The most and the least avoided consecutive patterns

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A new Catalan number

2,305,290
A new Catalan number

2,305,290: number of Catalans who voted yesterday in the informal poll for Catalan independence
Consecutive patterns

\[ \pi = \pi_1 \pi_2 \ldots \pi_n \in S_n, \quad \sigma \in S_m. \]

**Definition.** \( \pi \) contains \( \sigma \) as a consecutive pattern if it has a subsequence of adjacent entries order-isomorphic to \( \sigma \).
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**Examples:**

- 25134 avoids 132
- 42531 contains 132
Consecutive patterns

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**Definition.** \( \pi \) contains \( \sigma \) as a consecutive pattern if it has a subsequence of adjacent entries order-isomorphic to \( \sigma \).

**Examples:** 25134 avoids 132 42531 contains 132 15243 contains two occurrences of 132

In this talk, containment and avoidance will always refer to consecutive patterns.
Consecutive patterns

Consecutive patterns generalize basic combinatorial concepts:

- Occurrences of 21 are descents.
- Occurrences of 132 and 231 are peaks.
- Permutations avoiding 123 and 321 are alternating permutations.
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Consecutive patterns arise naturally in dynamical systems, and play a role in distinguishing deterministic from random sequences.
For a fixed pattern $\sigma$, let

$$P_\sigma(u, z) = \sum_{n \geq 0} \sum_{\pi \in S_n} u^{\#\{\text{occurrences of } \sigma \text{ in } \pi\}} \frac{z^n}{n!},$$
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$$P_{\sigma}(0, z) = \sum_{n \geq 0} \alpha_n(\sigma) \frac{z^n}{n!},$$

where $\alpha_n(\sigma) = \#\{\pi \in S_n : \pi \text{ avoids } \sigma\}$. 
Some questions being studied

- Exact enumeration: find $P_\sigma(u, z)$ or $P_\sigma(0, z)$.

In this talk: Formulas for $P_\sigma(u, z)$ for $\sigma$ of certain shapes.
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  **In this talk**: Formulas for $P_\sigma(u, z)$ for $\sigma$ of certain shapes.

- Classification of patterns according to \textit{c-Wilf-equivalence}.
  We write $\sigma \sim \tau$ if $P_\sigma(u, z) = P_\tau(u, z)$.

  **Example**: $1342 \sim 1432$.

  **In this talk**: Classification of patterns of length up to 6.
Some questions being studied

► Exact enumeration: find $P_{\sigma}(u,z)$ or $P_{\sigma}(0,z)$.

**In this talk:** Formulas for $P_{\sigma}(u,z)$ for $\sigma$ of certain shapes.

► Classification of patterns according to $c$-Wilf-equivalence.
We write $\sigma \sim \tau$ if $P_{\sigma}(u,z) = P_{\tau}(u,z)$.

**Example:** $1342 \sim 1432$.

**In this talk:** Classification of patterns of length up to 6.

► Comparison of $\alpha_n(\sigma)$ for different patterns.

**Example:** $\alpha_n(132) < \alpha_n(123)$ for $n \geq 4$.

**In this talk:** For which pattern $\sigma \in S_m$ is $\alpha_n(\sigma)$ largest.
Patterns of small length

**Length 3:** 2 c-Wilf classes (compare: 1 Wilf class in classical case)

123 $\sim$ 321
132 $\sim$ 231 $\sim$ 312 $\sim$ 213
Patterns of small length

Length 3: 2 c-Wilf classes (compare: 1 Wilf class in classical case)

123 \sim 321
132 \sim 231 \sim 312 \sim 213

Length 4: 7 c-Wilf classes (compare: 3 Wilf classes in classical case)

1234 \sim 4321
2413 \sim 3142
2143 \sim 3412
1324 \sim 4231
1423 \sim 3241 \sim 4132 \sim 2314
1342 \sim 2431 \sim 4213 \sim 3124 \sim 1432 \sim 2341 \sim 4123 \sim 3214
1243 \sim 3421 \sim 4312 \sim 2134

All \sim follow from reversal and complementation except for \sim^*.
Patterns of small length

**Length 3:** 2 c-Wilf classes (compare: 1 Wilf class in classical case)

123 ∼ 321
132 ∼ 231 ∼ 312 ∼ 213

**Length 4:** 7 c-Wilf classes (compare: 3 Wilf classes in classical case)

1234 ∼ 4321 enumeration solved
2413 ∼ 3142 enumeration unsolved
2143 ∼ 3412
1324 ∼ 4231
1423 ∼ 3241 ∼ 4132 ∼ 2314
1342 ∼ 2431 ∼ 4213 ∼ 3124 ∘ 1432 ∼ 2341 ∼ 4123 ∼ 3214
1243 ∼ 3421 ∼ 4312 ∼ 2134

All ∼ follow from reversal and complementation except for ∘.
We use an adaptation of the cluster method of Goulden and Jackson, based on inclusion-exclusion.

A $k$-cluster w.r.t. $\sigma \in S_m$ is a permutation filled with $k$ marked occurrences of $\sigma$ that overlap with each other.
Clustering

We use an adaptation of the cluster method of Goulden and Jackson, based on inclusion-exclusion.

A \textit{k-cluster} w.r.t. $\sigma \in S_m$ is a permutation filled with $k$ marked occurrences of $\sigma$ that overlap with each other.

Example: 142536879 is a 3-cluster w.r.t. 1324.
Let the EGF for clusters be

\[ C_\sigma(u, z) = \sum_{n,k} c_{n,k}^\sigma u^k \frac{z^n}{n!}, \]

where \( c_{n,k}^\sigma := \text{number of } k\text{-clusters of length } n \text{ w.r.t. } \sigma. \)
The cluster method

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**Theorem (Goulden-Jackson '79, adapted)**

\[ P_\sigma(u, z) = \frac{1}{1 - z - C_\sigma(u - 1, z)} \]
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where $c_{n,k}^\sigma :=$ number of $k$-clusters of length $n$ w.r.t. $\sigma$.

**Theorem (Goulden-Jackson ’79, adapted)**

$$P_{\sigma}(u, z) = \frac{1}{1 - z - C_{\sigma}(u - 1, z)} \overset{\text{def}}{=} \frac{1}{\omega_{\sigma}(u, z)}.$$

This reduces the computation of $P_{\sigma}(u, z)$ to the enumeration of clusters.
Clusters as linear extensions of posets

\[ \pi_1 \pi_2 \pi_3 \pi_4 \pi_5 \pi_6 \pi_7 \pi_8 \pi_9 \pi_{10} \pi_{11} \] is a cluster w.r.t. \( \sigma = 14253 \)

\[ \uparrow \]

\[ \pi_1 < \pi_3 < \pi_5 < \pi_2 < \pi_4 \]
\[ \pi_3 < \pi_5 < \pi_7 < \pi_4 < \pi_6 \]
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\( \pi \) is a linear extension of the poset given by these relations (called a cluster poset)
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\begin{align*}
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\end{align*}
\]

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Ex: 1 6 2 8 3 11 4 9 5 10 7
The pattern $\sigma = 12 \ldots m$ and generalizations

Theorem (Goulden-Jackson ’83, E.-Noy ’01)

For $\sigma = 12 \ldots m$, $\omega_\sigma(u, z)$ is the solution of

$$\omega^{(m-1)} + (1 - u)(\omega^{(m-2)} + \cdots + \omega' + \omega) = 0.$$
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It follows that $\omega_{12\ldots m}(0,z) = \sum_{j\geq 0} \left( \frac{z^{jm}}{(jm)!} - \frac{z^{jm+1}}{(jm + 1)!} \right)$. 
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$$

More generally, we get a similar differential equation for any $\sigma$ such that all its cluster posets are chains. For example,

$$
\sigma = 12 \ldots (s-1)(s+1)s(s+2)(s+3) \ldots m.
$$
Non-overlapping patterns

$\sigma \in S_m$ is non-overlapping if two occurrences of $\sigma$ can’t overlap in more than one position.

**Example:** 132, 1243, 1342, 21534, 34671285 are non-overlapping.
Non-overlapping patterns

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**Theorem (E.-Noy ’01)**

Let \( \sigma \in S_m \) be non-overlapping with \( \sigma_1 = 1, \sigma_m = b \). Then \( \omega_\sigma(u, z) \) is the solution of

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Similar arguments give differential equations for \( \sigma = 12534 \) and \( \sigma = 13254 \), which aren’t non-overlapping.
The pattern $134\ldots(s+1)2(s+2)(s+3)\ldots m$

Theorem (E.-Noy, Liese-Remmel, Dotsenko-Khoroshkin)

For $\sigma = 1324$, $\omega_{\sigma}(u, z)$ is the solution of

\[
zw^{(5)} - ((u-1)z-3)w^{(4)} - 3(u-1)(2z+1)w^{(3)} + (u-1)((4u-5)z-6)w''
+ (u - 1)(8(u - 1)z - 3)w' + 4(u - 1)^2zw = 0
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\]

The construction generalizes to patterns of the form

\[\sigma = 134\ldots(s + 1)2(s + 2)(s + 3)\ldots m.\]
Other patterns of length 4

For the remaining cases, 1423, 2143 and 2413, we do not know of similar differential equations for $\omega_\sigma(u, z)$. 
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Conjecture

For $\sigma = 1423$, $\omega_{1423}(0, z)$ is not D-finite.

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“Conjecture” (Noonan-Zeilberger ’96)

For every classical pattern $\sigma$ (i.e., where occurrences are not constrained to consecutive positions), the generating function for $\sigma$-avoiding permutations is D-finite.
Consecutive Wilf-equivalence

One can classify patterns of length up to 6 into c-Wilf-equivalence classes, proving four conjectures of Nakamura:

<table>
<thead>
<tr>
<th>n</th>
<th># of classes</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>6</td>
<td>92</td>
</tr>
</tbody>
</table>

Theorem (E.-Noy)

- $123546 \sim 124536 \rightarrow \text{solution of } \omega^{(5)} + (1 - u)(\omega' + \omega) = 0.$
- $123645 \sim 124635 \rightarrow \text{solution of } \omega^{(5)} + (1 - u)z(\omega'' + \omega') = 0.$
- $132465 \sim 142365 \rightarrow \text{solution of } \omega^{(5)} + (1 - u)(\omega'' + z\omega') = 0.$
- $154263 \sim 165243.$
Asymptotic behavior

Theorem (E. '05)

For every \( \sigma \), the limit

\[
\rho_\sigma := \lim_{n \to \infty} \left( \frac{\alpha_n(\sigma)}{n!} \right)^{1/n}
\]

exists.

The proof uses methods from spectral theory.
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This limit is known only for some patterns.
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This limit is known only for some patterns.

Theorem (Ehrenborg-Kitaev-Perry ’11)
For every $\sigma$,

$$\frac{\alpha_n(\sigma)}{n!} = \gamma_\sigma \rho_\sigma^n + O(\delta^n),$$

for some constants $\gamma_\sigma$ and $\delta < \rho_\sigma$.

The proof uses methods from spectral theory.
For what pattern $\sigma \in S_m$ is $\alpha_n(\sigma)$ largest?
The most avoided pattern

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**Theorem (E.)**

*For every $\sigma \in S_m$ there exists $n_0$ such that*

$$\alpha_n(\sigma) \leq \alpha_n(12\ldots m)$$

*for all $n \geq n_0$.*

Interestingly, the analogous result for classical patterns (i.e., without the adjacency requirement) is false.
The most avoided pattern

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The theorem is equivalent to $\rho_\sigma$ being largest for $\sigma = 12\ldots m$. 
Proof idea — 1. Singularity analysis

Let $\sigma \in S_m \setminus \{12 \ldots m, m \ldots 21\}$. Want to show: $\rho_\sigma < \rho_{12\ldots m}$. 
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Let $\sigma \in S_m \setminus \{12 \ldots m, m \ldots 21\}$. Want to show: $\rho_\sigma < \rho_{12\ldots m}$.

Recall: $\rho_\sigma$ is the growth rate of the coefficients of

$$P_\sigma(0, z) = \frac{1}{\omega_\sigma(0, z)} = \sum_{n \geq 0} \alpha_n(\sigma) \frac{z^n}{n!},$$

so $\rho_\sigma^{-1}$ is the smallest singularity of $P_\sigma(0, z)$.
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so $\rho_\sigma^{-1}$ is the smallest singularity of $P_\sigma(0, z)$.

One can show that $\omega_\sigma(z) := \omega_\sigma(0, z)$ is analytic near the origin, so

$\rho_\sigma^{-1}$ is the smallest zero of $\omega_\sigma(z)$,

$\rho_{12 \ldots m}^{-1}$ is the smallest zero of $\omega_{12 \ldots m}(z)$. 
Proof idea — 1. Singularity analysis

- \( \rho_\sigma^{-1} \) is the smallest zero of \( \omega_\sigma(z) \),
- \( \rho_{12\ldots m}^{-1} \) is the smallest zero of \( \omega_{12\ldots m}(z) \).
Proof idea — 1. Singularity analysis

- $\rho^{-1}_\sigma$ is the smallest zero of $\omega_\sigma(z)$,
- $\rho^{-1}_{12\ldots m}$ is the smallest zero of $\omega_{12\ldots m}(z)$.

To show that $\rho_\sigma < \rho_{12\ldots m}$, it is enough to show that

$$\omega_{12\ldots m}(z) < \omega_\sigma(z)$$

for $0 < z < 1.276$. 

![Graph](image.png)
Proof idea — 2. Comparing cluster numbers

We show that \( \omega_{12...m}(z) < \omega_\sigma(z) \) for \( 0 < z < 1.276 \):

\[
\omega_{12...m}(z) = \sum_{j \geq 0} \left( \frac{z^{jm}}{(jm)!} - \frac{z^{jm+1}}{(jm+1)!} \right) < 1 - z + \frac{z^m}{m!} - \frac{z^{m+1}}{(m+1)!} + \frac{z^{2m}}{(2m)!},
\]
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We show that $\omega_{12...m}(z) < \omega_\sigma(z)$ for $0 < z < 1.276$:

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$$\omega_\sigma(z) = 1 - z - \sum_{k \geq 1} (-1)^k \sum_{n} r_{n,k}^\sigma \frac{z^n}{n!}$$

$$\underbrace{s_k^\sigma(z)}_{s_k^\sigma(z)}$$
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$$
\omega_{\sigma}(z) = 1 - z - \sum_{k \geq 1} (-1)^k \sum_n r_{n,k}^{\sigma} \frac{z^n}{n!} > 1 - z + \frac{z^m}{m!} - s_2^{\sigma}(z).
$$

Key fact #1: The sequence $\{s_k^{\sigma}(z)\}_{k \geq 1}$ is decreasing.
Proof idea — 2. Comparing cluster numbers

We show that $\omega_{12...m}(z) < \omega_\sigma(z)$ for $0 < z < 1.276$:

$$\omega_{12...m}(z) = \sum_{j \geq 0} \left( \frac{z^{jm}}{(jm)!} - \frac{z^{jm+1}}{(jm + 1)!} \right) < 1 - z + \frac{z^m}{m!} - \frac{z^{m+1}}{(m + 1)!} + \frac{z^{2m}}{(2m)!},$$

and

$$\omega_\sigma(z) = 1 - z - \sum_{k \geq 1} (-1)^k \sum_{n} r_{n,k}^\sigma \frac{z^n}{n!} > 1 - z + \frac{z^m}{m!} - s_2^\sigma(z).$$

Key fact #1: The sequence $\{s_k^\sigma(z)\}_{k \geq 1}$ is decreasing.

Key fact #2: $s_2^\sigma(z) < \frac{z^{m+1}}{(m+1)!} - \frac{z^{2m}}{(2m)!}.$
The least avoided pattern

For what pattern $\sigma \in S_m$ is $\alpha_n(\sigma)$ smallest?
The least avoided pattern

For what pattern $\sigma \in S_m$ is $\alpha_n(\sigma)$ smallest?

**Theorem (E., conjectured by Nakamura)**

*For every $\sigma \in S_m$ there exists $n_0$ such that*

$$\alpha_n(123 \ldots (m-2)m(m-1)) \leq \alpha_n(\sigma)$$

*for all $n \geq n_0$.***

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Thank you