Central paths, reciprocal linear spaces, and the algebra behind them

Cynthia Vinzant, North Carolina State University

with Jesús De Loera, Bernd Sturmfels, and Mario Kummer
The Central Path of a Linear Program

Linear Program: Maximize_{x \in \mathbb{R}^n} \ c \cdot x \ s.t. \ A \cdot x = b \ and \ x \geq 0.
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Replace by: Maximize $x \in \mathbb{R}^n$ $f_\lambda(x)$ s.t. $A \cdot x = b$,

where $\lambda \in \mathbb{R}_+$ and $f_\lambda(x) := c \cdot x + \lambda \sum_{i=1}^n \log |x_i|$.
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where \( \lambda \in \mathbb{R}_+ \) and \( f_\lambda(x) := c \cdot x + \lambda \sum_{i=1}^n \log |x_i| \).

The maximum of the function \( f_\lambda \) is attained by a unique point \( x^*(\lambda) \) in the the open polytope \( \{ x \in (\mathbb{R}_{>0})^n : A \cdot x = b \} \).
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The central path is \ \{x^*(\lambda) : \lambda > 0\}. 
As \ \lambda \to 0 \ , \ the \ path \ leads \ from \ the \ analytic \ center \ of \ the \ polytope, \ x^*(\infty), \ to \ the \ optimal \ vertex, \ x^*(0).
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Bounds on curvature of the path \( \rightarrow \) bounds on \# Newton steps

The central path belongs to an algebraic variety and its rich real algebraic structure helps to bound the total curvature of the path.
The central curve $C$ is the Zariski closure of the central path. It contains the central paths of all polyhedra in the hyperplane arrangement $\{x_i = 0\}_{i=1,\ldots,n} \subset \{A \cdot x = b\}$.
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Goal: Study the nice algebraic geometry of this curve and its applications to the linear program.
History and Contributions

Motivating Question: What is the maximum total curvature of the central path given the size of the matrix $A$?

Bayer-Lagarias (1989) study the central path as an algebraic object and suggest the problem of identifying its defining equations.

Dedieu-Malajovich-Shub (2005) apply differential and algebraic geometry to bound the total curvature of the central path.


Conjecture: The total curvature of the central path is at most $O(n)$.

De Loera-Sturmfels-V. (2012) use results from algebraic geometry and matroid theory to find defining equations of the central curve and refine bounds on its degree and total curvature.

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Algebraic conditions for analytic centers

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where \( \mathcal{L}^{-1} \) denotes the coordinate-wise reciprocal of \( \mathcal{L} \):

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\mathcal{L}^{-1} := \left\{ (u_1^{-1}, \ldots, u_n^{-1}) \mid (u_1, \ldots, u_n) \in \mathcal{L} \right\}
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**Proposition.** All the intersection points of \( \mathcal{L}_A^{-1} \) with \( \{ A\mathbf{x} = \mathbf{b} \} \) are real. These are the analytic centers of all bounded regions of the hyperplane arrangement \( \{ x_i = 0 \}_{i=1}^{n} \subset \{ A\mathbf{x} = \mathbf{b} \} \).
The central curve is the union of the analytic centers of the arrangement
\[ \{x_i = 0\}_{i \in [n]} \text{ in } \{Ax = b, \; cx = c_0\} \]
in the level sets of the cost function \( c \).
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The algebraic equations of the analytic centers and the central curve come from reciprocal linear spaces \( \mathcal{L}^{-1} \).
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Using the matroid associated to $\mathcal{L}_{A,c}^{-1}$, they construct a simplicial complex containing combinatorial data of this ideal.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & 0 & 4 & 0 \end{pmatrix} \begin{array}{c} \{123, 1245, 1345, 2345\} \\ h = (1, 2, 2) \end{array}$$

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Proudfoot and Speyer (2006) determine the ideal of polynomials vanishing on $\mathcal{L}^{-1}_{A,c}$ and its Hilbert series.

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$\Rightarrow \deg(C) = \sum_{i=0}^{d} h_i$ and $\text{genus}(C) = 1 + \sum_{j=0}^{d} (j - 1) h_j$.
Classic differential geometry: The total curvature of any real algebraic curve $C$ in $\mathbb{R}^m$ is the arc length of its image under the Gauss map $\gamma : C \to S^{m-1}$. This quantity is bounded above by $\pi$ times the degree of the projective Gauss curve in $\mathbb{P}^{m-1}$. That is,

$$\text{total curvature of } C \leq \pi \cdot \deg(\gamma(C)).$$
**Total Curvature**

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**Theorem:** The degree of the projective Gauss curve of the central curve $C$ satisfies a bound in terms of matroid invariants:

$$\deg(\gamma(C)) \leq 2 \cdot \sum_{i=1}^{d} i \cdot h_i \leq 2 \cdot (n - d - 1) \cdot \binom{n-1}{d-1}.$$
A Tropical Counterexample

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\begin{align*}
\min \ v_0 & \quad \text{subject to} \\
u_0 & \leq t, \quad v_0 \leq t^2, \\
u_r & \geq 0, \quad v_r \geq 0, \\
v_i & \leq t^{1-1/2^i}(u_{i-1} + v_{i-1}), \\
u_i & \leq tu_{i-1}, \quad u_i \leq tv_{i-1}
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A polynomial \( f \in \mathbb{R}[x_1, \ldots, x_n]_d \) is hyperbolic with respect to a point \( e \in \mathbb{R}^n \) if for every \( v \in \mathbb{R}^n \) all the roots of \( f(te + v) \) are real.

Central curves in the plane are hyperbolic!
Hyperbolic Polynomials

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Ex: $\prod_i x_i$ is hyperbolic w.r.t $(1, \ldots, 1)$.
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Hyperbolic Varieties

A homogeneous variety $\mathcal{V} \in \mathbb{R}^n$ of codimension-$d$ is hyperbolic with respect to a $(d - 1)$-dimensional linear space $\mathcal{L}$ if for every $v \in \mathbb{R}^n$, all the intersection points of $\mathcal{V} \cap (\mathcal{L} + v)$ are real.
A homogeneous variety $\mathcal{V} \in \mathbb{R}^n$ of codimension-$d$ is hyperbolic with respect to a $(d - 1)$-dimensional linear space $\mathcal{L}$ if for every $v \in \mathbb{R}^n$, all the intersection points of $\mathcal{V} \cap (\mathcal{L} + v)$ are real.

Ex: For any linear space $\mathcal{L}$, the variety $\mathcal{L}^{-1}$ is hyperbolic w.r.t $\mathcal{L}^\perp$. 
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Ex: For any linear space $\mathcal{L}$, the variety $\mathcal{L}^{-1}$ is hyperbolic w.r.t $\mathcal{L}^\perp$.

The set of $(d-1)$-planes that intersect $\mathcal{V}$ nontrivially is a hyperbolic hypersurface in the Grassmannian.
Theorem (Kummer, V. (2015))

Let $\mathcal{L} \in \text{Gr}(d, n)$ be a linear space with non-zero Plücker coordinates $a_T$. A linear space $\mathcal{M} \in \text{Gr}(n - d, n)$ with Plücker coordinates $p_S$ intersects $\mathcal{L}^{-1}$ nontrivially if and only if

$$\det \left( \sum_{S \in \binom{[n]}{n-d}} (-1)^S \frac{p_S}{a_{S^c}} v_S v_S^T \right) = 0,$$

where $v_S \in \{-1, 0, 1\}^{(n-1)}$ for $S \in \binom{[n]}{n-d}$. 

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where $v_S \in \{-1, 0, 1\}^{(n-1)}$ for $S \in \binom{[n]}{n-d}$.

When $\mathcal{M} = \mathcal{L}^\perp$, $(-1)^S \frac{p_S}{a_S^c} = 1$ and this matrix is positive definite.
Example: \((d, n) = (2, 4)\)

\[
L = \text{rowspan} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}
\]

\[
\begin{align*}
a_{12} &= 1 \\
a_{13} &= 2 \\
a_{14} &= 3 \\
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\(\mathcal{L}^{-1}\) is a cubic 2-fold in \(\mathbb{R}^4\) \Rightarrow \(\mathbb{P}(\mathcal{L}^{-1})\) is a cubic curve in \(\mathbb{P}^3\).
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The line spanned by points \(v^1\) and \(v^2\) intersects \(\mathcal{L}^{-1}\) if and only if

\[
\det \begin{pmatrix}
\frac{p_{12}}{a_{34}} + \frac{p_{14}}{a_{23}} - \frac{p_{24}}{a_{13}} & \frac{p_{14}}{a_{23}} & \frac{p_{24}}{a_{13}} \\
\frac{p_{14}}{a_{23}} - \frac{p_{13}}{a_{24}} + \frac{p_{14}}{a_{23}} + \frac{p_{34}}{a_{12}} & \frac{p_{34}}{a_{12}} \\
\frac{p_{24}}{a_{13}} & \frac{p_{34}}{a_{12}} & \frac{p_{23}}{a_{14}} - \frac{p_{24}}{a_{13}} + \frac{p_{34}}{a_{12}}
\end{pmatrix} = 0
\]

where \(p_{ij}\) is the \(\{i, j\}\)th minor of \(\begin{pmatrix} v^1 \\ v^2 \end{pmatrix}\).
Conclusions

- Optimization can produce beautiful real algebraic objects.
- Their algebraic structure can reveal their behavior and answer questions in the theory of optimization.
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Thanks!