Collaborators

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Obliquely reflected Brownian motion

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Reflected Brownian motion
Obliquely reflected Brownian motion in fractal domains

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Reflected Brownian motion
Technical challenges

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Reflected Brownian motion
Classical Dirichlet form approach to Markov processes is limited to symmetric processes. Obliquely reflected Brownian motion is not symmetric.
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Non-symmetric Dirichlet form approach to obliquely reflected Brownian motion had limited success (Kim, Kim and Yun (1998) and Duarte (2012)).
$D$ – unit disc in $\mathbb{R}^2$

$\theta(x)$ – angle of reflection at $x \in \partial D$
Parametrization of obliquely reflected Brownian motions

$D$ – unit disc in $\mathbb{R}^2$

$\theta(x)$ – angle of reflection at $x \in \partial D$

**THEOREM (B and Marshall; 1993)**

For an arbitrary measurable $\theta$, obliquely reflected Brownian motion $X$ in $D$ with the oblique angle of reflection $\theta$ exists.
Parametrization of obliquely reflected Brownian motions

\( D \) – unit disc in \( \mathbb{R}^2 \)
\( \theta(x) \) – angle of reflection at \( x \in \partial D \)

**THEOREM (B and Marshall; 1993)**

For an arbitrary measurable \( \theta \), obliquely reflected Brownian motion \( X \) in \( D \) with the oblique angle of reflection \( \theta \) exists.

Lions and Sznitman (1984), Harrison, Landau and Shepp (1985), Varadhan and Williams (1985)
Jumps on the boundary
$D$ – unit disc in $\mathbb{R}^2$

$\theta(x)$ – angle of reflection at $x \in \partial D$
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$\theta \leftrightarrow (h, \mu)$
$D$ – unit disc in $\mathbb{R}^2$

$\theta(x)$ – angle of reflection at $x \in \partial D$

\[
\theta \leftrightarrow (h, \mu)
\]

$h(x)dx$ – stationary distribution
$D$ – unit disc in $\mathbb{R}^2$

$\theta(x)$ – angle of reflection at $x \in \partial D$

$\theta \leftrightarrow (h, \mu)$

$h(x)dx$ – stationary distribution

$\mu$ – rate of rotation
THEOREM (forthcoming; B, Chen, Marshall, Ramanan)

\[ h(z) = \frac{\Re \exp(\tilde{\theta}(z) - i\theta(z))}{\pi \Re(e^{-i\theta(0)})} = \frac{\Re \exp(\tilde{\theta}(z) - i\theta(z))}{\pi \cos \theta(0)}, \quad z \in D, \]

\[ \mu = \tan \theta(0) = \int_D \tan \theta(z) h(z) dz, \]

\[ \theta(z) = -\arg \left( h(z) + i\tilde{h}(z) - i\mu/\pi \right), \quad z \in D. \]
THEOREM (forthcoming; B, Chen, Marshall, Ramanan)

\[ h(z) = \frac{\Re \exp(\tilde{\theta}(z) - i\theta(z))}{\pi \Re(e^{-i\theta(0)})} = \frac{\Re \exp(\tilde{\theta}(z) - i\theta(z))}{\pi \cos \theta(0)}, \quad z \in D, \]

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Harrison, Landau and Shepp (1985): smooth \( \theta \)
THEOREM (forthcoming; B, Chen, Marshall, Ramanan)

1. Assume that \( \theta \) is \( C^2 \). Then, with probability 1, \( X \) is continuous and, therefore, \( \arg X_t \) is well defined for \( t > 0 \). The distributions of \( \frac{1}{t} \arg X_t - \mu \) converge to the Cauchy distribution when \( t \to \infty \).

2. Let \( \arg^* X_t \) be \( \arg X_t \) “without large excursions from \( \partial D \).” Then, a.s.,

\[
\lim_{t \to \infty} \frac{\arg^* X_t}{t} = \mu.
\]
Obliquely reflected Brownian motion in fractal domains

$D$ – simply connected bounded open set in $\mathbb{R}^2$

**THEOREM** (forthcoming; B, Chen, Marshall, Ramanan)

For every positive harmonic function $h$ in $D$ with $L^1$ norm equal to 1 and every real number $\mu$, there exists a (unique in distribution) obliquely reflected Brownian motion in $D$ with the stationary distribution $h(x)dx$ and rate of rotation $\mu$. 
Let $D$ be the unit disc and $\mu(z)$ be the rate of rotation around $z \in D$. In other words, the distributions of $\frac{1}{t} \arg(X_t - z) - \mu(z)$ converge to the Cauchy distribution when $t \to \infty$.
Let $D$ be the unit disc and $\mu(z)$ be the rate of rotation around $z \in D$. In other words, the distributions of $\frac{1}{t} \arg(X_t - z) - \mu(z)$ converge to the Cauchy distribution when $t \to \infty$.

**THEOREM (forthcoming; B, Chen, Marshall, Ramanan)**

The function $\mu(z)$ is harmonic in $D$. 
Rotation rate field

\[ D \text{ – unit disc in } \mathbb{R}^2 \]
\[ \theta(x) \text{ – angle of reflection at } x \in \partial D \]
\[ h(x)dx \text{ – stationary distribution} \]
\[ \mu(z) \text{ – rate of rotation around } z \]

\[ \theta \leftrightarrow (h, \mu(0)) \leftrightarrow \{\mu(z)\}_{z \in D} \]
Rotation rate field

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Arrows indicate one to one mappings. Are the mappings surjective?
Rotation rate field

\( D \) – unit disc in \( \mathbb{R}^2 \)
\( \theta(x) \) – angle of reflection at \( x \in \partial D \)
\( h(x) \) – stationary distribution
\( \mu(z) \) – rate of rotation around \( z \)

\[
\theta \iff (h, \mu(0)) \iff \{ \mu(z) \}_{z \in D}
\]

Arrows indicate one to one mappings. Are the mappings surjective? In the first case, yes.
Suppose that $\phi(z)$ is harmonic in the unit disc $D$. $\phi(z)$ does not have to be positive.
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For each $\phi$ there exists $b_0 \geq 0$ such that for all $a \in \mathbb{R}$ and $b \in [0, b_0]$, the function $\mu(z) = a + b\phi(z)$ is the rotation field of an obliquely reflected Brownian motion in $D$.

For $b > b_0$, the function $\mu(z) = a + b\phi(z)$ does not represent the rotation field of an obliquely reflected Brownian motion in $D$. 
Obliquely reflected Brownian motion in fractal domains
Smooth domain approximation

$D \subset \mathbb{R}^2$ – open bounded simply connected set
$D_k \subset D_{k+1}, \bigcup_k D_k = D$, $D_k$ have smooth boundaries
Smooth domain approximation

$D \subset \mathbb{R}^2$ – open bounded simply connected set

$D_k \subset D_{k+1}$, $\bigcup_k D_k = D$, $D_k$ have smooth boundaries

$\theta_k(x)$ – reflection angle; $x \in \partial D_k$

$X^k$ – obliquely reflected Brownian motion in $D_k$
Smooth domain approximation

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**THEOREM** (forthcoming; B, Chen, Marshall, Ramanan)

Suppose that $\theta_k$ converge as $k \to \infty$. Then obliquely reflected Brownian motions $X^k$ converge, as $k \to \infty$, to a process in $D$.

We apply conformal invariance of Brownian motion.
Technical challenges

Krzysztof Burdzy

Reflected Brownian motion
Technical challenges

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Reflected Brownian motion
Integrability of harmonic functions

Let $\delta_D(x) = \text{dist}(x, \partial D)$ and $x_0 \in D$. We say that $D$ is a John domain with John constant $c_J > 0$ if each $x \in D$ can be joined to $x_0$ by a rectifiable curve $\gamma$ such that $\delta_D(y) \geq c_J \ell(\gamma(x, y))$ for all $y \in \gamma$, where $\gamma(x, y)$ is the subarc of $\gamma$ from $x$ to $y$ and $\ell(\gamma(x, y))$ is the length of $\gamma(x, y)$.

**THEOREM (Aikawa, 2000)**

(i) If $D \subset \mathbb{R}^2$ is a bounded John domain with John constant $c_J \geq 7/8$ then all positive harmonic functions in $D$ are in $L^1(D)$.

(ii) If $D \subset \mathbb{R}^2$ is a bounded Lipschitz domain with constant $\lambda < 1$ then all positive harmonic functions in $D$ are in $L^1(D)$.

(iii) There exists a bounded Lipschitz domain $D$ with constant $\lambda = 1$ and a positive harmonic function $h$ in $D$ which is not in $L^1(D)$. 

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Reflected Brownian motion
Bounded harmonic functions

The modulus of continuity of $f$ is $\omega_f(a) = \sup_{|x-y|<a} |f(x) - f(y)|$ and $f$ is Dini continuous if $\int_0^b (\omega_f(a)/a) da < \infty$ for some $b > 0$.

THEOREM

(i) (Garnett, 2007) If $\theta$ is Dini continuous then $h$ is bounded.

(ii) Suppose that $\omega$ is an increasing continuous concave function on $[0, \pi/2]$ such that $\omega(0) = 0$, $\omega(\pi/2) = \pi/4$, and $\int_0^{\pi/2} (\omega(a)/a) da = \infty$. Then there exists $\theta$ such that $\omega_\theta(a) = \omega(a)$ for $a \leq \pi/2$ and $h$ is unbounded.
Myopic conditioning

\[ D \subset \mathbb{R}^d \] – open bounded connected set
\[ \varepsilon > 0 \]
\[ X^\varepsilon_t \] – a continuous process in \( D \)

**DEFINITION (Myopic Brownian motion)**

Given \( \{ X^\varepsilon_t, 0 \leq t \leq k\varepsilon \} \), the process \( \{ X^\varepsilon_t, k\varepsilon \leq t \leq (k + 1)\varepsilon \} \) is Brownian motion conditioned not to hit \( D^c \) (during the time interval \( [k\varepsilon, (k + 1)\varepsilon] \)).
$D \subset \mathbb{R}^d$ – open bounded connected set

$\varepsilon > 0$

$X^\varepsilon_t$ – a continuous process in $D$

**DEFINITION (Myopic Brownian motion)**

Given $\{X^\varepsilon_t, 0 \leq t \leq k\varepsilon\}$, the process $\{X^\varepsilon_t, k\varepsilon \leq t \leq (k + 1)\varepsilon\}$ is Brownian motion conditioned not to hit $D^c$ (during the time interval $[k\varepsilon, (k + 1)\varepsilon]$).

**THEOREM (B, Chen)**

Processes $X^\varepsilon$ converge weakly, as $\varepsilon \to 0$, to reflected Brownian motion in $D$. 
$D \subset \mathbb{R}^d$ – open bounded connected set, $\varepsilon > 0$

Given $\{X_t^\varepsilon, 0 \leq t \leq k\varepsilon\}$, the process $\{X_t^\varepsilon, k\varepsilon \leq t \leq (k + 1)\varepsilon\}$ is Brownian motion conditioned not to hit $D^c$ during the time interval $[k\varepsilon, (k + 1)\varepsilon]$. 
$D \subset \mathbb{R}^d$ – open bounded connected set, $\varepsilon > 0$

Given $\{X_t^\varepsilon, 0 \leq t \leq k\varepsilon\}$, the process $\{X_t^\varepsilon, k\varepsilon \leq t \leq (k+1)\varepsilon\}$ is Brownian motion conditioned not to hit $D^c$ during the time interval $[k\varepsilon, (k+1)\varepsilon]$.

$B$ – Brownian motion in $\mathbb{R}^d$, $\tau_D = \inf\{t \geq 0 : B_t \notin D\}$

$Y_k^\varepsilon = X_{k\varepsilon}^\varepsilon$, $k \geq 1$

$m_\varepsilon(dx) = P^x(\tau_D > \varepsilon)dx$
$D \subset \mathbb{R}^d$ – open bounded connected set, $\varepsilon > 0$

Given $\{X^\varepsilon_t, 0 \leq t \leq k\varepsilon\}$, the process $\{X^\varepsilon_t, k\varepsilon \leq t \leq (k + 1)\varepsilon\}$ is Brownian motion conditioned not to hit $D^c$ during the time interval $[k\varepsilon, (k + 1)\varepsilon]$.

$B$ – Brownian motion in $\mathbb{R}^d$, $\tau_D = \inf\{t \geq 0 : B_t \notin D\}$

$Y^\varepsilon_k = X^\varepsilon_{k\varepsilon}, \quad k \geq 1$

$m^\varepsilon(dx) = P^x(\tau_D > \varepsilon)dx$

**LEMMA (B, Chen)**

(i) $m^\varepsilon \rightarrow$ Lebesgue measure on $D$ as $\varepsilon \rightarrow 0$.

(ii) $m^\varepsilon(dx)$ is a reversible (stationary) measure for $Y^\varepsilon_k$. 
Increasing families of domains

\[ D \subset \mathbb{R}^d \] – open bounded connected set
\[ D_k \subset D_{k+1}, \bigcup_k D_k = D, \ D_k \text{ have smooth boundaries} \]
Increasing families of domains

$D \subset \mathbb{R}^d$ — open bounded connected set

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$X^k$ — reflected Brownian motion in $D_k$
Increasing families of domains

$D \subset \mathbb{R}^d$ – open bounded connected set
$D_k \subset D_{k+1}, \bigcup_k D_k = D$, $D_k$ have smooth boundaries
$X^k$ – reflected Brownian motion in $D_k$

**THEOREM (B, Chen; 1998)**

Reflected Brownian motions $X^k$ converge, as $k \to \infty$, to reflected Brownian motion in $D$. 
Invariance principle for reflected random walks

\( D \) – open connected bounded set
\( X^k \) – reflected random walk on \( D \cap (2^{-k} \mathbb{Z}^2) \)
\( X^k \) can jump along an edge if the edge is in \( D \)
Invariance principle for reflected random walks

$D$ – open connected bounded set
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**THEOREM (B, Chen; 2008)**

Assume that $D$ is an extension domain. Then reflected random walks $X^k$, with sped-up clocks, converge weakly to reflected Brownian motion in $D$, as $k \to \infty$. 
Invariance principle for reflected random walks

$D$ – open connected bounded set

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Examples of extension domains.

1. Smooth domains
2. Lipschitz domains
3. Uniform domains
4. NTA domains
5. Von Koch snowflake
Invariance principle in domains above graphs of continuous functions

$D$ – bounded domain
$\partial D$ is locally the graph of a continuous function
Invariance principle in domains above graphs of continuous functions

$D$ – bounded domain
\partial D$ is locally the graph of a continuous function

Fact: $D$ is an extension domain.

**COROLLARY (B, Chen; 2008)**

Assume that $D$ lies locally above the graph of a continuous function. Then reflected random walks $X^k$, with sped-up clocks, converge weakly to reflected Brownian motion in $D$, as $k \to \infty$. 
$X^k$ – reflected random walk on $D \cap (2^{-k}\mathbb{Z}^2)$

$X^k$ can jump along an edge if the edge is in $D$
$X^k$ – reflected random walk on $D \cap (2^{-k}\mathbb{Z}^2)$

$X^k$ can jump along an edge if the edge is in $D$

**THEOREM (B, Chen; 2008)**

There exists a bounded domain $D \subset \mathbb{R}^2$ such that reflected random walks $X^k$, with sped-up clocks, do not converge weakly to reflected Brownian motion in $D$, when $k \to \infty$. 
Invariance principle – a counterexample

$X^k$ – reflected random walk on $D \cap (2^{-k}\mathbb{Z}^2)$

$X^k$ can jump along an edge if the edge is in $D$

**THEOREM (B, Chen; 2008)**

There exists a bounded domain $D \subset \mathbb{R}^2$ such that reflected random walks $X^k$, with sped-up clocks, do not converge weakly to reflected Brownian motion in $D$, when $k \to \infty$.

Example: Remove suitable dust from a square.
Invariance principle (improved)

\( D \) – open connected bounded set
\( X^k \) – reflected random walk on \( D \cap (2^{-k}\mathbb{Z}^2) \)
\( X^k \) can jump along an edge if the edge is in \( D \)

**THEOREM (B, Chen; 2008)**

Assume that \( D \) is an extension domain. Then reflected random walks \( X^k \), with sped-up clocks, converge weakly to reflected Brownian motion in \( D \).
Invariance principle (improved)

$D$ – open connected bounded set
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THEOREM (B, Chen; 2008)
Assume that $D$ is an extension domain. Then reflected random walks $X^k$, with sped-up clocks, converge weakly to reflected Brownian motion in $D$.

$D$ – open connected bounded set
$D_k$ – subset of $D \cap (2^{-k} \mathbb{Z}^2)$; contains all vertices of the union of adjacent cubes in $D$
$X^k$ – reflected random walk on $D_k$
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Invariance principle (improved)

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\( X^k \) can jump along an edge if the edge is in \( D_k \)

**THEOREM (B, Chen; 2012)**

Reflected random walks \( X^k \) on \( D_k \), with sped-up clocks, converge weakly to reflected Brownian motion in \( D \), as \( k \to \infty \).
Two approximations
THEOREM (B, Chen; 2012)

Suppose that $D \subset \mathbb{R}^d$ is a domain with finite volume. There exists a countable sequence of bounded functions $\{\varphi_j\}_{j \geq 1} \subset W^{1,2}(D) \cap C^\infty(D)$ such that

1. $\{\varphi_j\}_{j \geq 1}$ is dense in $W^{1,2}(D)$,
2. $\{\varphi_j\}_{j \geq 1}$ separates points in $D$,
3. for each $j \geq 1$,

$$\limsup_{k \to \infty} 2^{k(2-d)} \sum_{xy \in D_k} (\varphi_j(x) - \varphi_j(y))^2 \leq 2 \int_D |\nabla \varphi_j(x)|^2 \, dx.$$
Robin problem

\[ \Delta u(x) = 0, \quad x \in D \setminus B, \]
\[ \frac{\partial u}{\partial n}(x) = cu(x), \quad x \in \partial D, \]
\[ u(x) = 1, \quad x \in \partial B. \]
Robin problem in fractal domains

Example: von Koch snowflake.

The normal vector does not exist at almost all boundary points.
Approximate the snowflake domain $D$ with an increasing sequence of smooth domains $D_k$, such that $\bigcup_k D_k = D$. Let $u_k$ be the solution to the Robin boundary problem in $D_k$, with the same $c$ (adsorption rate) for all $k$, and let $u(x) = \lim_{k \to \infty} u_k(x)$. Then $u$ satisfies the Dirichlet boundary conditions $u(x) = 0$ on $\partial D$. 
Approximate the snowflake domain $D$ with an increasing sequence of smooth domains $D_k$, such that $\bigcup_k D_k = D$.

Let $u_k$ be the solution to the Robin boundary problem in $D_k$, with the same $c$ (adsorption rate) for all $k$, and let

$$ u(x) = \lim_{k \to \infty} u_k(x). $$
Approximate the snowflake domain $D$ with an increasing sequence of smooth domains $D_k$, such that $\bigcup_k D_k = D$.

Let $u_k$ be the solution to the Robin boundary problem in $D_k$, with the same $c$ (adsorption rate) for all $k$, and let

$$u(x) = \lim_{k \to \infty} u_k(x).$$

Then $u$ satisfies the Dirichlet boundary conditions $u(x) = 0$ on $\partial D$. 
Assuming that $D$ is smooth, the Green-Gauss formula implies that for $u, v \in C^2(\overline{D})$,

$$
\int_D \nabla u(x) \cdot \nabla v(x) \, dx = -\int_D v(x) \Delta u(x) \, dx - \int_{\partial D} v(x) \frac{\partial u}{\partial n}(x) \sigma(dx),
$$

where $\sigma$ is the surface measure on $\partial D$. 
Assuming that $D$ is smooth, the Green-Gauss formula implies that for $u, v \in C^2(D)$,

$$\int_D \nabla u(x) \cdot \nabla v(x) \, dx = - \int_D v(x) \Delta u(x) \, dx - \int_{\partial D} v(x) \frac{\partial u}{\partial n}(x) \sigma(dx),$$

where $\sigma$ is the surface measure on $\partial D$.

A weak solution $u$ to the Robin problem is characterized by

$$\int_D \nabla u(x) \cdot \nabla v(x) \, dx = - \int_{\partial D} cu(x)v(x) \sigma(dx),$$

for every $v \in C^2(D)$ that vanishes on $B$. 
\( d = \frac{\log 4}{\log 3} \)

Let \( \mu \) be \( d \)-dimensional Hausdorff measure.
Solution to Robin problem in von Koch snowflake

\[ d = \frac{\log 4}{\log 3} \]

Let \( \mu \) be \( d \)-dimensional Hausdorff measure.

**DEFINITION**

We will say that a function \( u \) is a weak solution to the Robin problem in the snowflake domain if for all smooth \( v \),

\[
\int_D \nabla u(x) \cdot \nabla v(x) \, dx = - \int_{\partial D} cu(x)v(x) \mu(dx).
\]
Alternative representation

\( D \) – von Koch snowflake domain  
\( X \) – reflected Brownian motion in \( D \)  
\( \sigma_B \) – hitting time of \( B \)
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\[ D \] – von Koch snowflake domain
\[ X \] – reflected Brownian motion in \( D \)
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\[ L \] – “local time” on \( \partial D \), i.e., a continuous additive functional of \( X \) with Revuz measure \( \mu \)
Alternative representation

\( D \) – von Koch snowflake domain

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\( \sigma_B \) – hitting time of \( B \)

\( L \) – “local time” on \( \partial D \), i.e., a continuous additive functional of \( X \) with Revuz measure \( \mu \)

**THEOREM** (forthcoming; B, Chen)

- The continuous additive functional \( L \) with Revuz measure \( \mu \) exists.
Alternative representation

$D$ – von Koch snowflake domain
$X$ – reflected Brownian motion in $D$
$\sigma_B$ – hitting time of $B$
$L$ – “local time” on $\partial D$, i.e., a continuous additive functional of $X$ with Revuz measure $\mu$

**THEOREM** (forthcoming; B, Chen)

- The continuous additive functional $L$ with Revuz measure $\mu$ exists.
- The function

$$u(x) = \mathbb{E}_x \left[ \exp \left( -\frac{c}{2} \int_0^{\sigma_B} dL_s \right) \right], \quad x \in \overline{D} \setminus B,$$

is the unique weak solution to the Robin problem.