Slicing inequalities for measures of convex bodies.

Alexander Koldobsky

University of Missouri-Columbia
The Slicing Problem

Does there exist an absolute constant $C$ so that for any origin-symmetric convex body $K$ in $\mathbb{R}^n$

$$|K|^\frac{n-1}{n} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|$$

where $\xi^\perp$ is the central hyperplane in $\mathbb{R}^n$ perpendicular to $\xi$, and $|K|$ stands for volume of proper dimension.
The Slicing Problem

Does there exist an absolute constant $C$ so that for any origin-symmetric convex body $K$ in $\mathbb{R}^n$

$$|K|^{\frac{n-1}{n}} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^\perp| \ ?$$

where $\xi^\perp$ is the central hyperplane in $\mathbb{R}^n$ perpendicular to $\xi$, and $|K|$ stands for volume of proper dimension.

The best-to-date estimate $C \leq O(n^{1/4})$ is due to Klartag, who removed a logarithmic term from an earlier estimate of Bourgain.
The Slicing Problem

Does there exist an absolute constant $C$ so that for any origin-symmetric convex body $K$ in $\mathbb{R}^n$

$$|K|^\frac{n-1}{n} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^\perp| ?$$

where $\xi^\perp$ is the central hyperplane in $\mathbb{R}^n$ perpendicular to $\xi$, and $|K|$ stands for volume of proper dimension.

The best-to-date estimate $C \leq O(n^{1/4})$ is due to Klartag, who removed a logarithmic term from an earlier estimate of Bourgain.

For certain classes of bodies the question has been answered in affirmative.

- unconditional convex bodies (Bourgain),
- unit balls of subspaces of $L_p$ (Ball, Junge, E.Milman), $C \sim \sqrt{p}$, $p \to \infty$,
- intersection bodies (immediate from the Busemann-Petty problem),
- zonoids, duals of bodies with bounded volume ratio (V.Milman-Pajor),
- the Schatten classes (König, Meyer, Pajor), and others.
The Slicing Problem

Does there exist an absolute constant $C$ so that for any origin-symmetric convex body $K$ in $\mathbb{R}^n$

$$|K|^{\frac{n-1}{n}} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^\perp|$$

where $\xi^\perp$ is the central hyperplane in $\mathbb{R}^n$ perpendicular to $\xi$, and $|K|$ stands for volume of proper dimension.

The best-to-date estimate $C \leq O(n^{1/4})$ is due to Klartag, who removed a logarithmic term from an earlier estimate of Bourgain.

For certain classes of bodies the question has been answered in affirmative.

- unconditional convex bodies (Bourgain),
- unit balls of subspaces of $L_p$ (Ball, Junge, E.Milman), $C \sim \sqrt{p}$, $p \to \infty$,
- intersection bodies (immediate from the Busemann-Petty problem),
- zonoids, duals of bodies with bounded volume ratio (V.Milman-Pajor),
- the Schatten classes (König, Meyer, Pajor), and others.

$$|K|^{\frac{n-k}{n}} \leq C_k \max_{H \in Gr_{n-k}} |K \cap H|$$
The Slicing Problem

Does there exist an absolute constant $C$ so that for any origin-symmetric convex body $K$ in $\mathbb{R}^n$ 

$$|K|^\frac{n-1}{n} \leq C \max_{\xi \in S^{n-1}} |K \cap \xi^\perp| ?$$

where $\xi^\perp$ is the central hyperplane in $\mathbb{R}^n$ perpendicular to $\xi$, and $|K|$ stands for volume of proper dimension.

The best-to-date estimate $C \leq O(n^{1/4})$ is due to Klartag, who removed a logarithmic term from an earlier estimate of Bourgain.

For certain classes of bodies the question has been answered in affirmative.

- unconditional convex bodies (Bourgain),
- unit balls of subspaces of $L_p$ (Ball, Junge, E.Milman), $C \sim \sqrt{p}$, $p \to \infty$,
- intersection bodies (immediate from the Busemann-Petty problem),
- zonoids, duals of bodies with bounded volume ratio (V.Milman-Pajor),
- the Schatten classes (König, Meyer, Pajor), and others.

$$|K|^\frac{n-k}{n} \leq C^k \max_{H \in \text{Gr}_{n-k}} |K \cap H| ?$$

We prove this for $k \geq \lambda n$, $0 < \lambda < 1$ with $C = C(\lambda) = C_0 \sqrt{(1 - \log \lambda)^3/\lambda}$. 
Does there exist an absolute constant $C$ such that for every $n \in \mathbb{N}$, every origin-symmetric convex body $L$ in $\mathbb{R}^n$, and every measure $\mu$ with non-negative even continuous density $f$ in $\mathbb{R}^n$,

$$\mu(L) \leq C \max_{\xi \in S^{n-1}} \mu(L \cap \xi^\perp) |L|^{1/n} \quad ?$$

Here $\mu(B) = \int_B f$ for every Borel set $B$ in $\mathbb{R}^n$ or $H$. 
Does there exist an absolute constant $C$ such that for every $n \in \mathbb{N}$, every origin-symmetric convex body $L$ in $\mathbb{R}^n$, and every measure $\mu$ with non-negative even continuous density $f$ in $\mathbb{R}^n$,

$$
\mu(L) \leq C \max_{\xi \in S^{n-1}} \mu(L \cap \xi^\perp) |L|^{1/n} \ ?
$$

Here $\mu(B) = \int_B f$ for every Borel set $B$ in $\mathbb{R}^n$ or $H$.

More generally, does there exists an absolute constant $C$ so that for every $n, L, \mu$ and every integer $1 \leq k < n$,

$$
\mu(L) \leq C^k \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) |L|^{k/n},
$$

where $\text{Gr}_{n-k}$ is the Grassmanian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$?
A generalization to arbitrary measures in place of volume

Does there exist an absolute constant $C$ such that for every $n \in \mathbb{N}$, every origin-symmetric convex body $L$ in $\mathbb{R}^n$, and every measure $\mu$ with non-negative even continuous density $f$ in $\mathbb{R}^n$,

$$\mu(L) \leq C \max_{\xi \in S^{n-1}} \mu(L \cap \xi^\perp) \frac{|L|^{1/n}}{n}$$

Here $\mu(B) = \int_B f$ for every Borel set $B$ in $\mathbb{R}^n$ or $H$.

More generally, does there exist an absolute constant $C$ so that for every $n, L, \mu$ and every integer $1 \leq k < n$,

$$\mu(L) \leq C^k \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) \frac{|L|^{k/n}}{n},$$

where $\text{Gr}_{n-k}$ is the Grassmanian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$?

Denote by $c_{n,k} = |B_2^n|^{\frac{n-k}{n}} / |B_2^{n-k}| \in (e^{-k/2}, 1)$, where $B_2^n$ is the unit Euclidean ball in $\mathbb{R}^n$. Also, $1 \leq \frac{n}{n-k} \leq e^\frac{k}{n-k} \leq e^k$, the latter inequality is equivalent to

$$\mu(L) \leq C^k \frac{n}{n-k} c_{n,k} \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) \frac{|L|^{k/n}}{n}.$$
What is known?

\[ \mu(L) \leq C^k \frac{n}{n-k} c_{n,k} \max_{H \in Gr_{n-k}} \mu(L \cap H) |L|^{k/n} \]
What is known?

\[ \mu(L) \leq C^k \frac{n}{n-k} c_{n,k} \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) \left| L \right|^{k/n} \]

- Intersection bodies, with the best possible constant \( C = 1 \); K. 2012 \((k = 1)\), K.- Dan Ma 2013 (for all \( k \))
What is known?

\[ \mu(L) \leq C^k \frac{n}{n-k} c_{n,k} \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) |L|^{k/n} \]

- Intersection bodies, with the best possible constant \( C = 1 \);
  K. 2012 \((k = 1)\), K.- Dan Ma 2013 (for all \( k \))
- Holds with \( C = \sqrt{n} \) for all \( n, L, \mu, k \)
What is known?

\[ \mu(L) \leq C^k \frac{n}{n-k} c_{n,k} \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) |L|^{k/n} \]

- Intersection bodies, with the best possible constant \( C = 1 \); K. 2012 (\( k = 1 \)), K.- Dan Ma 2013 (for all \( k \))
- Holds with \( C = \sqrt{n} \) for all \( n, L, \mu, k \)
- For every \( \lambda \in (0, 1) \) there exists a constant \( C = C(\lambda) \) so that the inequality holds for arbitrary \( n, L, \mu \) and the codimension of sections \( k \geq \lambda n \);
What is known?

\[ \mu(L) \leq C^k \frac{n}{n-k} c_{n,k} \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) |L|^{k/n} \]

- Intersection bodies, with the best possible constant \( C = 1 \);
  K. 2012 \((k = 1)\), K.- Dan Ma 2013 (for all \( k \))
- Holds with \( C = \sqrt{n} \) for all \( n, L, \mu, k \)
- For every \( \lambda \in (0, 1) \) there exists a constant \( C = C(\lambda) \) so that the inequality holds for arbitrary \( n, L, \mu \) and the codimension of sections \( k \geq \lambda n \);
  \( C = C_0 \sqrt{(1-\log \lambda)^3 \over \lambda} \), \( C_0 \) absolute constant
**What is known?**

\[
\mu(L) \leq C^k \frac{n}{n-k} c_{n,k} \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) |L|^{k/n}
\]

- Intersection bodies, with the best possible constant \( C = 1 \);
  K. 2012 \((k = 1)\), K.- Dan Ma 2013 (for all \( k \))
- Holds with \( C = \sqrt{n} \) for all \( n, L, \mu, k \)
- For every \( \lambda \in (0,1) \) there exists a constant \( C = C(\lambda) \) so that the inequality holds for arbitrary \( n, L, \mu \) and the codimension of sections \( k \geq \lambda n \);
  \[
  C = C_0 \sqrt{(1-\log \lambda)^3}, \quad C_0 \text{ absolute constant}
  \]
- If \( L \) is the unit ball of an \( n \)-dimensional subspace of \( L_p \), \( p > 2 \), the constant can be improved to \( C = n^{1/2-1/p} \).
What is known?

\[ \mu(L) \leq C^k \frac{n}{n-k} c_{n,k} \max_{H \in Gr_{n-k}} \mu(L \cap H) \frac{|L|^{k/n}}{|H|^{k/n}} \]

- Intersection bodies, with the best possible constant \( C = 1 \); K. 2012 (\( k = 1 \)), K.- Dan Ma 2013 (for all \( k \))
- Holds with \( C = \sqrt{n} \) for all \( n, L, \mu, k \)
- For every \( \lambda \in (0, 1) \) there exists a constant \( C = C(\lambda) \) so that the inequality holds for arbitrary \( n, L, \mu \) and the codimension of sections \( k \geq \lambda n \);
  \[ C = C_0 \sqrt{\frac{(1-\log \lambda)^3}{\lambda}} \]
  \( C_0 \) absolute constant
- If \( L \) is the unit ball of an \( n \)-dimensional subspace of \( L_p \), \( p > 2 \), the constant can be improved to \( C = n^{1/2-1/p} \);
- \( m \)-intersection bodies, with \( C = C(m) \) depending only on \( m \);
What is known?

\[ \mu(L) \leq C^k \frac{n}{n-k} c_{n,k} \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) \left| L \right|^{k/n} \]

- Intersection bodies, with the best possible constant \( C = 1 \);
  - K. 2012 \((k = 1)\), K.- Dan Ma 2013 (for all \( k \))
- Holds with \( C = \sqrt{n} \) for all \( n, L, \mu, k \)
- For every \( \lambda \in (0, 1) \) there exists a constant \( C = C(\lambda) \) so that the inequality holds for arbitrary \( n, L, \mu \) and the codimension of sections \( k \geq \lambda n \);
  - \( C = C_0 \sqrt{(1 - \log \lambda)^3 / \lambda} \), \( C_0 \) absolute constant
- If \( L \) is the unit ball of an \( n \)-dimensional subspace of \( L_p \), \( p > 2 \), the constant can be improved to \( C = n^{1/2 - 1/p} \);
- \( m \)-intersection bodies, with \( C = C(m) \) depending only on \( m \);
- Unconditional convex bodies, with \( C = e \);
\[
\mu(L) \leq C^k \frac{n}{n-k} c_{n,k} \max_{H \in \mathcal{G}_{n-k}} \mu(L \cap H) |L|^{k/n}
\]

- Intersection bodies, with the best possible constant \( C = 1 \); K. 2012 \((k = 1)\), K.- Dan Ma 2013 (for all \( k \))
- Holds with \( C = \sqrt{n} \) for all \( n, L, \mu, k \)
- For every \( \lambda \in (0, 1) \) there exists a constant \( C = C(\lambda) \) so that the inequality holds for arbitrary \( n, L, \mu \) and the codimension of sections \( k \geq \lambda n \);
  \[
  C = C_0 \sqrt{\frac{(1-\log \lambda)^3}{\lambda}}, \quad C_0 \text{ absolute constant}
  \]
- If \( L \) is the unit ball of an \( n \)-dimensional subspace of \( L_p \), \( p > 2 \), the constant can be improved to \( C = n^{1/2-1/p} \);
- \( m \)-intersection bodies, with \( C = C(m) \) depending only on \( m \);
- Unconditional convex bodies, with \( C = e \);
- Duals of convex bodies with bounded volume ratio, with an absolute constant \( C \);
What is known?

\[ \mu(L) \leq C^k \frac{n}{n-k} c_{n,k} \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) |L|^{k/n} \]

- Intersection bodies, with the best possible constant \( C = 1 \);
  K. 2012 \((k = 1)\), K.- Dan Ma 2013 (for all \( k \))
- Holds with \( C = \sqrt{n} \) for all \( n, L, \mu, k \)
- For every \( \lambda \in (0,1) \) there exists a constant \( C = C(\lambda) \) so that the inequality holds for arbitrary \( n, L, \mu \) and the codimension of sections \( k \geq \lambda n \);
  \[ C = C_0 \sqrt{(1-\log \lambda)^3 \lambda} \]
  \( C_0 \) absolute constant
- If \( L \) is the unit ball of an \( n \)-dimensional subspace of \( L_p \), \( p > 2 \), the constant can be improved to \( C = n^{1/2-1/p} \);
- \( m \)-intersection bodies, with \( C = C(m) \) depending only on \( m \);
- Unconditional convex bodies, with \( C = e \);
- Duals of convex bodies with bounded volume ratio, with an absolute constant \( C \);
- If \( k = 1 \) and \( \mu \) is log concave, the inequality holds with \( C \sim n^{1/4} \);
A closed bounded set $K$ in $\mathbb{R}^n$ is called a **star body** if every straight line passing through the origin crosses the boundary of $K$ at exactly two points different from the origin, the origin is an interior point of $K$, and the **Minkowski functional** of $K$ defined by $\|x\|_K = \min\{a \geq 0 : x \in aK\}$ is a continuous function on $\mathbb{R}^n$. 

The **radial function** of a star body $K$ is defined by $r_K(x) = \|x\| - 1_K$, $x \in \mathbb{R}^n$. If $x \in S^{n-1}$ then $r_K(x)$ is the radius of $K$ in the direction of $x$. The class of intersection bodies was introduced by Lutwak. We say that a star body $K$ in $\mathbb{R}^n$ is the intersection body of another star body $L$ for every $\xi \in S^{n-1}$, $r_K(\xi) = \|\xi\| - 1_K = \frac{1}{n} \int_{S^{n-1} \cap \xi^\perp} \|\theta\| - n + 1_L \, d\theta = \frac{1}{n} R(\|\cdot\| - n + 1_L)(\xi)$, where $R : C(S^{n-1}) \to C(S^{n-1})$ is the spherical Radon transform $Rf(\xi) = \int_{S^{n-1} \cap \xi^\perp} f(x) \, dx$. 

**Intersection bodies**
A closed bounded set $K$ in $\mathbb{R}^n$ is called a **star body** if every straight line passing through the origin crosses the boundary of $K$ at exactly two points different from the origin, the origin is an interior point of $K$, and the **Minkowski functional** of $K$ defined by $\|x\|_K = \min\{a \geq 0 : x \in aK\}$ is a continuous function on $\mathbb{R}^n$.

The **radial function** of a star body $K$ is defined by $r_K(x) = \|x\|^{-1}_K$, $x \in \mathbb{R}^n$. If $x \in S^{n-1}$ then $r_K(x)$ is the radius of $K$ in the direction of $x$. 

**Intersection bodies**

The class of intersection bodies was introduced by Lutwak. We say that a star body $K$ in $\mathbb{R}^n$ is the intersection body of another star body $L$ for every $\xi \in S^{n-1}$, $r_K(\xi) = \|\xi\|^{-1}_K = |L \cap \xi^\perp| = \frac{1}{n}n^{-1}\int_{S^{n-1} \cap \xi^\perp} \|\theta\| - 1_L \, d\theta = \frac{1}{n}R(\|\cdot\|^{-1}_L)(\xi)$, where $R : C(S^{n-1}) \to C(S^{n-1})$ is the spherical Radon transform: $Rf(\xi) = \int_{S^{n-1} \cap \xi^\perp} f(x) \, dx$. 

**Slicing inequalities for measures of convex bodies.**
A closed bounded set $K$ in $\mathbb{R}^n$ is called a **star body** if every straight line passing through the origin crosses the boundary of $K$ at exactly two points different from the origin, the origin is an interior point of $K$, and the **Minkowski functional** of $K$ defined by $\|x\|_K = \min\{a \geq 0 : x \in aK\}$ is a continuous function on $\mathbb{R}^n$.

The **radial function** of a star body $K$ is defined by $r_K(x) = \|x\|_K^{-1}$, $x \in \mathbb{R}^n$. If $x \in S^{n-1}$ then $r_K(x)$ is the radius of $K$ in the direction of $x$.

The class of intersection bodies was introduced by Lutwak. We say that a star body $K$ in $\mathbb{R}^n$ is the **intersection body of** another star body $L$ for every $\xi \in S^{n-1}$,

$$r_K(\xi) = \|\xi\|_K^{-1} = |L \cap \xi^\perp|$$
A closed bounded set $K$ in $\mathbb{R}^n$ is called a **star body** if every straight line passing through the origin crosses the boundary of $K$ at exactly two points different from the origin, the origin is an interior point of $K$, and the **Minkowski functional** of $K$ defined by $\|x\|_K = \min\{a \geq 0 : x \in aK\}$ is a continuous function on $\mathbb{R}^n$.

The **radial function** of a star body $K$ is defined by $r_K(x) = \|x\|_K^{-1}$, $x \in \mathbb{R}^n$. If $x \in S^{n-1}$ then $r_K(x)$ is the radius of $K$ in the direction of $x$.

The class of intersection bodies was introduced by Lutwak. We say that a star body $K$ in $\mathbb{R}^n$ is the **intersection body of** another star body $L$ for every $\xi \in S^{n-1}$,

$$r_K(\xi) = \|\xi\|_K^{-1} = |L \cap \xi^\perp|$$

$$= \frac{1}{n-1} \int_{S^{n-1} \cap \xi^\perp} \|\theta\|_L^{-n+1} d\theta = \frac{1}{n-1} R \left( \| \cdot \|^{-n+1}_L \right)(\xi),$$

where $R : C(S^{n-1}) \to C(S^{n-1})$ is the **spherical Radon transform**:

$$Rf(\xi) = \int_{S^{n-1} \cap \xi^\perp} f(x) dx.$$
If $\mu$ is a finite Borel measure on $S^{n-1}$, then $R\mu$ is defined by

$$(R\mu, f) = (\mu, Rf) = \int_{S^{n-1}} Rf(x) d\mu(x), \quad \forall f \in C(S^{n-1}).$$

A star body $K$ in $\mathbb{R}^n$ is called an **intersection body** if $\|\cdot\|^{-1}_K = R\mu$ for some measure $\mu$, i.e.

$$\int_{S^{n-1}} \|x\|^{-1}_K f(x) dx = \int_{S^{n-1}} Rf(x) d\mu(x), \quad \forall f \in C(S^{n-1}).$$
If $\mu$ is a finite Borel measure on $S^{n-1}$, then $R\mu$ is defined by

$$(R\mu, f) = (\mu, Rf) = \int_{S^{n-1}} Rf(x) d\mu(x), \quad \forall f \in C(S^{n-1}).$$

A star body $K$ in $\mathbb{R}^n$ is called an **intersection body** if $\|\cdot\|^{-1}_K = R\mu$ for some measure $\mu$, i.e.

$$\int_{S^{n-1}} \|x\|^{-1}_K f(x) dx = \int_{S^{n-1}} Rf(x) d\mu(x), \quad \forall f \in C(S^{n-1}).$$

For $1 \leq k \leq n-1$, the **$(n-k)$-dimensional spherical Radon transform** $R_{n-k} : C(S^{n-1}) \to C(Gr_{n-k})$ is a linear operator defined by

$$R_{n-k} g(H) = \int_{S^{n-1}\cap H} g(x) \, dx, \quad \forall H \in Gr_{n-k}$$

for every function $g \in C(S^{n-1})$. 

**Generalized intersection bodies**

Alexander Koldobsky
Slicing inequalities for measures of convex bodies.
If $\mu$ is a finite Borel measure on $S^{n-1}$, then $R\mu$ is defined by

$$(R\mu, f) = (\mu, Rf) = \int_{S^{n-1}} Rf(x) d\mu(x), \quad \forall f \in C(S^{n-1}).$$

A star body $K$ in $\mathbb{R}^n$ is called an intersection body if $\| \cdot \|_K^{-1} = R\mu$ for some measure $\mu$, i.e.

$$\int_{S^{n-1}} \| x \|_K^{-1} f(x) dx = \int_{S^{n-1}} Rf(x) d\mu(x), \quad \forall f \in C(S^{n-1}).$$

For $1 \leq k \leq n-1$, the $(n-k)$-dimensional spherical Radon transform $R_{n-k} : C(S^{n-1}) \to C(Gr_{n-k})$ is a linear operator defined by

$$R_{n-k}g(H) = \int_{S^{n-1} \cap H} g(x) \, dx, \quad \forall H \in Gr_{n-k}$$

for every function $g \in C(S^{n-1})$.

Following Zhang, we say that an origin symmetric star body $K$ in $\mathbb{R}^n$ is a generalized $k$-intersection body ($K \in BP^n_k$) if there exists a finite Borel non-negative measure $\mu$ on $Gr_{n-k}$ so that for every $g \in C(S^{n-1})$

$$\int_{S^{n-1}} \| x \|_K^{-k} g(x) \, dx = \int_{Gr_{n-k}} R_{n-k}g(H) d\mu(H).$$
Stability


Suppose that \(1 \leq k \leq n-1\), \(K\) is a generalized \(k\)-intersection body in \(\mathbb{R}^n\), \(f\) is an even continuous function on \(K\), \(f \geq 0\) everywhere on \(K\), and \(\varepsilon > 0\). If

\[
\int_{K \cap H} f \leq \varepsilon, \quad \forall H \in \text{Gr}_{n-k},
\]

then

\[
\int_K f \leq \frac{n}{n-k} c_{n,k} |K|^{k/n} \varepsilon.
\]

Suppose that $1 \leq k \leq n-1$, $K$ is a generalized $k$-intersection body in $\mathbb{R}^n$, $f$ is an even continuous function on $K$, $f \geq 0$ everywhere on $K$, and $\varepsilon > 0$. If

$$\int_{K \cap H} f \leq \varepsilon, \quad \forall H \in \text{Gr}_{n-k},$$

then

$$\int_K f \leq \frac{n}{n-k} c_{n,k} \frac{|K|^{k/n}}{\varepsilon}.$$

For a convex body $L$ in $\mathbb{R}^n$ and $1 \leq k < n$, denote by

$$\text{o.v.r.}(L, \mathcal{B}\mathcal{P}_k^n) = \inf \left\{ \left( \frac{|K|}{|L|} \right)^{1/n} : L \subset K, \ K \in \mathcal{B}\mathcal{P}_k^n \right\}$$

the outer volume ratio distance from a body $K$ to the class $\mathcal{B}\mathcal{P}_k^n$. 
Corollary

Let $L$ be an origin-symmetric star body in $\mathbb{R}^n$. Then for any measure $\mu$ with even continuous density on $L$ we have

$$
\mu(L) \leq \left(\text{o.v.r.}(L, \mathcal{B} \mathcal{P}^n_k)\right)^k \frac{n}{n-k} c_{n,k} \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) \frac{|L|^k}{n}.
$$
Corollary

Let $L$ be an origin-symmetric star body in $\mathbb{R}^n$. Then for any measure $\mu$ with even continuous density on $L$ we have

$$
\mu(L) \leq \left(\text{o.v.r.}(L, \mathcal{B} \mathcal{P}_k^n)\right)^k \frac{n}{n-k} c_{n,k} \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) |L|^{k/n}.
$$

**Proof:** Let $C > \text{o.v.r.}(L, \mathcal{B} \mathcal{P}_k^n)$, then there exists a body $K$ in $\mathcal{B} \mathcal{P}_k^n$ such that $L \subset K$ and $|K|^{1/n} \leq C |L|^{1/n}$. 

Corollary

Let $L$ be an origin-symmetric star body in $\mathbb{R}^n$. Then for any measure $\mu$ with even continuous density on $L$ we have

$$\mu(L) \leq \left( \text{o.v.r.}(L, \mathcal{B}^n_P) \right)^k \frac{n}{n-k} c_{n,k} \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) |L|^{k/n}.$$ 

Proof: Let $C > \text{o.v.r.}(L, \mathcal{B}^n_P)$, then there exists a body $K$ in $\mathcal{B}^n_P$ such that $L \subset K$ and $|K|^{1/n} \leq C |L|^{1/n}$.

Let $g$ be the density of the measure $\mu$, and define a function $f$ on $K$ by $f = g \chi_L$, where $\chi_L$ is the indicator function of $L$. Clearly, $f \geq 0$ everywhere on $K$. Put

$$\varepsilon = \max_{H \in \text{Gr}_{n-k}} \int_{K \cap H} f = \max_{H \in \text{Gr}_{n-k}} \int_{L \cap H} g = \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H)$$

and apply Stability Theorem to $f, K, \varepsilon$ ($f$ is not continuous, but we can do an easy approximation).
Corollary

Let $L$ be an origin-symmetric star body in $\mathbb{R}^n$. Then for any measure $\mu$ with even continuous density on $L$ we have

$$\mu(L) \leq \left(\text{o.v.r.}(L, \mathcal{B} \mathcal{P}_k^n)\right)^k \frac{n}{n-k} c_{n,k} \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) |L|^{k/n}.$$

**Proof:** Let $C > \text{o.v.r.}(L, \mathcal{B} \mathcal{P}_k^n)$, then there exists a body $K$ in $\mathcal{B} \mathcal{P}_k^n$ such that $L \subset K$ and $|K|^{1/n} \leq C |L|^{1/n}$.

Let $g$ be the density of the measure $\mu$, and define a function $f$ on $K$ by $f = g \chi_L$, where $\chi_L$ is the indicator function of $L$. Clearly, $f \geq 0$ everywhere on $K$. Put

$$\varepsilon = \max_{H \in \text{Gr}_{n-k}} \int_{K \cap H} f = \max_{H \in \text{Gr}_{n-k}} \int_{L \cap H} g = \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H)$$

and apply Stability Theorem to $f, K, \varepsilon$ ($f$ is not continuous, but we can do an easy approximation).

We have

$$\mu(L) = \int_L g = \int_K f \leq \frac{n}{n-k} c_{n,k} |K|^{k/n} \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H)$$

$$\leq C^k \frac{n}{n-k} c_{n,k} |L|^{k/n} \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H).$$

The result follows by sending $C$ to $\text{o.v.r.}(L, \mathcal{B} \mathcal{P}_k^n)$.\[\square\]
Let $L$ be an origin-symmetric convex body in $\mathbb{R}^n$. By John’s theorem, there exists an origin-symmetric ellipsoid $K$ such that $\frac{1}{\sqrt{n}}K \subset L \subset K$. 

Every ellipsoid in $\mathbb{R}^n$ is a generalized $k$-intersection body for every $1 \leq k < n$ (Grinberg-Zhang, 1999).

We get that for every $k$-o.

$v$. $r$. $(L, BP_n^k) \leq \sqrt{n}$.

By Corollary, $\mu(L) \leq \frac{n^k}{2^n} n^n - k c n$.
Let $L$ be an origin-symmetric convex body in $\mathbb{R}^n$. By John’s theorem, there exists an origin-symmetric ellipsoid $K$ such that $\frac{1}{\sqrt{n}}K \subset L \subset K$.

Every ellipsoid in $\mathbb{R}^n$ is a generalized $k$-intersection body for every $1 \leq k < n$ (Grinberg-Zhang, 1999).
Let $L$ be an origin-symmetric convex body in $\mathbb{R}^n$. By John’s theorem, there exists an origin-symmetric ellipsoid $K$ such that $\frac{1}{\sqrt{n}}K \subset L \subset K$.

Every ellipsoid in $\mathbb{R}^n$ is a generalized $k$-intersection body for every $1 \leq k < n$ (Grinberg-Zhang, 1999).

We get that for every $k$

$$\text{o.v.r.}(L, \mathcal{B}_k^n) \leq \sqrt{n}.$$
Let $L$ be an origin-symmetric convex body in $\mathbb{R}^n$. By John’s theorem, there exists an origin-symmetric ellipsoid $K$ such that $\frac{1}{\sqrt{n}}K \subset L \subset K$.

Every ellipsoid in $\mathbb{R}^n$ is a generalized $k$-intersection body for every $1 \leq k < n$ (Grinberg-Zhang, 1999).

We get that for every $k$

$$\text{o.v.r.}(L, \mathcal{BP}_k^n) \leq \sqrt{n}.$$ 

By Corollary,

$$\mu(L) \leq n^{k/2} \frac{n}{n-k} c_{n,k} \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) |L|^{k/n}.$$
Let $e_i$, $1 \leq i \leq n$, be the standard basis of $\mathbb{R}^n$. A convex body $K$ in $\mathbb{R}^n$ is called unconditional if for every choice of real numbers $x_i$ and $\delta_i = \pm 1$, $1 \leq i \leq n$ we have $\| \sum_{i=1}^{n} \delta_i x_i e_i \|_K = \| \sum_{i=1}^{n} x_i e_i \|_K$. 

By a result of Lozanovskii, there exists a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ so that $T(\mathbb{B}_n^\infty) \subset L \subset nT(\mathbb{B}_n^1)$, where $\mathbb{B}_n^1$ and $\mathbb{B}_n^\infty$ are the unit balls of the spaces $\ell_n^1$ and $\ell_n^\infty$, respectively.

Let $K = nT(\mathbb{B}_n^1)$. By K. 1998, $\mathbb{B}_n^1$ is an intersection body. A linear transformation of an intersection body is an intersection body, so the body $K$ is an intersection body in $\mathbb{R}^n$. By a result of Grinberg and Zhang, $K$ is a generalized $k$-intersection body for every $1 \leq k < n$.

Since $|\mathbb{B}_n^1| = 2^n/n!$, we have $|K|^{1/n} \leq 2e|\det T|^{1/n}$. On the other hand, $|T(\mathbb{B}_n^\infty)| = 2^n |\det T|$, and $T(\mathbb{B}_n^\infty) \subset L$, so $|K|^{1/n} \leq e|\mu|^{1/n}$.

Therefore, by Corollary, $\mu(L) \leq e^k n^{n-k} c_n$, $k_{\text{max}} H \in \text{Gr}_{n-k} \mu(L) \leq |L|^{k/n}$. 

Alexander Koldobsky
Slicing inequalities for measures of convex bodies.
Unconditional bodies

Let $e_i$, $1 \leq i \leq n$, be the standard basis of $\mathbb{R}^n$. A convex body $K$ in $\mathbb{R}^n$ is called unconditional if for every choice of real numbers $x_i$ and $\delta_i = \pm 1$, $1 \leq i \leq n$ we have $\| \sum_{i=1}^{n} \delta_i x_i e_i \|_K = \| \sum_{i=1}^{n} x_i e_i \|_K$.

By a result of Lozanovskii, there exists a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$T(B^n_{\infty}) \subset L \subset nT(B^n_1),$$

where $B^n_1$ and $B^n_{\infty}$ are the unit balls of the spaces $\ell^n_1$ and $\ell^n_{\infty}$, respectively.
Let $e_i, 1 \leq i \leq n$, be the standard basis of $\mathbb{R}^n$. A convex body $K$ in $\mathbb{R}^n$ is called unconditional if for every choice of real numbers $x_i$ and $\delta_i = \pm 1, 1 \leq i \leq n$ we have $\|\sum_{i=1}^{n} \delta_i x_i e_i\|_K = \|\sum_{i=1}^{n} x_i e_i\|_K$.

By a result of Lozanovskii, there exists a linear operator $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ so that

$$T(B_1^n) \subset L \subset nT(B_1^n),$$

where $B_1^n$ and $B_\infty^n$ are the unit balls of the spaces $\ell_1^n$ and $\ell_\infty^n$, respectively.

Let $K = nT(B_1^n)$. By K. 1998, $B_1^n$ is an intersection body. A linear transformation of an intersection body is an intersection body, so the body $K$ is an intersection body in $\mathbb{R}^n$. By a result of Grinberg and Zhang, $K$ is a generalized $k$-intersection body for every $1 \leq k < n$. 
Let $e_i$, $1 \leq i \leq n$, be the standard basis of $\mathbb{R}^n$. A convex body $K$ in $\mathbb{R}^n$ is called unconditional if for every choice of real numbers $x_i$ and $\delta_i = \pm 1$, $1 \leq i \leq n$ we have $\| \sum_{i=1}^{n} \delta_i x_i e_i \|_K = \| \sum_{i=1}^{n} x_i e_i \|_K$.

By a result of Lozanovskii, there exists a linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ so that

$$T(B_1^n) \subset L \subset nT(B_1^n),$$

where $B_1^n$ and $B_\infty^n$ are the unit balls of the spaces $\ell_1^n$ and $\ell_\infty^n$, respectively.

Let $K = nT(B_1^n)$. By K. 1998, $B_1^n$ is an intersection body. A linear transformation of an intersection body is an intersection body, so the body $K$ is an intersection body in $\mathbb{R}^n$. By a result of Grinberg and Zhang, $K$ is a generalized $k$-intersection body for every $1 \leq k < n$.

Since $|B_1^n| = 2^n/n!$, we have $|K|^{1/n} \leq 2e|\det T|^{1/n}$. On the other hand, $|T(B_\infty^n)| = 2^n|\det T|$, and $T(B_\infty^n) \subset L$, so $|K|^{1/n} \leq e|L|^{1/n}$. Therefore, $o.v.r(L, B_P^n) \leq e$. 

Unconditional bodies
Unconditional bodies

Let \( e_i, 1 \leq i \leq n \), be the standard basis of \( \mathbb{R}^n \). A convex body \( K \) in \( \mathbb{R}^n \) is called unconditional if for every choice of real numbers \( x_i \) and \( \delta_i = \pm 1, 1 \leq i \leq n \) we have \( \| \sum_{i=1}^{n} \delta_i x_i e_i \|_K = \| \sum_{i=1}^{n} x_i e_i \|_K \).

By a result of Lozanovskii, there exists a linear operator \( T : \mathbb{R}^n \to \mathbb{R}^n \) so that

\[ T(B_1^n) \subset L \subset nT(B_1^n), \]

where \( B_1^n \) and \( B_\infty^n \) are the unit balls of the spaces \( \ell_1^n \) and \( \ell_\infty^n \), respectively.

Let \( K = nT(B_1^n) \). By K. 1998, \( B_1^n \) is an intersection body. A linear transformation of an intersection body is an intersection body, so the body \( K \) is an intersection body in \( \mathbb{R}^n \). By a result of Grinberg and Zhang, \( K \) is a generalized \( k \)-intersection body for every \( 1 \leq k < n \).

Since \( |B_1^n| = 2^n / n! \), we have \( |K|^{1/n} \leq 2e|\det T|^{1/n} \). On the other hand, \( |T(B_\infty^n)| = 2^n |\det T| \), and \( T(B_\infty^n) \subset L \), so \( |K|^{1/n} \leq e |L|^{1/n} \). Therefore, \( o.v.r(L, B^P_k) \leq e \). By Corollary,

\[ \mu(L) \leq e^k \frac{n}{n-k} c_{n,k} \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) |L|^{k/n}. \]
Theorem (K., Paouris and Zymonopoulou, 2011)

Let $L$ be an origin-symmetric convex body in $\mathbb{R}^n$, and let $1 \leq k \leq n - 1$. Then

$$\operatorname{o.v.r.}(L, \mathcal{B}\mathcal{P}_k^n) \leq C_0 \sqrt{\frac{n}{k}} \left( \log \left( e \frac{n}{k} \right) \right)^{3/2},$$

where $C_0$ is an absolute constant.
Lower dimensional sections

**Theorem (K., Paouris and Zymonopoulou, 2011)**

Let $L$ be an origin-symmetric convex body in $\mathbb{R}^n$, and let $1 \leq k \leq n - 1$. Then

$$o.v.r. (L, B\mathcal{P}_k^n) \leq C_0 \sqrt{\frac{n}{k}} \left( \log \left( \frac{e n}{k} \right) \right)^{3/2},$$

where $C_0$ is an absolute constant.

**Corollary**

If the codimension of sections $k$ satisfies $\lambda n \leq k < n$, for some $\lambda \in (0, 1)$, then for every origin-symmetric convex body $L$ in $\mathbb{R}^n$ and every measure $\mu$ with continuous non-negative density in $\mathbb{R}^n$,

$$\mu(L) \leq \left( C_0 \sqrt{\frac{(1 - \log \lambda)^3}{\lambda}} \right)^k \frac{n}{n - k} c_{n,k} \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) |L|^{k/n},$$

where $C_0$ is an absolute constant.
Theorem (K., Paouris and Zymonopoulou, 2011)

Let $L$ be an origin-symmetric convex body in $\mathbb{R}^n$, and let $1 \leq k \leq n-1$. Then

$$\text{o.v.r.}(L, \mathcal{B}^n_k) \leq C_0 \sqrt{\frac{n}{k}} \left( \log \left( \frac{e}{\sqrt{k}} \right) \right)^{3/2},$$  

where $C_0$ is an absolute constant.

Corollary

If the codimension of sections $k$ satisfies $\lambda n \leq k < n$, for some $\lambda \in (0, 1)$, then for every origin-symmetric convex body $L$ in $\mathbb{R}^n$ and every measure $\mu$ with continuous non-negative density in $\mathbb{R}^n$,

$$\mu(L) \leq \left( C_0 \sqrt{\frac{(1-\log \lambda)^3}{\lambda}} \right)^k \frac{n}{n-k} c_{n,k} \max_{H \in \text{Gr}_{n-k}} \mu(L \cap H) |L|^{k/n},$$

where $C_0$ is an absolute constant. In particular, if $\lambda n \leq k$,

$$|L|^{\frac{n-k}{n}} \leq C(\lambda)^k \max_{H \in \text{Gr}_{n-k}} |L \cap H|.$$
If $\mu$ is a measure on $K$ with even continuous density $f$, then

$$\mu(K) = \int_{K} f = \int_{S^{n-1}} \left( \int_{0}^{\|\theta\|_{K}^{-1}} r^{n-1} f(r\theta) \, dr \right) \, d\theta.$$ 

and

$$\mu(K \cap H) = \int_{K \cap H} f = \int_{S^{n-1} \cap H} \left( \int_{0}^{\|\theta\|_{K}^{-1}} r^{n-k-1} f(r\theta) \, dr \right) \, d\theta$$

$$= R_{n-k} \left( \int_{0}^{\|\cdot\|_{K}^{-1}} r^{n-k-1} f(r \cdot) \, dr \right) (H).$$
Proof of the Stability Theorem. Part 2

Stability Theorem: Suppose that $1 \leq k \leq n-1$, $K$ is a generalized $k$-intersection body in $\mathbb{R}^n$, $f$ is an even continuous function on $K$, $f \geq 0$ everywhere on $K$, and $\varepsilon > 0$. If
\[ \int_{K \cap H} f \leq \varepsilon, \quad \forall H \in \text{Gr}_{n-k}, \]
then
\[ \int_K f \leq \frac{n}{n-k} c_{n,k} |K|^{k/n} \varepsilon. \]
Proof of the Stability Theorem. Part 2

Stability Theorem: Suppose that $1 \leq k \leq n-1$, $K$ is a generalized $k$-intersection body in $\mathbb{R}^n$, $f$ is an even continuous function on $K$, $f \geq 0$ everywhere on $K$, and $\varepsilon > 0$. If

$$
\int_{K \cap H} f \leq \varepsilon, \quad \forall H \in \text{Gr}_{n-k},
$$

then

$$
\int_K f \leq \frac{n}{n-k} c_{n,k} |K|^{k/n} \varepsilon.
$$

Proof: By the polar formula

$$
R_{n-k} \left( \int_0^{\|\cdot\|^n_K^{-1}} r^{n-k-1} f(r \cdot) \, dr \right) (H) \leq \varepsilon.
$$

Integrate both sides with respect to the measure $\mu$ on $\text{Gr}_{n-k}$ that corresponds to $K$ as a generalized $k$-intersection body. Recall that for any $g \in C(S^{n-1})$

$$
\int_{S^{n-1}} \|x\|_K^{-k} g(x) \, dx = \int_{\text{Gr}_{n-k}} R_{n-k} g(H) d\mu(H).
$$
We get

\[
\int_{S^{n-1}} \|\theta\|_{K}^{-k} \left( \int_{0}^{\|\theta\|_{K}^{-1}} r^{n-k-1} f(r \theta) \, dr \right) \, d\theta \leq \varepsilon \mu(Gr_{n-k}).
\]
We get

\[ \int_{S^{n-1}} \|\theta\|_K^{-k} \left( \int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta) \, dr \right) d\theta \leq \varepsilon \mu(Gr_{n-k}). \]

Estimate the integral in the left-hand side from below using \( f \geq 0 \).

\[ \int_{K} f = \int_{S^{n-1}} \left( \int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta) \, dr \right) d\theta \]

\[ \leq \int_{S^{n-1}} \left( \int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta) \, dr \right) d\theta \]

\[ + \int_{S^{n-1}} \left( \int_0^{\|\theta\|_K^{-1}} (\|\theta\|_K^{-k} - r^k) r^{n-k-1} f(r\theta) \, dr \right) d\theta \]

\[ = \int_{S^{n-1}} \|\theta\|_K^{-k} \left( \int_0^{\|\theta\|_K^{-1}} r^{n-k-1} f(r\theta) \, dr \right) d\theta. \]
Now we estimate $\mu(Gr_{n-k})$ from above. We use $1 = R_{n-k}1(H)/|S^{n-k-1}|$ for every $H \in Gr_{n-k}$.
Now we estimate $\mu(Gr_{n-k})$ from above. We use $1 = R_{n-k}1(H)/|S^{n-k-1}|$ for every $H \in Gr_{n-k}$.

$$\mu(Gr_{n-k}) = \frac{1}{|S^{n-k-1}|} \int_{Gr_{n-k}} R_{n-k}1(H) d\mu(H)$$

$$= \frac{1}{|S^{n-k-1}|} \int_{S^{n-1}} \|\theta\|_{K}^{-k} d\theta$$

$$\leq \frac{1}{|S^{n-k-1}|} |S^{n-1}|^{\frac{n-k}{n}} \left( \int_{S^{n-1}} \|\theta\|_{K}^{-n} d\theta \right)^{\frac{k}{n}}$$

$$= \frac{1}{|S^{n-k-1}|} |S^{n-1}|^{\frac{n-k}{n}} n^{k/n} |K|^{k/n} = \frac{n}{n-k} c_{n,k} |K|^{k/n}.$$
Now we estimate $\mu(Gr_{n-k})$ from above. We use $1 = R_{n-k}1(H)/|S^{n-k-1}|$ for every $H \in Gr_{n-k}$.

$$
\mu(Gr_{n-k}) = \frac{1}{|S^{n-k-1}|} \int_{Gr_{n-k}} R_{n-k}1(H)d\mu(H)
$$

$$
= \frac{1}{|S^{n-k-1}|} \int_{S^{n-1}} \|\theta\|^{-k}_K d\theta
$$

$$
\leq \frac{1}{|S^{n-k-1}|} |S^{n-1}|^{-\frac{n-k}{n}} \left( \int_{S^{n-1}} \|\theta\|^{-n}_K d\theta \right)^{\frac{k}{n}}
$$

$$
= \frac{1}{|S^{n-k-1}|} |S^{n-1}|^{-\frac{n-k}{n}} n^{k/n} |K|^{k/n} = \frac{n}{n-k} c_{n,k} |K|^{k/n}.
$$

Combining the estimates,

$$
\int_K f \leq \frac{n}{n-k} c_{n,k} |K|^{k/n} \varepsilon.
$$