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On log-concave functions

Analytic Tools in Probability and Applications
IMA – University of Minnesota

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Outline of the talk
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- Log-concave functions and their addition
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- The Prékopa-Leindler inequality and its infinitesimal form
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- Log-concave functions and their addition
- The Prékopa-Leindler inequality and its infinitesimal form
- Functional forms of Blaschke-Santaló and Rogers-Shephard inequalities
- Valuations on log-concave functions
Log-concave functions

A function $f : \mathbb{R}^n \rightarrow \mathbb{R}^+$ is said to be log-concave if $f = e^{-u}$ where $u : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex ($e^{-\infty} = 0$). We will always assume $\lim_{|x| \rightarrow +\infty} u(x) = +\infty \Rightarrow \lim_{|x| \rightarrow +\infty} f(x) = 0$, and set $C_n = \{f \text{ log-concave on } \mathbb{R}^n : \lim_{|x| \rightarrow +\infty} f(x) = 0\}$.
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\[ C^n = \{ f \text{ log-concave on } \mathbb{R}^n : \lim_{|x| \to +\infty} f(x) = 0 \}. \]
Examples

- The Gaussian function:
  \[ f(x) = e^{-\|x\|^2/2} \]

- The characteristic function of a convex body \( K \) (compact convex subset of \( \mathbb{R}^n \)):
  \[ f(x) = \chi_K(x) = \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{if } x \not\in K. \end{cases} \]

Note that \( f = e^{-I_K} \) where \( I_K(x) = \begin{cases} 0 & \text{if } x \in K, \\ +\infty & \text{if } x \not\in K. \end{cases} \)
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= indicatrix function of \(K\).
Motivations

Log-concave functions are natural objects to study, in analysis and in probability. ▶ Probability measures with a log-concave density.

Isoperimetry; spectral gap (or Poincaré) inequalities; log-Sobolev inequalities; concentration phenomena; central limit theorems...

▶ Geometrization of analysis.

Study log-concave functions in parallel with the theory of convex bodies (convex geometry). Functional forms of classical geometric inequalities in convex geometry; duality; functional versions of mixed and intrinsic volumes; valuations...

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Addition of log-concave functions

For $f, g \in \mathcal{C}_n$ we set

$$(f \oplus g)(z) = \sup \left\{ f(x)g(y) \mid x + y = z \right\}$$

Consistently, for $t > 0$ we define

$$(t \cdot f)(z) = f(tz)$$

(so that, e.g., $f \oplus f = 2 \cdot f$).

$\mathcal{C}_n$ is closed with respect to these operations.
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Minkowski addition and $\oplus$

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In other words, $\oplus$ extends the Minkowski addition at the level of log-concave functions.
More on ⊕

$$(f \oplus g)(z) = \sup \{f(x) g(y) | x + y = z\}$$

If we write $f = e^{-u}$ and $g = e^{-v}$ then $f \oplus g = e^{-w}$ where $w(z) = \inf \{u(x) + v(y) | x + y = z\} = \inf\text{-convolution of } u \text{ and } v = \text{conv}\langle u, v \rangle$ (extensively studied in Rockafellar's monograph Convex analysis).

Note that $\text{epi}(\text{conv}\langle u, v \rangle) = \text{epi}(u) + \text{epi}(v)$. 
More on $\oplus$

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Note that

$$\text{epi}(u \boxplus v) = \text{epi}(u) + \text{epi}(v).$$
The role of the conjugate function

Another property of inf-convolution:

\[(u \boxplus v) = u^* + v^*\]

where \(u^*\) is the usual conjugate of convex functions:

\[u^*(y) = \sup_{x \in \mathbb{R}^n} \left( (x, y) - u(x) \right)\]

Summarizing,

\[f \boxplus g = e^{-u} \boxplus e^{-v} = e^{-u \boxplus v} = e^{-u^* + v^*} = e^{-u^*} \boxplus e^{-v^*},\]

i.e. \(\boxplus\) coincides with the usual addition applied to the conjugate of the exponent (with sign changed) of log-concave functions.
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The total mass of a log-concave function

\[ I(f) = \int_{\mathbb{R}^n} f(x) \, dx. \]

It is not difficult to see that our assumptions imply \( 0 \leq I(f) < \infty \) \( \forall f \).

\( I(f) \) is the counterpart of the volume (Lebesgue measure) of a convex body \( K \), in convex geometry, denoted by \( V_n(K) \):

\[ V_n(K) = \text{volume of } K. \]
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The Prékopa-Leindler inequality

For every \( f, g \) log-concave and for every \( t \in [0, 1] \)
\[
\int_{\mathbb{R}^n} \left[ (1-t) \cdot f \oplus t \cdot g \right] \, dz \geq \left( \int_{\mathbb{R}^n} f \, dx \right)^{1-t} \left( \int_{\mathbb{R}^n} g \, dy \right)^{t} \quad \text{(PL)}
\]
(Prékopa, 1971; Leindler 1972).

Equality holds (up to some degenerate cases) iff
\( g(y) = f(\alpha(y-y_0)), \forall y \in \mathbb{R}^n \),
for some \( y_0 \in \mathbb{R}^n \) and \( \alpha > 0 \) (Dubuc, 1979).

\( f \to \log(I(f)) \) is concave.
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$$f \rightarrow \log(I(f)) \quad \text{is concave.}$$
Prékopa-Leindler and Brunn-Minkowski inequality

\[ \int_{\mathbb{R}^n} \left[ (1-t) \cdot f \oplus t \cdot g \right] \, dz \geq \left( \int_{\mathbb{R}^n} f \, dx \right)^{1-t} \left( \int_{\mathbb{R}^n} g \, dy \right)^{t} \]  

For \( f = \chi_K \) and \( g = \chi_L \) (\( K \) and \( L \) convex bodies) we get:

\[ V_n((1-t)K + tL) \geq V_n(K)^{1-t}V_n(L)^t, \]  

and then, by a simple argument based on homogeneity of volume,

\[ V_n((1-t)K + tL)^{1/n} \geq (1-t)V_n(K)^{1/n} + tV_n(L)^{1/n}. \]  

(BM) is among the most important results in convex geometry. It is connected to the isoperimetric inequality and to the family of Aleksandrov-Fenchel inequalities. See the exhaustive survey by Gardner: The Brunn-Minkowski inequality, 2002.
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More remarks about (PL)

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It can be viewed as a reverse form of the Hölder inequality, recalling that

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(1 - t) \cdot f \oplus t \cdot g (z) = \sup_{(1 - t) \cdot x + ty = z} f^{1 - t} (x) \cdot g^t (y).
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It is a special case of a family of inequalities proved by Barthe, which reverse the Borell-Brascamp-Lieb inequalities. It is also a limit case of the reverse Young inequalities for convolutions.
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From (PL) to Poincaré and Sobolev inequalities
Bobkov and Ledoux (2000) showed how to obtain a Poincaré type inequality due to Brascamp and Lieb (see next slide), and the log-Sobolev inequality starting from the (PL) inequality. In 2007 they also provided a proof of the Sobolev inequality via the Brunn-Minkowski inequality.
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Bobkov and Ledoux (2000) showed how to obtain a Poincaré type inequality due to Brascamp and Lieb (see next slide), and the log-Sobolev inequality starting from the (PL) inequality. In 2007 they also provided a proof of the Sobolev inequality via the Brunn-Minkowski inequality.

In general it is now well understood that concavity inequalities like (PL) and (BM) can be used to prove integral inequalities involving derivatives.
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In general it is now well understood that concavity inequalities like (PL) and (BM) can be used to prove integral inequalities involving derivatives.

More ambitiously, we would like to see that (PL) is in fact equivalent to a class of integral inequalities of Poincaré type (precisely those of Brascamp and Lieb).
A Poincaré inequality for log-concave measures
by Brascamp and Lieb
Thm. (Brascamp-Lieb, ’76). Let $\mu$ be a probability measure on $\mathbb{R}^n$ with density $f$. Assume that

$$f = e^{-u}, \quad u \in C^2(\mathbb{R}^n),$$

Then for every $\phi \in C^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} (\phi - \int_{\mathbb{R}^n} \phi \, d\mu)^2 \, d\mu \leq \int_{\mathbb{R}^n} \left( (D^2 u - 1) \nabla \phi, \nabla \phi \right) \, d\mu.$$
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**Remark.** This result applies to the Gaussian measure $\gamma_n$ and provides the standard Poincaré inequality for this measure.
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The infinitesimal form of (PL)

This is equivalent to say that

\[ J: \mathbb{C}^n \to \mathbb{R}, \quad J(f) = \log (\int_{\mathbb{R}^n} f dx) \]

is concave; hence \( J''(f) \leq 0, \forall f \).

Claim:

\[ J''(f) \leq 0 \iff (BL) \text{ for the measure } \mu: d\mu = f \text{ (whenever } f = e^{-u}, D^2u > 0). \]
The infinitesimal form of (PL)

\[
\int_{\mathbb{R}^n} [(1 - t) \cdot f \oplus t \cdot g] dz \geq \left( \int_{\mathbb{R}^n} f dx \right)^{1-t} \left( \int_{\mathbb{R}^n} g dy \right)^t
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A formal computation of $J''(f)$
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\( J''(f) \) can be thought as a bilinear quadratic form acting on test functions \( h \), defined as follows

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(J''(f)h, h) = \left. \frac{d^2}{dt^2} J(f \oplus t \cdot h) \right|_{t=0}.
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A formal computation of $J''(f)$

$J''(f)$ can be thought as a bilinear quadratic form acting on test functions $h$, defined as follows

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The condition

$$(J''(f)h, h) \leq 0 \quad \forall h$$

turns out to be exactly the Poincaré inequality of Brascamp and Lieb.
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In practice: write $l(f)$ in terms of $u^*$

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\nabla u = (\nabla u^*)^{-1}, \quad D^2u(x) = [D^2u^*(\nabla u(x))]^{-1}.
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$$u(x) = x \cdot \nabla u(x) - u^*(\nabla u(x)).$$
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In practice: write $I(f)$ in terms of $u^*$

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$$= \int_{\mathbb{R}^n} e^{u^*(y) - y \cdot \nabla u^*(y)} \det(D^2u^*(y)) dy.$$
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for an arbitrary test function \( \psi \). This condition plus the reverse change of variable \( y = \nabla u(x) \) lead to the inequality by Brascamp and Lieb.
Remarks

- A similar procedure shows that the Brunn-Minkowski inequality is equivalent to a family of weighted Poincaré inequalities on the unit sphere ([C. ’08]).
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- Kolesnikov and Milman (’14) extended these inequalities to a certain class of Riemannian manifolds and used their result to establish a (BM) inequality in this setting.
- Many inequalities of (BM) type (proved or conjectured) admit an infinitesimal version, typically consisting of a family of functional inequalities of Poincaré type, but in general it is difficult to use them to prove (BM).
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Many inequalities of (BM) type (proved or conjectured) admit an infinitesimal version, typically consisting of a family of functional inequalities of Poincaré type, but in general it is difficult to use them to prove (BM). At this regard see the recent result by Livshyts, Marsiglietti, Nayar and Zvavitch about the conjectured dimensional version of (BM) in Gauss space, for symmetric convex bodies.
The functional version of Blaschke-Santaló inequality

Let $f = e^{-u}$ be a log-concave function and assume that $f$ is even. Then
$$\left( \int_{\mathbb{R}^n} e^{-u} \, dx \right) \cdot \left( \int_{\mathbb{R}^n} e^{-u^*} \, dx \right) \leq (2\pi)^n.$$ Equality holds if and only if $f$ is a Gaussian function.

Thm. (Blaschke-Santaló inequality).

Let $K$ be a centrally symmetric convex body, and let $K^\circ$ be its polar:
$$K^\circ = \{ y \in \mathbb{R}^n : x \cdot y \leq 1 \ \forall \ x \in K \}.$$ Then
$$V_n(K) V_n(K^\circ) \leq V_n(B_n)^2,$$ where $B_n$ is the unit ball. Equality holds iff $K$ is an ellipsoid.
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Remarks

The proof of the functional version of (BL) is due to Ball. The result was extended to the general (non-even) case by Artstein, Klartag and Milman. Further contributions were given by Fradelizi-Gordon-Reisner, Meyer, Lehec, Rotem.

For both theorems, to find an optimal lower bound for the corresponding products (of integrals or of volumes) is an open problem, for $n \geq 3$. 
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- For both theorems, to find an optimal lower bound for the corresponding products (of integrals or of volumes) is an open problem, for $n \geq 3$. 
The difference body inequality

Given a convex body $K$ in $\mathbb{R}^n$, its difference body $D_K$ is

$$D_K = K + (-K) = \{x - y : x, y \in K\}.$$  

Note that $D_K$ is centrally symmetric.

By the Brunn-Minkowski inequality

$$V_n(D_K) \geq 2^n V_n(K).$$

Thm. (Rogers-Shephard inequality).

$$V_n(D_K) \leq \left(\frac{2^n}{n}\right)^n V_n(K).$$

Equality holds iff $K$ is a symplex.
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$$V_n(DK) \leq \binom{2n}{n} V_n(K).$$

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A functional version

Let $f$ be a log-concave function and set $\bar{f}(z) = f(-z)$, for every $z \in \mathbb{R}^n$.

Define $Df(z) = \left(\frac{1}{2} \cdot f \oplus \frac{1}{2} \bar{f}\right)(z) = \sup_{x+y=2z} \sqrt{f(x)f(-y)}$.

This is an even function. By the (PL) inequality:

$$\int_{\mathbb{R}^n} Df \, dz \geq \int_{\mathbb{R}^n} f \, dx.$$
A functional version

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By the (PL) inequality:

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A functional version

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Thm. (C. '06).

\[ \int_{\mathbb{R}^n} Df \, dz \leq 2^n \int_{\mathbb{R}^n} f \, dx. \]
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Equality holds iff (up to linear transformations)
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f(x_1, \ldots, x_n) = \begin{cases} 
  e^{-(x_1+\cdots+x_n)} & \text{if } x_i \geq 0 \text{ for every } i, \\
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Extensions and other functional versions of the Rogers-Shephard inequality have been recently found by Artstein-Avidan, Einhorn, Florentin, Ostrover (2015), and Alonso-Gutiérrez, González, Jiménez, Villa (2015).
Valuations on convex bodies

Let \( K_n = \{ \text{convex bodies in } \mathbb{R}^n \} \). A mapping \( \mu : K_n \to \mathbb{R} \) is a valuation if
\[
\mu(K \cup L) + \mu(K \cap L) = \mu(K) + \mu(L)
\]
for every \( K \) and \( L \) s.t. \( K \cup L \in K_n \).

The study of valuations, especially with continuity and invariance properties, is one of the most active and prolific branches of convex geometry, after the seminal contributions by Hadwiger, McMullen and Alesker.

Thm. (Volume theorem). Let \( \mu \) be a rigid motion invariant and simple valuation on \( K_n \), which is also either monotone or continuous. Then it is a multiple of the volume.
Valuations on convex bodies

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**Thm.** (Volume theorem). Let $\mu$ be a rigid motion invariant and simple valuation on $\mathcal{K}^n$, which is also either monotone or continuous. Then it is a multiple of the volume.
Valuations on spaces of functions

Let $X$ be a space of functions. A mapping $\sigma: X \rightarrow \mathbb{R}$ is a valuation if

$$\sigma(f \vee g) + \sigma(f \wedge g) = \sigma(f) + \sigma(g)$$

for every $f, g \in X$ s.t. $f \vee g$ and $f \wedge g$ are in $X$. 

$\exists X = \{\text{definable functions}\}; \text{Wright (’11), and Baryshnikov, Ghrist, Wright (’13).}$

$\exists X = L^p(\mathbb{R}^n)$ or $X = \text{Orlicz space}; \text{Tsang (’10, ’12) and Kone (’14).}$

$\exists X = W^{1,p}(\mathbb{R}^n)$ or $X = BV(\mathbb{R}^n); \text{Ludwig (’11, ’12, ’13), Ma (’15), Wang (’13).}$
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A result for log-concave functions

Thm. (Cavallina, C., '15)

Let $\sigma$ be a valuation defined on the space of log-concave functions, which is:
▶ rigid motion invariant, i.e.
$\mu(f) = \mu(f \circ T) \forall T$ rigid motion of $\mathbb{R}^n$;
▶ simple (vanishes on functions which are a.e. zero);
▶ monotone decreasing;
▶ continuous (w.r.t. a suitable topology).

Then $\mu$ can be written in the form
$\mu(f) = \int_{\mathbb{R}^n} G(f) \, dx \forall f$
for a suitable function $G$ (continuous and monotone).
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