, \ldots, G_t\}$ of graphs, each containing at most $o(n^3)$ triangles, such that for every triangle-free graph $H$ there is an $i \in [t]$ such that $H \subseteq G_i$.

- For each $F_n$ triangle-free graph there is an $i$ that $F_n \subset G_i$.
- $e(G_i) \leq n^2/4 + o(n^2)$.
- Number of choices for $F_n$ is $t \cdot 2^{n^2/4+o(n^2)} = 2^{n^2/4+o(n^2)}$. 

---

**Szemerédi container lemma**

There is a $t < 2^{O(\log n \cdot n^{3/2})}$ and a set $\{G_1, \ldots, G_t\}$ of graphs, each containing at most $o(n^3)$ triangles, such that for every triangle-free graph $H$ there is an $i \in [t]$ such that $H \subseteq G_i$. 

- For each $F_n$ triangle-free graph there is an $i$ that $F_n \subset G_i$.
- $e(G_i) \leq n^2/4 + o(n^2)$.
- Number of choices for $F_n$ is $t \cdot 2^{n^2/4+o(n^2)} = 2^{n^2/4+o(n^2)}$. 

---
The number of triangle-free graphs: ‘New approach’

**Theorem (Erdős–Kleitman–Rothschild [1976])**

The number of triangle-free graphs is $2^{n^2/4+o(n^2)}$.

**Balogh–Morris–Samotij, Saxton–Thomason [2012+]**

There is a $t < 2^{O(\log n \cdot n^3/2)}$ and a set $\{G_1, \ldots, G_t\}$ of graphs, each containing at most $o(n^3)$ triangles, such that for every triangle-free graph $H$ there is an $i \in [t]$ such that $H \subseteq G_i$.

- For each $F_n$ triangle-free graph there is an $i$ that $F_n \subset G_i$.
- $e(G_i) \leq n^2/4 + o(n^2)$.
- Number of choices for $F_n$ is $t \cdot 2^{n^2/4+o(n^2)} = 2^{n^2/4+o(n^2)}$.

**Szemerédi container lemma**

There is a $t = 2^{o(n^2)}$ and a set $\{G_1, \ldots, G_t\}$ of graphs, each containing at most $o(n^3)$ triangles, such that for every triangle-free graph $H$ there is $i \in [t]$ such that $H \subseteq G_i$. 
The number of maximal triangle-free graphs

Balogh–Petříčková [2014+]

There are at most $2^{n^2/8+o(n^2)}$ maximal triangle-free graphs on $n$ vertices.
The number of maximal triangle-free graphs

<table>
<thead>
<tr>
<th>Balogh–Petříčková [2014+]</th>
</tr>
</thead>
<tbody>
<tr>
<td>There are at most $2^{n^2/8+o(n^2)}$ maximal triangle-free graphs on $n$ vertices.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Balogh–Morris–Samotij, Saxton–Thomason [2012+]</th>
</tr>
</thead>
<tbody>
<tr>
<td>There is a $t &lt; 2^{O(\log n \cdot n^{3/2})}$ and a set ${G_1, \ldots, G_t}$ of graphs, each containing at most $o(n^3)$ triangles, such that for every triangle-free graph $H$ there is an $i \in [t]$ such that $H \subseteq G_i$. Note $e(G_i) \leq n^2/4 + o(n^2)$.</td>
</tr>
</tbody>
</table>
### The number of maximal triangle-free graphs

<table>
<thead>
<tr>
<th>Source</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>Balogh–Petříčková [2014+]</td>
<td>There are at most $2^{n^2/8+o(n^2)}$ maximal triangle-free graphs on $n$ vertices.</td>
</tr>
<tr>
<td>Balogh–Morris–Samotij, Saxton–Thomason [2012+]</td>
<td>There is a $t &lt; 2^{O\left(\log n \cdot n^{3/2}\right)}$ and a set ${G_1, \ldots, G_t}$ of graphs, each containing at most $o(n^3)$ triangles, such that for every triangle-free graph $H$ there is an $i \in [t]$ such that $H \subseteq G_i$. Note $e(G_i) \leq n^2/4 + o(n^2)$.</td>
</tr>
<tr>
<td>Ruzsa–Szemerédi (1976)</td>
<td>Any graph $G_n$ with at most $o(n^3)$ triangles can be made triangle-free by removing at most $o(n^2)$ edges.</td>
</tr>
</tbody>
</table>
The number of maximal triangle-free graphs

**Balogh–Petříčková [2014+]**
There are at most \(2^{n^2/8+o(n^2)}\) maximal triangle-free graphs on \(n\) vertices.

**Balogh–Morris–Samotij, Saxton–Thomason [2012+]**
There is a \(t < 2^{O(\log n \cdot n^{3/2})}\) and a set \(\{G_1, \ldots, G_t\}\) of graphs, each containing at most \(o(n^3)\) triangles, such that for every triangle-free graph \(H\) there is an \(i \in [t]\) such that \(H \subseteq G_i\). Note \(e(G_i) \leq n^2/4 + o(n^2)\).

**Ruzsa–Szemerédi (1976)**
Any graph \(G_n\) with at most \(o(n^3)\) triangles can be made triangle-free by removing at most \(o(n^2)\) edges.

**Hujter–Tuza (1993)**
Any triangle-free graph \(T_N\) has at most \(2^{N/2}\) maximal independent sets. Sharpness is by a perfect matching.
There are at most $2^{n^2/8 + o(n^2)}$ maximal triangle-free graphs on $n$ vertices.
The number of maximal triangle-free graphs

Balogh–Petříčková [2014+]

There are at most $2^{n^2/8+o(n^2)}$ maximal triangle-free graphs on $n$ vertices.

- For $F_n$ triangle-free graph there is a $G_i$ containing it. $2^{O(\log n \cdot n^{3/2})}$ choices.
The number of maximal triangle-free graphs

Balogh–Petříčková [2014+]

There are at most \(2^{n^2/8+o(n^2)}\) maximal triangle-free graphs on \(n\) vertices.

- For \(F_n\) triangle-free graph there is a \(G_i\) containing it. \(2^{O(\log n \cdot n^{3/2})}\) choices.
- Fix a \(T_i \subset E(G_i)\) that \(|T_i| = o(n^2)\) and \(E(G_i) - T_i\) is triangle-free.
The number of maximal triangle-free graphs

Balogh–Petříčková [2014+]

There are at most \( 2^{n^2/8 + o(n^2)} \) maximal triangle-free graphs on \( n \) vertices.

- For \( F_n \) triangle-free graph there is a \( G_i \) containing it. \( 2^{O(\log n \cdot n^{3/2})} \) choices.
- Fix a \( T_i \subset E(G_i) \) that \( |T_i| = o(n^2) \) and \( E(G_i) - T_i \) is triangle-free. Decide on \( T_i \cap E(F_n) \). Number of choices is \( 2^{o(n^2)} \).
The number of maximal triangle-free graphs

Balogh–Petříčková [2014+]

There are at most $2^{n^2/8+o(n^2)}$ maximal triangle-free graphs on $n$ vertices.

- For $F_n$ triangle-free graph there is a $G_i$ containing it. $2^{O(\log n \cdot n^{3/2})}$ choices.
- Fix a $T_i \subset E(G_i)$ that $|T_i| = o(n^2)$ and $E(G_i) - T_i$ is triangle-free. Decide on $T_i \cap E(F_n)$. Number of choices is $2^{o(n^2)}$.
- Form auxiliary graph: $V := E(G_i) - T_i$, $E = \{ef : \exists g \in T_i \cap E(F_n), \text{ that } efg \text{ is a triangle.}\}$
The number of maximal triangle-free graphs

Balogh–Petříčková [2014+]

There are at most $2^{n^2/8+o(n^2)}$ maximal triangle-free graphs on $n$ vertices.

- For $F_n$ triangle-free graph there is a $G_i$ containing it.  
  $2^{O(\log n \cdot n^{3/2})}$ choices.
- Fix a $T_i \subset E(G_i)$ that $|T_i| = o(n^2)$ and $E(G_i) - T_i$ is triangle-free.  
  Decide on $T_i \cap E(F_n)$. Number of choices is $2^{o(n^2)}$.
- Form auxiliary graph: $V := E(G_i) - T_i$,  
  $E = \{ef : \exists g \in T_i \cap E(F_n), \text{ that } efg \text{ is a triangle}\}$.
- This graph is triangle-free;
The number of maximal triangle-free graphs

Balogh–Petříčková [2014+]

There are at most $2^{n^2/8+o(n^2)}$ maximal triangle-free graphs on $n$ vertices.

- For $F_n$ triangle-free graph there is a $G_i$ containing it. $2^{O(\log n \cdot n^{3/2})}$ choices.

- Fix a $T_i \subseteq E(G_i)$ that $|T_i| = o(n^2)$ and $E(G_i) - T_i$ is triangle-free. Decide on $T_i \cap E(F_n)$. Number of choices is $2^{o(n^2)}$.

- Form auxiliary graph: $V := E(G_i) - T_i$, $E = \{ef : \text{if } \exists g \in T_i \cap E(F_n), \text{that } efg \text{ is a triangle.}\}$

- This graph is triangle-free;

- Number of choices for $(F_n \cap G_i) - T_i$ is at most the number of maximal independent sets in the auxiliary graph.
The number of maximal triangle-free graphs

Balogh–Petříčková [2014+]

There are at most $2^{n^2/8+o(n^2)}$ maximal triangle-free graphs on $n$ vertices.

For $F_n$ triangle-free graph there is a $G_i$ containing it. $2^{O(\log n \cdot n^{3/2})}$ choices.

Fix a $T_i \subset E(G_i)$ that $|T_i| = o(n^2)$ and $E(G_i) - T_i$ is triangle-free. Decide on $T_i \cap E(F_n)$. Number of choices is $2^{o(n^2)}$.

Form auxiliary graph: $V := E(G_i) - T_i$, $E = \{ef : \text{if } \exists \ g \in T_i \cap E(F_n), \text{that } efg \text{ is a triangle.}\}$

This graph is triangle-free;

Number of choices for $(F_n \cap G_i) - T_i$ is at most the number of maximal independent sets in the auxiliary graph.

$|V| \leq n^2/4$; Hujter–Tuza gives $\leq 2^{n^2/8}$ choices.
Example (Erdős–Turán problem)

- $V = \{1, \ldots, n\}$,
- $\mathcal{H} = k$-term APs in $[n]$. 

Example (Turán problem)

- $V = \text{edges of } K_n$,
- $\mathcal{H} = \text{edge-sets of copies of } K_k \text{ in } K_n$.

Example (sum-free sets)

- $V = \text{an Abelian group}$,
- $\mathcal{H} = \text{sets of the form } \{x, y, z\} \text{ with } x+y=z \text{ (Schur triples)}$. 

Example (Erdős–Turán problem)
- \( V = \{1, \ldots, n\} \),
- \( \mathcal{H} = k\)-term APs in \([n]\).  

Example (Turán problem)
- \( V = \) edges of \( K_n \),
- \( \mathcal{H} = \) edge-sets of copies of \( K_k \) in \( K_n \).
Example (Erdős–Turán problem)

- $V = \{1, \ldots, n\}$,
- $\mathcal{H} = k$-term APs in $[n]$.

Example (Turán problem)

- $V =$ edges of $K_n$,
- $\mathcal{H} =$ edge-sets of copies of $K_k$ in $K_n$.

Example (sum-free sets)

- $V =$ an Abelian group,
- $\mathcal{H} =$ sets of the form $\{x, y, z\}$ with $x + y = z$ (Schur triples).
Transference Theorem

**Theorem (Balogh–Morris–S. [2012+])**

For every \( k, c, \varepsilon \) there is a \( C \) that the following holds. Let \( \mathcal{H} \subseteq \binom{V}{k} \) such that for \( \ell \in [k], p \in [0, 1] \)

\[
\Delta_\ell(\mathcal{H}) \leq c \cdot p^{\ell-1} \frac{e(\mathcal{H})}{v(\mathcal{H})}.
\]
**Theorem (Balogh–Morris–S. [2012+])**

For every $k$, $c$, $\varepsilon$ there is a $C$ that the following holds. Let $\mathcal{H} \subseteq \binom{V}{k}$ such that for $\ell \in [k]$, $p \in [0, 1]$

$$\Delta_\ell(\mathcal{H}) \leq c \cdot p^{\ell-1} \frac{e(\mathcal{H})}{v(\mathcal{H})}.$$ 

Let $\mathcal{F} = \{A \subseteq V : |\mathcal{H}[A]| \geq \varepsilon \cdot e(\mathcal{H})\}$. 

---

**Example of triangle-free graphs.**
Transference Theorem

Theorem (Balogh–Morris–S. [2012+])

For every $k, c, \varepsilon$ there is a $C$ that the following holds. Let $\mathcal{H} \subseteq \binom{V}{k}$ such that for $\ell \in [k], p \in [0, 1]$

$$\Delta_\ell(\mathcal{H}) \leq c \cdot p^{\ell-1} \frac{e(\mathcal{H})}{v(\mathcal{H})}.$$ 

Let $\mathcal{F} = \{A \subseteq V : |\mathcal{H}[A]| \geq \varepsilon \cdot e(\mathcal{H})\}$. Then there are:

- a very small family $\mathcal{S} \subseteq \binom{V(\mathcal{H})}{\leq Cp \cdot v(\mathcal{H})}$ of labels,
Theorem (Balogh–Morris–S. [2012+])

For every $k, c, \varepsilon$ there is a $C$ that the following holds. Let $\mathcal{H} \subseteq \binom{V}{k}$ such that for $\ell \in [k]$, $p \in [0, 1]$

$$\Delta_\ell(\mathcal{H}) \leq c \cdot p^{\ell-1} \frac{e(\mathcal{H})}{v(\mathcal{H})}.$$ 

Let $\mathcal{F} = \{A \subseteq V : |\mathcal{H}[A]| \geq \varepsilon \cdot e(\mathcal{H})\}$. Then there are:

- a very small family $\mathcal{S} \subseteq \binom{V(\mathcal{H})}{\leq Cp \cdot v(\mathcal{H})}$ of labels,
- $f : \mathcal{S} \rightarrow \mathcal{F}^c$ (maps each label to a set that is sparse in $\mathcal{H}$),

Similar result was obtained independently by Saxton and Thomason.

Explain: Example of triangle-free graphs.
Transference Theorem

Theorem (Balogh–Morris–S. [2012+])

For every $k, c, \varepsilon$ there is a $C$ that the following holds. Let $\mathcal{H} \subseteq \binom{V}{k}$ such that for $\ell \in [k]$, $p \in [0, 1]$

$$\Delta_\ell(\mathcal{H}) \leq c \cdot p^{\ell-1} \frac{e(\mathcal{H})}{v(\mathcal{H})}.$$ 

Let $\mathcal{F} = \{ A \subseteq V : |\mathcal{H}[A]| \geq \varepsilon \cdot e(\mathcal{H}) \}$. Then there are:

- a very small family $S \subseteq \binom{V(\mathcal{H})}{\leq Cp \cdot v(\mathcal{H})}$ of labels,
- $f : S \rightarrow \mathcal{F}^c$ (maps each label to a set that is sparse in $\mathcal{H}$),
- a labeling function $g : I(\mathcal{H}) \rightarrow S$, 

Similar result was obtained independently by Saxton and Thomason.

Explain: Example of triangle-free graphs.
Transference Theorem

Theorem (Balogh–Morris–S. [2012+])

For every \( k, c, \varepsilon \) there is a \( C \) that the following holds. Let \( \mathcal{H} \subseteq \binom{V}{k} \) such that for \( \ell \in [k], \ p \in [0, 1] \)

\[
\Delta_\ell(\mathcal{H}) \leq c \cdot p^{\ell-1} \frac{e(\mathcal{H})}{v(\mathcal{H})}.
\]

Let \( \mathcal{F} = \{ A \subseteq V : |\mathcal{H}[A]| \geq \varepsilon \cdot e(\mathcal{H}) \} \). Then there are:

- a very small family \( S \subseteq \binom{V(\mathcal{H})}{\leq Cp \cdot v(\mathcal{H})} \) of labels,
- \( f : S \rightarrow \mathcal{F}^c \) (maps each label to a set that is sparse in \( \mathcal{H} \)),
- a labeling function \( g : \mathcal{I}(\mathcal{H}) \rightarrow S \),

such that for every \( l \in \mathcal{I}(\mathcal{H}) \),
Theorem (Balogh–Morris–S. [2012+])

For every $k, c, \varepsilon$ there is a $C$ that the following holds. Let $\mathcal{H} \subseteq \binom{V}{k}$ such that for $\ell \in [k]$, $p \in [0, 1]$

$$\Delta_\ell(\mathcal{H}) \leq c \cdot p^{\ell-1} \frac{e(\mathcal{H})}{v(\mathcal{H})}.$$ 

Let $\mathcal{F} = \{ A \subseteq V : |\mathcal{H}[A]| \geq \varepsilon \cdot e(\mathcal{H}) \}$. Then there are:

- a very small family $\mathcal{S} \subseteq \binom{V(\mathcal{H})}{\leq Cp \cdot v(\mathcal{H})}$ of labels,
- $f : \mathcal{S} \rightarrow \mathcal{F}^c$ (maps each label to a set that is sparse in $\mathcal{H}$),
- a labeling function $g : \mathcal{I}(\mathcal{H}) \rightarrow \mathcal{S}$,

such that for every $I \in \mathcal{I}(\mathcal{H})$,

$$g(I) \subseteq I \quad \text{and} \quad I \setminus g(I) \subseteq f(g(I)).$$
Transference Theorem

Theorem (Balogh–Morris–S. [2012+])

For every $k, c, \varepsilon$ there is a $C$ that the following holds. Let $\mathcal{H} \subseteq \binom{V}{k}$ such that for $\ell \in [k], p \in [0, 1]$

$$\Delta_\ell(\mathcal{H}) \leq c \cdot p^{\ell-1} \frac{e(\mathcal{H})}{v(\mathcal{H})}.$$ 

Let $\mathcal{F} = \{A \subseteq V : |\mathcal{H}[A]| \geq \varepsilon \cdot e(\mathcal{H})\}$. Then there are:

- a very small family $S \subseteq \binom{V(\mathcal{H})}{\leq Cp \cdot v(\mathcal{H})}$ of labels,
- $f : S \rightarrow \mathcal{F}^c$ (maps each label to a set that is sparse in $\mathcal{H}$),
- a labeling function $g : \mathcal{I}(\mathcal{H}) \rightarrow S$,

such that for every $I \in \mathcal{I}(\mathcal{H})$,

$$g(I) \subseteq I \quad \text{and} \quad I \setminus g(I) \subseteq f(g(I)).$$

Similar result was obtained independently by Saxton and Thomason.

Explain: Example of triangle-free graphs.
Transference Theorem: — illustration

- Dense sets
- Independent sets
- Small sets (labels)
Transference Theorem: — illustration

- Dense sets
- Covering sets
- Independent sets
- Small sets (labels)

\[ f(S) \]

\[ \mathcal{I}(\mathcal{H}) \]

\[ \mathcal{F} \]
Transference Theorem: — illustration

- Dense sets
- Covering sets
- Independent sets
- Small sets (labels)
- $\mathcal{F}$
- $f(S)$
- $\mathcal{I}(\mathcal{H})$
Transference Theorem: — illustration

- dense sets
- covering sets
- independent sets
- small sets (labels)

\[ \mathcal{F} \]

\[ f(\mathcal{S}) \]

\[ \mathcal{I}(\mathcal{H}) \]

\[ g \]

\[ g(I) \]
Transference Theorem: — illustration

\[ f(g(l)) \]

\[ f(S) \]

\[ f(\mathcal{H}) \]

dense sets
covering sets
independent sets
small sets (labels)
Transference Theorem: — illustration

- dense sets
- covering sets
- independent sets
- small sets (labels)

\[ f(g(I)) \]

\[ f(S) \]

\[ \mathcal{F} \]

\[ \mathcal{I}(H) \]
How to use Transference Theorem?

Let $V(\mathcal{H}) = E(K_n)$, $E(\mathcal{H}) = \text{copies of } K_{r+1}$.
How to use Transference Theorem?

- Let $V(\mathcal{H}) = E(K_n)$, $E(\mathcal{H}) = \text{copies of } K_{r+1}$.
- An $I$ independent set in $\mathcal{H}$ is a $K_{r+1}$-free graph.
How to use Transference Theorem?

- Let $V(\mathcal{H}) = E(K_n)$, $E(\mathcal{H}) = \text{copies of } K_{r+1}$.
- An independent set in $\mathcal{H}$ is a $K_{r+1}$-free graph.
- Let $t = \left( \frac{n^2}{2} \right)$. There are $G_1, \ldots G_t$ graphs that for any $H$ $K_{r+1}$-free graph there is an $i$ that $H \subset G_i$. 
How to use Transference Theorem?

- Let $V(\mathcal{H}) = E(K_n)$, $E(\mathcal{H}) = \text{copies of } K_{r+1}$.
- An $I$ independent set in $\mathcal{H}$ is a $K_{r+1}$-free graph.
- Let $t = \binom{n^2/2}{Cn^2-1/r}$. There are $G_1, \ldots G_t$ graphs that for any $H$ $K_{r+1}$-free graph there is an $i$ that $H \subset G_i$.
- The number of $K_{r+1}$ in each $G_i$ is $o(n^{r+1})$. 

Super-saturation implies that for each $i$:

$$e(G_i) < (1 - \frac{1}{r} + o(1)) \frac{n^2}{2}.$$ 

Super-saturation – Stability theorems implies that each $G_i$ is almost $r$-partite or

$$e(G_i) < (1 - \frac{1}{r} - c) \frac{n^2}{2}.$$ 

Computation gives: Almost all $K_{r+1}$-free graph is almost $r$-partite.

$$\left(\frac{n^2/2}{Cn^2-1/r}\right)^2 \left(1 - \frac{1}{r} - c\right) \frac{n^2}{2} \ll 2^{\left(1 - \frac{1}{r}\right) \frac{n^2}{2}}.$$
How to use Transference Theorem?

- Let $V(\mathcal{H}) = E(K_n)$, $E(\mathcal{H}) =$ copies of $K_{r+1}$.
- An $I$ independent set in $\mathcal{H}$ is a $K_{r+1}$-free graph.
- Let $t = \binom{n^2/2}{cn^{2-1/r}}$. There are $G_1, \ldots, G_t$ graphs that for any $H$ $K_{r+1}$-free graph there is an $i$ that $H \subset G_i$.
- The number of $K_{r+1}$ in each $G_i$ is $o(n^{r+1})$.
- Super-saturation implies that for each $i:
  \[ e(G_i) < (1 - \frac{1}{r} + o(1)) \frac{n^2}{2}. \]
How to use Transference Theorem?

- Let $V(\mathcal{H}) = E(K_n)$, $E(\mathcal{H}) =$ copies of $K_{r+1}$.
- An $I$ independent set in $\mathcal{H}$ is a $K_{r+1}$-free graph.
- Let $t = \binom{n^2/2}{Cn^2-1/r}$. There are $G_1, \ldots, G_t$ graphs that for any $H$ $K_{r+1}$-free graph there is an $i$ that $H \subset G_i$.
- The number of $K_{r+1}$ in each $G_i$ is $o(n^{r+1})$.
- Super-saturation implies that for each $i$:
  
  $$e(G_i) < (1 - \frac{1}{r} + o(1)) \frac{n^2}{2}.$$ 

- Super-saturation – Stability theorems implies that each $G_i$ is
How to use Transference Theorem?

- Let $V(\mathcal{H}) = E(K_n)$, $E(\mathcal{H}) = \text{copies of } K_{r+1}$.
- An $I$ independent set in $\mathcal{H}$ is a $K_{r+1}$-free graph.
- Let $t = \left(\frac{n^2}{2Cn^2 - \frac{1}{r}}\right)$. There are $G_1, \ldots G_t$ graphs that for any $H$ $K_{r+1}$-free graph there is an $i$ that $H \subset G_i$.
- The number of $K_{r+1}$ in each $G_i$ is $o(n^{r+1})$.
- Super-saturation implies that for each $i$:
  \[ e(G_i) < (1 - \frac{1}{r} + o(1)) \frac{n^2}{2}. \]
- Super-saturation – Stability theorems implies that each $G_i$ is either almost $r$-partite or
How to use Transference Theorem?

- Let $V(\mathcal{H}) = E(K_n)$, $E(\mathcal{H}) = \text{copies of } K_{r+1}$.
- An $I$ independent set in $\mathcal{H}$ is a $K_{r+1}$-free graph.
- Let $t = \left(\frac{n^2}{2Cn^2 - 1/r}\right)$. There are $G_1, \ldots G_t$ graphs that for any $H$ $K_{r+1}$-free graph there is an $i$ that $H \subset G_i$.
- The number of $K_{r+1}$ in each $G_i$ is $o(n^{r+1})$.
- Super-saturation implies that for each $i$:
  \[ e(G_i) < (1 - \frac{1}{r} + o(1)) \frac{n^2}{2}. \]
- Super-saturation – Stability theorems implies that each $G_i$ is either almost $r$-partite or
  \[ e(G_i) < (1 - \frac{1}{r} - c) \frac{n^2}{2}. \]
How to use Transference Theorem?

- Let \( V(\mathcal{H}) = E(K_n) \), \( E(\mathcal{H}) = \) copies of \( K_{r+1} \).
- An \( I \) independent set in \( \mathcal{H} \) is a \( K_{r+1} \)-free graph.
- Let \( t = \left( \frac{n^2/2}{Cn^2-1/r} \right) \). There are \( G_1, \ldots G_t \) graphs that for any \( H \) \( K_{r+1} \)-free graph there is an \( i \) that \( H \subset G_i \).
- The number of \( K_{r+1} \) in each \( G_i \) is \( o(n^{r+1}) \).
- Super-saturation implies that for each \( i \):
  \[ e(G_i) < (1 - \frac{1}{r} + o(1)) \frac{n^2}{2}. \]
- Super-saturation – Stability theorems implies that each \( G_i \) is either almost \( r \)-partite or
  \[ e(G_i) < (1 - \frac{1}{r} - c) \frac{n^2}{2}. \]
- Computation gives: Almost all \( K_{r+1} \)-free graph is almost \( r \)-partite.

\[
\left( \frac{n^2/2}{Cn^2-1/r} \right) 2^{(1-1/r-c)n^2/2} \ll 2^{(1-1/r)n^2/2}.
\]