Integral versions of Helly’s theorem and more!!

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R. La Haye - D. Oliveros - E. Roldán (coming soon!)
Edouard Helly (1884-1943):
HELLY’s THEOREM (1914)
Given a finite family $H$ of convex sets in $\mathbb{R}^n$. If every $n+1$ of its elements have a common intersection point, then all elements in $H$ has a non-empty intersection.
My favorite special case

Given $m$ Linear Inequalities over $R$ we want to find a solution, i.e.:

find $x_1, x_2, \ldots, x_n$, satisfying:

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,d}x_n \leq b_1$$
$$a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,d}x_n \leq b_2$$
$$\vdots$$
$$a_{m,1}x_1 + a_{k,2}x_2 + \cdots + a_{k,d}x_n \leq b_m$$

IF the system has no solutions then there is a CERTIFICATE of only $n + 1$ inequalities that have no solution!!
HELLY’s NEEDS THE MAGIC NUMBER $n + 1$!

**WARNING:** Helly’s theorem needs that the subfamilies intersecting have the right size!!
Nice, but why do I care?

- Fast randomized algorithms for solving Linear programs
- Finding the smallest circle containing a given set of planar points.
- Finding a line transversal for a some special classes of objects.
- Finding the separating hyperplane for a family of points
Many Many Many Versions of Helly’s Theorem!!
Quantitative versions of Helly’s Theorem

Large-volume Helly Theorem
Given a family $F$ of $m \geq 2n$ convex sets in $\mathbb{R}^n$. If every $2n$ members of $F$ has volume at least 1, then the intersection of all the members of $F$ has volume at least $\frac{1}{n^2 n^2}$.

Large-diameter Helly Theorem
Given a family $F$ of $m \geq 2n$ convex sets in $\mathbb{R}^n$. If every $2n$ members of $F$ has diameter at least 1, then the intersection of all the members of $F$ has diameter at least $\frac{1}{2n^2 n^2}$.
DOIGNON’S theorem

Given a finite collection $D$ of convex sets in $\mathbb{R}^n$, if every $2^n$ of its elements contain a lattice point in the intersection, then all the sets in $D$ have a common point with integer coordinates.
David E. Bell & Herbert Scarf (1977) Rediscovered Doignon’s Theorem in Integer Optimization
WHY did they care?
Solvability of **Integer Linear Programming Problem:**

maximize \( C_1 x_1 + C_2 x_2 + \cdots + C_d x_d \)

among all \( x_1, x_2, \ldots, x_d \), satisfying:

\[
a_{1,1} x_1 + a_{1,2} x_2 + \cdots + a_{1,d} x_d \leq b_1 \\
a_{2,1} x_1 + a_{2,2} x_2 + \cdots + a_{2,d} x_d \leq b_2 \\
\vdots \\
a_{k,1} x_1 + a_{k,2} x_2 + \cdots + a_{k,d} x_d \leq b_k
\]

where \( x_i \) has to be integer.

Using the Doignon-Bell-Scarf theorem gives the fastests (randomized) algorithms for Integer Optimization!!
Theorem of Doignon-Bell-Scarf

**Theorem** Let $A$ be a $m \times n$ matrix and $b$ a vector of $\mathbb{Q}^m$. If the problem $P_A(b) = \{x : Ax \leq b, x \in \mathbb{Z}^n\}$ has no integer solution, then there is a subset $S$ of the $m$ rows of $A$ of cardinality no more than $2^n$, so that the smaller system has no integer solution either.
Can one do a Quantitive Integral version of Helly’s Theorem ??
Theorem (Iskander Aliev, JDL, Quentin Louveaux)

Let $n$, $k$ non-negative integers, there exists a magic number $c(k, n)$, depending only on $k$ and $n$, such that

If a rational polytope $P_A(b) = \{ x : Ax \leq b \}$ has exactly $k$ integral solutions, then there is a subset of the inequalities of $Ax \leq b$, of cardinality no more than $c(k, n)$, such that the smaller subproblem has exactly the same $k$ integer solutions as $P_A(b)$.

**ORIGINAL DOIGNON-BELL-SCARF** is case of $k = 0$. 
Corollary
For $n$, $k$ non-negative integers, there exists a magic number $c(k, n)$, determined by $k$ and $n$, such that

- For any system of inequalities $\{x : Ax \leq b\}$ in $\mathbb{R}^n$, if every subset of the constraints of cardinality $c(k, n)$ has at least $k$ integer solutions, then the entire system of inequalities must have at least $k$ integral solutions.

- Let $(X_i)_{i \in \Lambda}$ be a collection of convex sets in $\mathbb{R}^n$, where at least one of these sets is compact.
  If exactly $k$ integer points are in $\bigcap_{i \in \Lambda} X_i$, then there is a subcollection of size less than or equal to $c(n, k)$ with exactly the same integer points in their intersection.

But, **WHAT IS THE MAGIC NUMBER $c(n, k)$?**

When $k = 0$ we knew $c(0, n) = 2^n$. 
MAIN THEOREM 2: Bound for $c(n, k)$

- **Theorem 2** For $n, k$ non-negative integers

  $$c(k, n) \leq \lceil 2(k + 1)/3 \rceil 2^n - 2 \lceil 2(k + 1)/3 \rceil + 2$$

  and the bound is tight for $c(0, n)$ and $c(1, n)$.

- **Example** For $c(1, 2) = 6$ and $c(1, 3) = 14$, but $c(3, 2) = 6$ (but 8 is our bound!)

- **OPEN PROBLEM** Find the exact value of the $c(k, n)$!!!
MORE Helly’s Theorems!

- Original Helly’s theorem is over the real numbers or $\mathbb{R}^n$
- Doignon-Bell-Scarf is a Helly theorem integer numbers or $\mathbb{Z}^n$
- Theorem (Averkov-Weismantel): Helly theorem over $\mathbb{Z}^{n_1} \times \mathbb{R}^{n_2}$.
- QUESTION For which subsets $S$ of $\mathbb{R}^n$ is a Helly-type Theorem possible?
- $h(S)$ is the $S$-Helly number when the sets are required to intersect at points in $S$. E.g., $h(\mathbb{R}^n) = d + 1$ and $h(\mathbb{Z}^n) = 2^n$.
- Warning Not all subsets of $\mathbb{R}^n$ have a finite Helly number!!!
Helly’s theorems over subgroups of $\mathbb{R}^n$

Theorem (R. La Haye, D. Oliveros, E. Roldan)

- If $S$ is an additive subgroup of $\mathbb{R}^n$ then $h(S) \leq 2^n$.
- Moreover $h(S) = 2^n$ when $S$ is a lattice and $h(S) \leq 2^{d-1} + 2$ if $S$ is not a lattice.
- Let $G$ be a dense subgroup of the real numbers, if $S \subset \mathbb{R}^d$ is a $G$-module then $h(d) \leq 2d$. 
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<table>
<thead>
<tr>
<th>The group $S$</th>
<th>Bound for $h(S)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{F}^n$ with $\mathbb{F} \subset \mathbb{R}$ a field.</td>
<td>$n + 1$</td>
</tr>
<tr>
<td>The lattice $\mathbb{Z}^n$.</td>
<td>$2^n$</td>
</tr>
<tr>
<td>A group of the form $\mathbb{Z}^{n-k} \times \mathbb{R}^k$.</td>
<td>$2^{n-k}(k + 1)$</td>
</tr>
<tr>
<td>A subgroup of $\mathbb{R}^n$ that is not a lattice</td>
<td>$2^{n-1} + 2$</td>
</tr>
<tr>
<td>A $G$-module with $G$ a dense subgroup of $\mathbb{R}$.</td>
<td>$2n$</td>
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</tbody>
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Table: SUMMARY: Upper bounds for the $S$-Helly number.
IDEAS & INSPIRATION

(if I have time!!)
GRACIAS !!