Non-homogeneous T1 theorem for Singular integrals with general Calderón-Zygmund kernels

Ana Grau de la Herrán – University of Helsinki

(joint work with T. Hytönen)

Goal

$$\|Tf(x)\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}$$
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Method

1. To solve the problem for \( p = 2 \).
2. Tool = To test how the operator behaves locally.
   - Constant function 1.
   - An accretive function \( b \).
   - Locally adapted functions.
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Ana Grau de la Herrán – University of Helsinki
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**Singular integral operators**

\[ Tf(x) = \int K(x, y)f(y)d\mu(y) \]
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Calderón-Zygmund kernel operators

\[ |K(x, y)| \leq \frac{C}{|x-y|^\alpha} \]

\[ |K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq C \frac{|x-x'|^\alpha}{|x-y|^{d+\alpha}} \]
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Examples
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Singular integral operators

\[ Tf(x) = \int K(x, y)f(y)d\mu(y) \]

Examples

- Hilbert Transform

\[ Hf(x) = \int_{\mathbb{R}} \frac{f(y)}{x - y} dy \]
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Singular integral operators

\[ Tf(x) = \int K(x, y)f(y)d\mu(y) \]

Examples

- Riesz Transform

\[ R_jf(x) = \int \frac{x_j - y_j}{|x - y|^{d+1}} f(y)dy \]
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\[ \| Tf(x) \|_{L^p(\mathbb{R}^d)} \leq C \| f \|_{L^p(\mathbb{R}^d)} \]

Singular integral operators

\[ Tf(x) = \int K(x, y)f(y)d\mu(y) \]

Examples

- Calderón commutators

\[ C_A^k f(x) = \text{p.v.} \frac{i}{2\pi} \int_{\mathbb{R}} \left[ \frac{A(x) - A(y)}{x - y} \right]^k \frac{1}{x - y} f(y)dy \]
Pioneer work: Zygmund, Calderón and Mikhlin. (concerning the Calderón program as developed by Coifman and Meyer).
Key results:

- Calderón Commutator theorem
- $L^2$ bounds on the higher commutators.
- $L^2$ bounds on the Cauchy integral on Lipchitz curves.
- The solution of the Painlevé problem on analytic capacity.
- The solution of the Kato square root problem for elliptic operators.
- (...)

For classical theory on SIO refer to [St] and [Ch].
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$L^2$ boundedness criteria

1. The convolution case: Fourier theory
2. The Non-convolution case:
   - First Calderón commutator.
   - The Second Calderón commutator.
   - Higher commutators.
   - Cauchy integral on a Lipschitz graph with small Lipschitz constant.
   - Cauchy integral on all Lipschitz graphs

NEEDED general $L^2$ boundedness criteria.
T1 Theorem [David and Journé]

Suppose that T is a singular integral operator associated to a standard Calderón-Zygmund kernel $K(x, y)$. Then T extends to a bounded operator on $L^2$ if and only if T satisfies WBP, and $T1, T^*1 \in BMO$. 
The convolution case: Fourier theory

The Non-convolution case:
- First Calderón commutator.
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\( L^2 \) boundedness criteria

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Tb theorem [McM] [DJS]

Let $T$ be a SIO associated to a standard Calderón-Zygmund kernel and suppose that $b_1, b_2 \in L^\infty$ are accretive. If $b_2 \mathcal{T} b_1$ satisfies the WBP and $\mathcal{T} b_1, \mathcal{T}^* b_2 \in BMO$ then $T$ extends to a bounded operator on $L^2$.

**Remark.**- The accretivity condition can be relaxed to pseudo-accretivity or even para-accretivity. An accretive function $b$ is an $L^\infty$ function that satisfies $\Re e(b) \geq c > 0$.
Applications:

- Cauchy integral.
- n-dimensional analogues of Cauchy-integral.
- Double layer potential operators.
- Derivatives of Single layer potentials.
- Allowed the use of potential theory to solve BVP on harmonic analysis.

Still in some applications it was not evident the existence of a single accretive or pseudo-accretive function $b$ such that $Tb$ was well-behaved.
Local Tb theorem [Christ]

Suppose that $T$ is a SIO associated to a standard CZ kernel, which in addition we assume to be in $L^\infty$. Suppose also that there are constants $\delta > 0$ and $C_0 < \infty$, and pseudo-accretive systems $\{b_Q^1\}$, $\{b_Q^2\}$, with $\text{supp} b_Q^i \subseteq Q$, $i = 1, 2$, such that for each dyadic cube $Q$.

- $\|b_Q^1\|_{L^\infty(Q)} + \|b_Q^2\|_{L^\infty(Q)} \leq C_0$
- $\|Tb_Q^1\|_{L^\infty(Q)} + \|T^*b_Q^2\|_{L^\infty(Q)} \leq C_0$
- $\min\{|\int_Q b_Q^1|, |\int_Q b_Q^2|\} \geq \delta |Q|$.

Then $T$ extends to a bounded operator on $L^2$. 
Singular Integral operators

- Conditions have been relaxed on the theorem to

\[(i) \int_Q |b_Q^i|^p \leq C_0|Q|,\]
\[(ii) \int_Q |Tb_Q^1|^{p'} \leq C_0|Q|, \int_Q |T^*b_Q^2|^{p'} \leq C_0|Q|\]

for \(1 < p < \infty\).

- More general kernels.
- Non-homogeneous setting, product spaces, bi-parameter SIO.
General Calderón-Zygmund kernels

\[ Tf(x) = \int_{\mathbb{R}^d} K(x, y)f(y)d\mu(y), \quad x \notin \text{supp}f \]
General Calderón-Zygmund kernels

\[ Tf(x) = \int_{\mathbb{R}^d} K(x, y) f(y) d\mu(y), \quad x \notin \text{supp} f \]

where \( \mu \) is a Borel measure on \( \mathbb{R}^d \) satisfying that there exists \( n \in (0, d] \) such that \( \forall x \in \mathbb{R}^d \) and \( \forall r > 0, \mu(B(x, r)) \leq C \cdot r^n \).
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\[ |K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \frac{C}{|x-y|^n} \omega\left(\frac{|x-x'|}{|x-y|}\right) \]

whenever \( |x - x'| \leq 1/2|x - y| \)
General Calderón-Zygmund kernels

\[ Tf(x) = \int_{\mathbb{R}^d} K(x, y)f(y) d\mu(y), \ x \notin \text{supp}f \]

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General Calderón-Zygmund kernel operators

\[ |K(x, y)| \leq \frac{C}{|x-y|^n} \]

\[ |K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \frac{C}{|x-y|^n} \omega \left( \frac{|x-x'|}{|x-y|} \right) \]

whenever \( |x - x'| \leq 1/2|x - y| \)

Before \( \omega(t) = t^\alpha \ for \ \alpha > 0 \).
Modulus of continuity

\[ \omega(0) = 0, \quad \omega(1) = 1. \]

\[ t \mapsto \frac{\omega(t)}{t} \text{ with } \lim_{t \to 0} \frac{\omega(t)}{t} = \infty. \]

\[ \int_{0}^{1} \omega(t) \, dt \leq C \]

Dini condition
Modulus of continuity

- Strictly increasing.

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\[ t \mapsto \frac{\omega(t)}{t} \]

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Modulus of continuity

- Strictly increasing.
- $\omega(0) = 0$, $\omega(1) = 1$.
- $t \mapsto \omega(t)/t$ with $\lim_{t \to 0} \omega(t)/t = \infty$. 

$m \leq C$ Dini condition
Modulus of continuity

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- \( t \mapsto \omega(t)/t \) with \( \lim_{t \to 0} \omega(t)/t = \infty. \)

\[
\int_0^1 \omega(t) \, dt \leq C \text{ Dini condition}
\]

\( \frac{dt}{t} \leq C \text{ Dini condition} \)
Modulus of continuity

- Strictly increasing.
- \( \omega(0) = 0, \omega(1) = 1. \)
- \( t \mapsto \omega(t)/t \) with \( \lim_{t \to 0} \omega(t)/t = \infty. \)
- 
  \[
  \int_0^1 \omega(t) \log(1/t)^{1/2} \frac{dt}{t} \leq C
  \]
Let $T$ be a general Calderón-Zygmund operator satisfying

$$
\begin{align*}
\|T1_Q\|_{L^2(Q)} & \leq C\mu^{1/2}(Q), \\
\|T^*1_Q\|_{L^2(Q)} & \leq C\mu^{1/2}(Q).
\end{align*}
$$

In case that $\mu$ is a finite Borel measure we further require that

$$
\begin{align*}
\|T1\|_{L^2(\mathbb{R}^d)} & \leq C\mu^{1/2}(\mathbb{R}^d), \\
\|T^*1\|_{L^2(\mathbb{R}^d)} & \leq C\mu^{1/2}(\mathbb{R}^d).
\end{align*}
$$

Then $T$ is a bounded Calderón-Zygmund operator, i.e.,

$T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.
We define the **Haar functions** as a collection of functions \( \{ \varphi_n^I \}_{n=1}^{2^d-1} \) satisfying

\[
\text{supp}(\varphi_n^I) \subseteq I \\
\int_I \varphi_n^I \, d\mu = 0 \\
\|\varphi_n^I\|_\infty \cdot \|\varphi_n^I\|_1 \leq C \]
We define the **Haar functions** as a collection of functions \( \{ \varphi^n_l \}_{n=1}^{2^{d-1}} \) satisfying

- \( \text{supp}(\varphi^n_l) \subseteq l \)
**Haar functions**

**Definition**

We define the **Haar functions** as a collection of functions \( \{ \varphi^n_I \}_{n=1}^{2d-1} \) satisfying

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Haar functions

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- \( D_I f = \sum_{n=1}^{2^d-1} \langle f, \varphi^n_I \rangle \varphi^n_I \)
We define the **Haar functions** as a collection of functions \( \{ \varphi^n_l \}_{n=1}^{2^d-1} \) satisfying:

- \( \text{ supp}(\varphi^n_l) \subseteq l \)
- \( \int \varphi^n_l d\mu = 0 \)
- \( \| \varphi^n_l \|_\infty \cdot \| \varphi^n_l \|_1 \leq C \)
- \( D_l f = \sum_{n=1}^{2^d-1} \langle f, \varphi^n_l \rangle \varphi^n_l \)

\[
E_l f(x) = \frac{1_l(x)}{\mu(l)} \int_l f(y) d\mu(y)
\]

\[
D_l f(x) = \sum_{l' \in \text{ch}(l)} E_{l'} f(x) - E_l f(x)
\]
Definition

We define the **Haar functions** as a collection of functions \( \{ \varphi^n_I \}_{n=1}^{2^d-1} \) satisfying

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E_I f(x) = \frac{1_I(x)}{\mu(I)} \int_I f(y) \, d\mu(y)
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\]

**Note.** \( \sum_I D_I f(x) = f(x) \)
Haar functions $n = 1$

$$\varphi_I(x) \begin{cases} 
C_l/\mu(l^-) & \text{if } x \in l^- \\
-C_l/\mu(l^+) & \text{if } x \in l^+
\end{cases}$$

where $C_l^2 = \frac{\mu(l^+)\mu(l^-)}{\mu(l)}$. 
Haar functions $n = 1$

$$\varphi_I(x) \begin{cases} C_I/\mu(I^-) & \text{if } x \in I^- \\ -C_I/\mu(I^+) & \text{if } x \in I^+ \end{cases}$$

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Frequently added properties
Haar functions $n = 1$

\[ \varphi_I(x) \begin{cases} 
\frac{C_I}{\mu(I^-)} & \text{if } x \in I^- \\
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\end{cases} \]

where $C_I^2 = \frac{\mu(I^+)\mu(I^-)}{\mu(I)}$. 

Frequently added properties

- $\varphi_I$ is constant for any $Q \in ch(I)$. 

Haar functions $n = 1$

$$
\varphi_l(x) = \begin{cases} 
\frac{C_l}{\mu(l^-)} & \text{if } x \in l^- \\
-\frac{C_l}{\mu(l^+)} & \text{if } x \in l^+ 
\end{cases}
$$

where $C_l^2 = \frac{\mu(l^+)\mu(l^-)}{\mu(l)}$.

**Frequently added properties**

- $\varphi_l$ is constant for any $Q \in ch(l)$.
- $\{\varphi_l\}$ is an orthogonal family of functions.
Remember!!!
We want to bound the function using the local information that we have.
Haar functions

Definition

We define the **Haar functions** as a collection of functions \( \{\varphi^n_I\}_{n=1}^{2^d-1} \) satisfying

- \( \text{supp}(\varphi^n_I) \subseteq I \)
- \( \int \varphi^n_I \, d\mu = 0 \)
- \( \|\varphi^n_I\|_{\infty} \cdot \|\varphi^n_I\|_1 \leq C \)
- \( D_I f = \sum_{n=1}^{2^d-1} \langle f, \varphi^n_I \rangle \varphi^n_I \)

\[ E_I f(x) = \frac{1_I(x)}{\mu(I)} \int_I f(y) \, d\mu(y) \]

\[ D_I f(x) = \sum_{I' \in \text{ch}(I)} E_{I'} f(x) - E_I f(x) \]

**Note.** \( \sum_I D_I f(x) = f(x) \)
New dyadic Representation Theorem

\[ \langle T f, g \rangle := \sum_{I,J} \langle T D_I f, D_J g \rangle \]

Separation of cases

We separate the case when \( \ell(I) \sim \ell(J) \) and otherwise.

Only keep cubes that are “good.”
New dyadic Representation Theorem

\[ \langle Tf, g \rangle := \sum_{I,J} \langle TD_I f, D_J g \rangle \]

\[ \langle Tf, g \rangle := \sum_{I,J} \sum_{n_1, n_2} \langle f, \varphi_I^{n_1} \rangle \langle T \varphi_I^{n_1}, \varphi_J^{n_2} \rangle \langle g, \varphi_J^{n_2} \rangle \]
New dyadic Representation Theorem

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**Separation of cases**

- We separate the case when \( \ell(I) \sim \ell(J) \) and otherwise.
- Only keep cubes that are “good”.
Let $r \in \mathbb{N}$ and $r \geq 2$, we say that a cube $I$ is $r$–good if $\text{dist}(I, \partial I^{(r)}) \geq \frac{1}{4} \ell(I^{(r)})$. 

**Figure:** $r$-goodness
Dyadic representation theorem

New representation theorem

\[ \langle Tf, g \rangle = \frac{1}{\pi_{\text{good}}} \mathbb{E}_{\sigma} \left[ \sum_{k=2}^{\infty} \omega(2^{-k}) \langle S_k f, g \rangle + \sum_{k=2}^{\infty} \omega(2^{-k}) \langle \tilde{S}_k f, g \rangle ight. \\
+ \sum_{k=1}^{\infty} \omega(2^{-k}) \langle R_k f, g \rangle + \sum_{k=1}^{\infty} \omega(2^{-k}) \langle \tilde{R}_k f, g \rangle \\
\left. + \langle \Pi T_1 f, g \rangle + \langle \Pi^* T_1 f, g \rangle \right] \]
Dyadic representation theorem

New representation theorem

\[ \langle Tf, g \rangle = \frac{1}{\pi_{\text{good}}} E_\sigma \left[ \sum_{k=2}^{\infty} \omega(2^{-k}) \langle S_k f, g \rangle + \sum_{k=2}^{\infty} \omega(2^{-k}) \langle \tilde{S}_k f, g \rangle \right. \]
\[ \left. + \sum_{k=1}^{\infty} \omega(2^{-k}) \langle R_k f, g \rangle + \sum_{k=1}^{\infty} \omega(2^{-k}) \langle \tilde{R}_k f, g \rangle + \langle \Pi_{T_1} f, g \rangle + \langle \Pi^*_{T^*_1} f, g \rangle \right] \]

Representation theorem thru dyadic shifts

\[ \langle Tf, g \rangle = C_1 E \left[ \sum_{i,j=0}^{\max\{i,j\}>0} \tau(i, j) \langle S^{ij} f, g \rangle \right. \]
\[ \left. + C_2 E \left[ \langle S^{00} f, g \rangle + \langle \Pi_{T_1} f, g \rangle + \langle \Pi^*_{T^*_1} f, g \rangle \right] \right] \]
Dyadic representation theorem

New representation theorem

\[ \langle S_k f, g \rangle = \sum_{K \in \mathcal{D}} \langle A_K^{(k)} f, g \rangle, \quad A_K^{(k)} f = \int a_K^{(k)}(x, y) f(y) dy \]

\[ \langle R_k f, g \rangle = \sum_{K \in \mathcal{D}} \langle B_K^{(k)} f, g \rangle, \quad B_K^{(k)} f = \int b_K^{(k)}(x, y) f(y) dy \]

Representation theorem thru dyadic shifts

\[ \langle S^{ij} f, g \rangle = \sum_{K \in \mathcal{D}} \langle C_K^{ij} f, g \rangle, \quad C_K^{ij} f = \sum_{\substack{I^{(i)} = K \\ j(I) = K \\ J^{(j)} = K}} c_{IKJ} \langle f, h_I \rangle h_J \]
Dyadic representation theorem

New representation theorem

\[ |a_K^{(k)}(x, y)| \leq C \left[ \frac{1_K(x)1_K(y)}{\ell(K)^n} + \sum_{H: H^{(k)}=K} \frac{1_H(x)1_H(y)}{\mu(H)} \right] \text{ for } k \geq 2 \]

\[ \langle R_k f, g \rangle = \sum_{K \in \mathcal{D}} \langle B_K^{(k)} f, g \rangle, \quad B_K^{(k)} f = \int b_k^{(k)}(x, y)f(y)dy \]

Representation theorem thru dyadic shifts

\[ \langle S_{ij}^{ij} f, g \rangle = \sum_{K \in \mathcal{D}} \langle C_K^{ij} f, g \rangle, \quad C_K^{ij} f = \sum_{l(i)=K} c_{lJK} \langle f, h_l \rangle h_J \]
Dyadic representation theorem

New representation theorem

\[ |a_{K}^{(k)}(x, y)| \leq C \left[ \frac{1_K(x)1_K(y)}{\ell(K)^n} + \sum_{H: H^{(k)}=K \text{ k-good}} \frac{1_H(x)1_H(y)}{\mu(H)} \right] \text{ for } k \geq 2 \]

\[ |b_{K}^{(k)}(x, y)| \leq C \left[ \frac{1_K(x)1_K(y)}{\ell(K)^n} \right] \text{ for } k \geq 2 \]

Representation theorem thru dyadic shifts

\[ \langle S^{ij}f, g \rangle = \sum_{K \in \mathcal{D}} \langle C_{K}^{ij}f, g \rangle, \quad C_{K}^{ij}f = \sum_{I^{(i)}=K} c_{IJK} \langle f, h_I \rangle h_J \]
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New representation theorem

\[
|a_K^{(k)}(x, y)| \leq C \left[ \frac{1_K(x)1_K(y)}{\ell(K)^n} + \sum_{H: H^{(k)}=K \text{ and } H \text{ $k$-good}} \frac{1_H(x)1_H(y)}{\mu(H)} \right] \quad \text{for } k \geq 2
\]

\[
|b_K^{(k)}(x, y)| \leq C \left[ \frac{1_K(x)1_K(y)}{\ell(K)^n} \right] \quad \text{for } k \geq 2
\]

Representation theorem thru dyadic shifts

\[
C_K^{ij}f = \sum_{\substack{I^{(i)}=K \\ J^{(i)}=K}} c_{IJK} \langle f, h_I \rangle h_J , \quad \sum_{\substack{I^{(i)}=K \\ J^{(i)}=K}} |c_{IJK}|^2 \leq 1.
\]
Dyadic representation theorem

New representation theorem

\[ \|S_k\|_{2\to2} \leq Ck^{1/2} \]
\[ \|R_k\|_{2\to2} \leq C \]

Representation theorem thru dyadic shifts

\[ \|S^{ij}\|_{2\to2} \leq C \cdot k. \]
other properties New representation theorem

\begin{align*}
\forall k \geq 2, \quad &
\|S_k\|_{L^1 \to L^1, \infty} + \|R_k\|_{L^1 \to L^1, \infty} \\
&\leq C \cdot k
\end{align*}

Control by Dyadic shifts in the homogeneous setting

$S_k, R_k$ can be written as a sum of $O(k)$ dyadic shifts of complexity $O(k)$. 

Ana Grau de la Herrán – University of Helsinki

Workshop for Women in Analysis and PDE. IMA 2015.
Orthogonality

\[
A^{(k)}_K = \sum_{j=0}^{k-1} D^{(j)}_k A^{(k)}_K D^{(k+r)}_K \\
B^{(k)}_K = \sum_{s=0}^{r} B^{(k,s)}_K, \quad B^{(k,s)}_K = D^{(r+k)}_k B^{(k,s)}_K D^{(s+k)}_K
\]
Orthogonality

\[
A^{(k)}_K = \sum_{j=0}^{k-1} D^{(j)}_k A^{(k)}_K D^{(k+r)}_K
\]

\[
B^{(k)}_K = \sum_{s=0}^{r} B^{(k,s)}_K, \quad B^{(k,s)}_K = D^{(r+k)}_k B^{(k,s)}_K D^{(s+k)}_K
\]

Weak (1,1) property in the homogeneous setting

\[
\|S_k\|_{L^1 \to L^{1,\infty}} + \|R_k\|_{L^1 \to L^{1,\infty}} \leq C \cdot k \text{ for } k \geq 2
\]
Orthogonality

\[ A_K^{(k)} = \sum_{j=0}^{k-1} D_k^{(j)} A_K^{(k)} D_{K}^{(k+r)} \]

\[ B_K^{(k)} = \sum_{s=0}^{r} B_K^{(k,s)}, \quad B_K^{(k,s)} = D_k^{(r+k)} B_K^{(k,s)} D_K^{(s+k)} \]

Weak (1,1) property in the homogeneous setting

\[ \|S_k\|_{L^1 \to L^{1,\infty}} + \|R_k\|_{L^1 \to L^{1,\infty}} \leq C \cdot k \text{ for } k \geq 2 \]

Control by Dyadic shifts in the homogeneous setting

\( S_k, R_k \) can be written as a sum of \( O(k) \) dyadic shifts of complexity \( O(k) \).
Thank you!!!