Selection of defective components in unknown backgrounds

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Goal: Determine defects or defective components in a complex and unknown medium from multi-static measurements of scattered waves at a given frequency.

Constraints:

- The background is unknown and cannot be accurately reconstructed.
- The background components diameters are comparable to the wavelength.

But: We have access to differential measurements: measurements with and without defects.
The original motivation
Detection of defects in a concrete like material using ultrasounds

An example of a concrete structure
The original motivation
Detection of defects in a concrete like material using ultrasounds

A simulation using the Linear Sampling Method (without differential measurements)

An example of synthetic background

A filtered background + crack
The original motivation
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Reconstructed crack using the exact background
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Reconstructed crack using homogeneous background: weak perturbation
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A filtered background + crack

Reconstructed crack using the exact background

Reconstructed crack using homogeneous background: strong perturbation
Outline

▶ A model problem (not for cracks)
▶ The Linear Sampling Method revisited
▶ Application to the case of differential measurements
▶ Numerical results and perspectives
A simple model problem
Scalar acoustic equation for inhomogeneous media

The background index $n_0 : n_0 = 1$ in $\mathbb{R}^d \setminus D_0$ and $\mathbb{R}^d \setminus D_0$ is connected.

The modified index $n : n = 1$ in $\mathbb{R}^d \setminus D$ and $\mathbb{R}^d \setminus D$ is connected.

The total fields $u_0 \in H^1_{loc}(\mathbb{R}^d)$ and $u \in H^1_{loc}(\mathbb{R}^d)$

$$\Delta u_0 + k^2 n_0 u_0 = 0 \text{ and } \Delta u + k^2 n u = 0 \text{ in } \mathbb{R}^d$$

We assume that the field is generated by incident plane waves:

$$u^i(\theta, x) := e^{i k x \cdot \theta} \quad \theta \in S^{d-1}$$

The scattered fields

$$u^s_0(\theta, \cdot) = u_0 - u^i(\theta, \cdot) \text{ and } u^s(\theta, \cdot) = u - u^i(\theta, \cdot) \text{ in } \mathbb{R}^d,$$

satisfy the Sommerfeld radiation condition.
A simple model problem
Scalar acoustic equation for inhomogeneous media

The background index \( n_0 : \ n_0 = 1 \) in \( \mathbb{R}^d \setminus D_0 \) and \( \mathbb{R}^d \setminus D_0 \) is connected.

The modified index \( n : \ n = 1 \) in \( \mathbb{R}^d \setminus D \) and \( \mathbb{R}^d \setminus D \) is connected.

Our data is formed by (noisy measurements of) so-called farfield patterns

\[ u_0^\infty(\theta, \hat{x}) \text{ and } u^\infty(\theta, \hat{x}) \] for all \( (\theta, \hat{x}) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \)
A simple model problem
Scalar acoustic equation for inhomogeneous media

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Our data is formed by (noisy measurements of) so-called farfield patterns

$$u_0^\infty(\theta, \hat{x}) \text{ and } u^\infty(\theta, \hat{x}) \text{ for all } (\theta, \hat{x}) \in S^{d-1} \times S^{d-1}$$

Recall that with $\hat{x} := x/|x|$, 

$$u_0^s(\theta, x) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}} \left(u_0^\infty(\theta, \hat{x}) + O(1/|x|)\right)$$

$$u^s(\theta, x) = \frac{e^{ik|x|}}{|x|^{(d-1)/2}} \left(u^\infty(\theta, \hat{x}) + O(1/|x|)\right)$$

as $|x| \to \infty$.
A simple model problem
Scalar acoustic equation for inhomogeneous media

The background index \( n_0 : n_0 = 1 \) in \( \mathbb{R}^d \setminus D_0 \) and \( \mathbb{R}^d \setminus D_0 \) is connected.

The modified index \( n : n = 1 \) in \( \mathbb{R}^d \setminus D \) and \( \mathbb{R}^d \setminus D \) is connected.

Our data is formed by (noisy measurements of) so-called farfield patterns

\[
\begin{align*}
u_0^\infty(\theta, \hat{x}) & \quad \text{and} \quad u^\infty(\theta, \hat{x})
\end{align*}
\]

for all \( (\theta, \hat{x}) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \)

**Goal:** Assuming that \( D_0 \subset D \) and would like to reconstruct (an approxima-

\[
\Omega \equiv \text{supp}(n - n_0)
\]

without knowing (or approximating) \( n \) and \( n_0 \).
A simple model problem
Scalar acoustic equation for inhomogeneous media

The background index $n_0 : n_0 = 1$ in $\mathbb{R}^d \setminus D_0$ and $\mathbb{R}^d \setminus D_0$ is connected.

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Our data is formed by (noisy measurements of) so-called farfield patterns $u_0^\infty(\theta, \hat{x})$ and $u^\infty(\theta, \hat{x})$ for all $(\theta, \hat{x}) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}.$

**Goal:** Assuming that $D_0 \subset D$ and would like to reconstruct (an approximation of) $\Omega \equiv \text{supp}(n - n_0)$ without knowing (or approximating) $n$ and $n_0.$

**Algorithm:** introduce a filtered difference between the indicator functions provided by a modified version of the Linear Sampling Method (LSM) applied to each set of data separately.
A generalized version of LSM

▶ A version based on a new exact characterization of the scatterer geometry in terms of the farfields.

▶ A version capable of answering the imaging problem for differential measure: explicit link with solutions of the interior transmission problem.

▶ A flexible setting that can be generalized to limited aperture or/and near-field data (ongoing).

Outline of LSM

Farfield Operator: \( F : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1}) \), defined by

\[
Fg(\hat{x}) := \int_{\mathbb{S}^{d-1}} u^\infty(\theta, \hat{x})g(\theta)ds(\theta).
\]

Let us define for \( \psi \in L^2(D) \), the unique function \( w \in H^1_{\text{loc}}(\mathbb{R}^d) \) satisfying

\[
\begin{cases}
\Delta w + nk^2w = k^2(1 - n)\psi \text{ in } \mathbb{R}^d, \\
\lim_{r \to \infty} \int_{|x|=r} |\frac{\partial w}{\partial r} - ikw|^2 \, ds = 0.
\end{cases}
\]

(1)

Remark

\[
\psi = u^i(\theta, \cdot) \Rightarrow w = u^s(\theta, \cdot) \Rightarrow w^\infty = u^\infty(\theta, \cdot)
\]

\( \Rightarrow Fg \) is nothing but \( w^\infty \) for \( w \) solution of (1) with \( \psi = v_g \) in \( D \), where

\[
v_g(x) := \int_{\mathbb{S}^{d-1}} u^i(\theta, x)g(\theta)ds(\theta), \, g \in L^2(\mathbb{S}^{d-1}), \, x \in \mathbb{R}^d.
\]
Outline of LSM

Farfield Operator: $F : L^2(S^{d-1}) \to L^2(S^{d-1})$, defined by

$$Fg(\hat{x}) := \int_{S^{d-1}} u^\infty(\theta, \hat{x}) g(\theta) d\theta.$$  

⇒ Considering the (compact) operator $H : L^2(S^{d-1}) \to L^2(D)$ defined by

$$Hg := v_g|_D,$$  

and the (compact) operator $G : \mathcal{R}(H) \subset L^2(D) \to L^2(S^{d-1})$, defined by

$$G\psi := w^\infty,$$

then clearly:

$$F = G \circ H.$$
Main ingredient of LSM

**Theorem:** Assume that ITP is well posed. With \( \phi_z^\infty(\hat{x}) = e^{-ik\hat{x} \cdot z} \) we have: \( \phi_z^\infty \in \mathcal{R}(G) \) if and only if \( z \in D \).

Main ingredients of the proof:

- \( \mathcal{R}(H) = \{ v \in L^2(D); \Delta v + k^2 v = 0 \text{ in } D \} \).
- \( \phi_z^\infty \) is the farfield of \( \Phi(\cdot, z) \), radiating solution of \( \Delta \Phi + k^2 \Phi = -\delta_z \).

**Interior Transmission Problem** (ITP): \( (u, v) \in L^2(D) \times L^2(D) \) such that \( u - v \in H^2(D) \) and

\[
\begin{align*}
\Delta u + k^2 nu &= 0 \quad \text{in } D, \\
\Delta v + k^2 v &= 0 \quad \text{in } D, \\
(u - v) &= f \quad \text{on } \partial D, \\
\frac{\partial}{\partial n}(u - v) &= g \quad \text{on } \partial D,
\end{align*}
\]

for given \( f \in H^{3/2}_2(\partial D) \) and \( g \in H^{1/2}_1(\partial D) \).

**Remark:** A well posed ITP requires \( n \neq 1 \) in any neighborhood of \( \partial D \).
Main theorem of LSM

Farfield Operator: \( F : L^2(S^{d-1}) \to L^2(S^{d-1}) \), defined by

\[
Fg(\hat{x}) := \int_{S^{d-1}} u^\infty(\theta, \hat{x}) g(\theta) ds(\theta).
\]

**Theorem:** Assume that ITP is well posed. Then the operator \( F \) is injective with dense range. Moreover, the following holds.

- If \( z \in D \) then there exists \( g^\varepsilon_z \) such that \( \| F g^\varepsilon_z - \phi_z \|_{L^2(S^{d-1})} \leq \varepsilon \) and \( \limsup_{\varepsilon \to 0} \| H g^\varepsilon_z \|_{L^2(D)} < \infty \).

- If \( z \notin D \) then for all \( g^\varepsilon_z \) such that \( \| F g^\varepsilon_z - \phi_z \|_{L^2(S^{d-1})} \leq \varepsilon \), \( \lim_{\varepsilon \to 0} \| H g^\varepsilon_z \|_{L^2(D)} = \infty \).

⇒ Gives a "characterization" of \( D \) in terms of a nearby solutions of \( F g^\varepsilon_z \approx \phi_z \).

Problems: This is not constructive...

- We do not know how to construct \( g^\varepsilon_z \). In practice we use a regularization scheme.

- We cannot compute \( \| H g^\varepsilon_z \|_{L^2(D)} \). In practice we use \( \| g^\varepsilon_z \|_{L^2(S^{d-1})} \).
Main theorem of LSM

**Farfield Operator:** $F : L^2(\mathbb{S}^{d-1}) \rightarrow L^2(\mathbb{S}^{d-1})$, defined by

$$Fg(\hat{x}) := \int_{\mathbb{S}^{d-1}} u^\infty(\theta, \hat{x}) g(\theta) ds(\theta).$$

**Theorem:** Assume that ITP is well posed. Then the operator $F$ is injective with dense range. Moreover, the following holds.

- If $z \in D$ then there exists $g_z^\epsilon$ such that $\|Fg_z^\epsilon - \phi_z\|_{L^2(\mathbb{S}^{d-1})} \leq \epsilon$ and
  $$\limsup_{\epsilon \rightarrow 0} \|Hg_z^\epsilon\|_{L^2(D)} < \infty.$$

- If $z \notin D$ then for all $g_z^\epsilon$ such that $\|Fg_z^\epsilon - \phi_z\|_{L^2(\mathbb{S}^{d-1})} \leq \epsilon$, 
  $$\lim_{\epsilon \rightarrow 0} \|Hg_z^\epsilon\|_{L^2(D)} = \infty.$$

\Rightarrow\text{ Gives a “characterization” of } D \text{ in terms of a nearby solutions of } Fg_z^\epsilon \simeq \phi_z.

**Problems:** This is not constructive...

- We do not know how to construct $g_z^\epsilon$. In practice we use a regularization scheme.

- We cannot compute $\|Hg_z^\epsilon\|_{L^2(D)}$. In practice we use $\|g_z^\epsilon\|_{L^2(\mathbb{S}^{d-1})}$. 

A robust formulation of LSM

**Idea:** Reconstruct a nearby solution of the LSM by using a least squares misfit functional with a penalty term that controls $\|Hg^\epsilon\|_{L^2(D)}^2$. 

$$w_\infty(\hat{x}) = -\int_D e^{-iky}\hat{x}(1-n)k^2(\psi(y) + w(y))dy,$$

$$G = H^\ast \psi \text{ where } H^\ast : L^2(D) \rightarrow L^2(Sd^{-1}) \text{ is the adjoint of } H \text{ given by } H^\ast \phi(\hat{x}) = \int_D e^{-iky}\hat{x}\phi(y)dy, \phi \in L^2(D), \hat{x} \in Sd^{-1},$$

$$F = H^\ast \circ T \circ H.$$
A robust formulation of LSM

Idea: Reconstruct a nearby solution of the LSM by using a least squares misfit functional with a penalty term that controls $\|Hg_\varepsilon\|_{L^2(D)}^2$.

We exploit the (second) Factorization:

$$w^\infty(\hat{x}) = -\int_D e^{-iky \cdot \hat{x}} (1 - n) k^2 (\psi(y) + w(y)) dy,$$

$\Rightarrow G = H^* T \psi$ where $H^* : L^2(D) \rightarrow L^2(S^{d-1})$ is the adjoint of $H$ given by

$$H^* \varphi(\hat{x}) := \int_D e^{-iky \cdot \hat{x}} \varphi(y) dy, \quad \varphi \in L^2(D), \quad \hat{x} \in S^{d-1},$$

and where $T : L^2(D) \rightarrow L^2(D)$ is defined by

$$T \psi := -k^2 (1 - n)(\psi + w), \quad (4)$$

$$F = H^* \circ T \circ H$$
A robust formulation of LSM

**Idea:** Reconstruct a nearby solution of the LSM by using a least squares misfit functional with a penalty term that controls $\|Hg^e\|_{L^2(D)}^2$.

$$F = H^* \circ T \circ H$$

**Theorem:** Assume that (ITP) is well posed and there exists $n_0 > 0$ and $\alpha > 0$ such that

1. $1 - \Re(n(x)) + \alpha \Im(n(x)) \geq n_0$ for a.e. $x \in D$
2. $\Re(n(x)) - 1 + \alpha \Im(n(x)) \geq n_0$ for a.e. $x \in D$.

Then: $\|(T\psi, \psi)_{L^2(D)}\| \geq c\|\psi\|_{L^2(D)}^2$ for all $\psi \in \mathcal{R}(H)$.

$\Rightarrow \|(Fg, g)_{L^2(S^{d-1})}\| \geq c\|Hg\|_{L^2(D)}^2$

$\Rightarrow \|(Fg, g)_{L^2(S^{d-1})}\| \text{ is equivalent to } \|Hg\|_{L^2(D)}^2$
Abstract setting for a Generalized LSM (GLSM)

We consider two bounded linear operators $F : X \to X^*$ and $B : X \to X^*$

\[ F = GH \quad \text{and} \quad B = H^* TH \]

$H : X \to Y$, $T : Y \to Y^*$ and $G : \overline{\mathcal{R}(H)} \subset Y \to X^*$ are bounded.
Abstract setting for a Generalized LSM (GLSM)

We consider two bounded linear operators $F : X \rightarrow X^*$ and $B : X \rightarrow X^*$

$$F = GH \quad \text{and} \quad B = H^* TH$$

$H : X \rightarrow Y$, $T : Y \rightarrow Y^*$ and $G : \overline{\mathcal{R}(H)} \subset Y \rightarrow X^*$ are bounded.

For $\alpha > 0$ be a given parameter and $\phi \in X^*$ we consider:

$$J_\alpha(\phi; g) := \alpha |\langleBg, g\rangle| + \|Fg - \phi\|^2 \quad \forall g \in X.$$  

**Remark** This functional has not a minimizer in general!

Assume that $F$ has dense range. Then for all $\phi \in X^*$,

$$j_\alpha(\phi) := \inf_{g \in X} J_\alpha(\phi; g) \rightarrow 0 \text{ as } \alpha \rightarrow 0.$$
Abstract setting for a Generalized LSM (GLSM)

We consider two bounded linear operators $F : X \to X^*$ and $B : X \to X^*$

\[
F = GH \quad \text{and} \quad B = H^* TH
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$H : X \to Y$, $T : Y \to Y^*$ and $G : \overline{\mathcal{R}(H)} \subset Y \to X^*$ are bounded. For $\alpha > 0$ be a given parameter and $\phi \in X^*$ we consider:

\[
J_\alpha(\phi; g) := \alpha |\langle Bg, g \rangle| + \|Fg - \phi\|^2 \quad \forall g \in X.
\]

Remark This functional has not a minimizer in general!

Assume that $F$ has dense range. Then for all $\phi \in X^*$,

\[
j_\alpha(\phi) := \inf_{g \in X} J_\alpha(\phi; g) \to 0 \text{ as } \alpha \to 0.
\]

$\Rightarrow$ Nearby solutions (of the farfield equation) are given by $g_\alpha \in X$ such that

\[
J_\alpha(\phi; g_\alpha) \leq j_\alpha(\phi) + p(\alpha).
\]

where $p(\alpha) > 0$ is such that $p(\alpha) \to 0$ as $\alpha \to 0$
Main theorem of GLSM (for noise free)

\[ F : X \to X^*, \ B : X \to X^* \text{ and } F = GH \quad \text{and} \quad B = H^* TH \]

\[ J_\alpha(\phi; g) := \alpha |\langle Bg, g \rangle| + \|Fg - \phi\|^2 \quad \forall g \in X. \]

**Theorem:** We assume in addition that

- \( G \) is compact and \( F = GH \) has dense range.
- \( T \) satisfies: \(|\langle T \varphi, \varphi \rangle| > \mu \|\varphi\|^2 \quad \forall \varphi \in \mathcal{R}(H)\).

Consider for \( \alpha > 0 \) and \( \phi \in X^* \), \( g_\alpha \in X \) such that

\[ J_\alpha(\phi; g_\alpha) \leq j_\alpha(\phi) + p(\alpha) \text{ and } p(\alpha) \leq C_\alpha. \]

- \( \phi \in \mathcal{R}(G) \Rightarrow \limsup_{\alpha \to 0} |\langle Bg_\alpha, g_\alpha \rangle| < \infty. \)
- \( \phi \notin \mathcal{R}(G) \Rightarrow \lim_{\alpha \to 0} |\langle Bg_\alpha, g_\alpha \rangle| = \infty. \)
Main theorem of GLSM (for noise free)

\[ F : X \rightarrow X^*, \quad B : X \rightarrow X^* \quad \text{and} \quad F = GH \quad \text{and} \quad B = H^* TH \]

\[ J_\alpha(\phi; g) := \alpha |\langle Bg, g \rangle| + \|Fg - \phi\|^2 \quad \forall g \in X. \]

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- \( G \) is compact and \( F = GH \) has dense range.
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Consider for \( \alpha > 0 \) and \( \phi \in X^*, \ g_\alpha \in X \) such that

\[ J_\alpha(\phi; g_\alpha) \leq j_\alpha(\phi) + p(\alpha) \quad \text{and} \quad p(\alpha) \leq C\alpha. \]

- \( \phi \in \mathcal{R}(G) \Rightarrow \limsup_{\alpha \to 0} |\langle Bg_\alpha, g_\alpha \rangle| < \infty. \)
- \( \phi \notin \mathcal{R}(G) \Rightarrow \lim_{\alpha \to 0} |\langle Bg_\alpha, g_\alpha \rangle| = \infty. \)

**Application:** \( \mathcal{R}(G) \) characterizes the inclusion \( D \Rightarrow F \) and \( B \) uniquely determine \( D. \)
Main theorem of GLSM (for noise free)

\( F : X \rightarrow X^* \), \( B : X \rightarrow X^* \) and \( F = GH \) and \( B = H^* TH \)

\[
J_\alpha(\phi; g) := \alpha |\langle Bg, g \rangle| + \|Fg - \phi\|^2 \quad \forall g \in X.
\]

**Theorem:** We assume in addition that

- \( G \) is compact and \( F = GH \) has dense range.
- \( T \) satisfies:
  \[
  |\langle T \varphi, \varphi \rangle| > \mu \|\varphi\|^2 \quad \forall \varphi \in \mathcal{R}(H).
  \]

Consider for \( \alpha > 0 \) and \( \phi \in X^* \), \( g_\alpha \in X \) such that

\[
J_\alpha(\phi; g_\alpha) \leq j_\alpha(\phi) + p(\alpha) \quad \text{and} \quad p(\alpha) \leq C\alpha.
\]

- \( \phi \in \mathcal{R}(G) \Rightarrow \limsup_{\alpha \to 0} |\langle Bg_\alpha, g_\alpha \rangle| < \infty. \)
- \( \phi \notin \mathcal{R}(G) \Rightarrow \lim_{\alpha \to 0} |\langle Bg_\alpha, g_\alpha \rangle| = \infty. \)

Application with the natural choice: $B = F$

For $z \in \mathbb{R}^d$ we consider $g^z_\alpha \in L^2(S^{d-1})$ such that

$$J_\alpha(\phi_\alpha^\infty; g^z_\alpha) \leq j_\alpha(\phi_\alpha^\infty) + p(\alpha) \text{ and } p(\alpha) \leq C\alpha.$$ 

$\phi_\alpha^\infty(\hat{x}) = e^{-i k \hat{x} \cdot z}$

**Theorem:** Assume that there exists $n_* > 0$ and $\alpha > 0$ such that

1. $1 - \Re(n(x)) + \alpha \Im(n(x)) \geq n_*$ for a.e. $x \in D$ or
2. $\Re(n(x)) - 1 + \alpha \Im(n(x)) \geq n_*$ for a.e. $x \in D$.

Then, except for a countable set of $k$ (without finite accumulation points),

$$\begin{align*}
\forall z \in D & \Rightarrow \limsup_{\alpha \to 0} |\langle Fg^z_\alpha, g^z_\alpha \rangle| < \infty. \\
\forall z \notin D & \Rightarrow \lim_{\alpha \to 0} |\langle Fg^z_\alpha, g^z_\alpha \rangle| = \infty.
\end{align*}$$
Application with the natural choice: $B = F$

For $z \in \mathbb{R}^d$ we consider $g^z_\alpha \in L^2(S^{d-1})$ such that

$$J_\alpha (\phi^\infty_z; g^z_\alpha) \leq j_\alpha (\phi^\infty_z) + p(\alpha) \text{ and } p(\alpha) \leq C \alpha.$$

$$\phi^\infty_z (\hat{x}) = e^{-ik\hat{x} \cdot z}$$

**Theorem:** Assume that there exists $n_* > 0$ and $\alpha > 0$ such that

$$1 - \Re(n(x)) + \alpha \Im(n(x)) \geq n_* \text{ for a.e. } x \in D \text{ or }$$

$$\Re(n(x)) - 1 + \alpha \Im(n(x)) \geq n_* \text{ for a.e. } x \in D.$$

Then, except for a countable set of $k$ (without finite accumulation points),

- $z \in D \Rightarrow \limsup_{\alpha \to 0} |\langle Fg^z_\alpha, g^z_\alpha \rangle| < \infty.$
- $z \notin D \Rightarrow \lim_{\alpha \to 0} |\langle Fg^z_\alpha, g^z_\alpha \rangle| = \infty.$

⇒ An **indicator** of some approximation of $D$ is given by

$$z \to 1/|\langle Fg^z_\alpha, g^z_\alpha \rangle|.$$
Application with the natural choice: $B = F$

⇒ An indicator of some approximation of $D$ is given by

$$z \to 1/|\langle Fg^z_\alpha, g^z_\alpha \rangle|.$$ 

Remarks

▶ In this case

$$\lim_{\alpha \to 0} |\langle Fg^z_\alpha, g^z_\alpha \rangle| = \lim_{\alpha \to 0} |\langle \phi^\infty_z, g^z_\alpha \rangle| = \lim_{\alpha \to 0} |v_{g^z_\alpha}(z)|$$

⇒ We obtain a similar indicator function as the one proposed by Arens (2004), Arens-Lechleiter (2009), to justify LSM using the $(F^*F)^{1/4}$ of Kirsch (1997) in the case $\mathcal{S}n = 0$.

▶ However this turns out to be a bad indicator function for noisy measurements.
On other possible choices for $B$
Under more restrictive assumptions on the refractive index

- If $\Im n > n_0 > 0$ in $D$ then we can use

$$B = \Im(F) = \frac{1}{2i}(F - F^*)$$

- If $\Re(e^{it(n-1)}) > n_0|n-1| > n_1 > 0$ in $D$ for some $t$

$$B = F\# := |e^{it}F + e^{-it}F^*| + \Im(F)$$

(Using the Factorization theorem of Kirsch-Grinberg)

Remarks

- For these cases the functional $J_\alpha$ is convex.
- In these cases we also have (Kirsch-Grinberg)

$$z \in D \text{ iff } \phi_z \in \mathcal{R}(B^{1/2}).$$
Main theorem of GLSM for noisy operators

$B^\delta$ and $F^\delta$ compact operators corresponding with noisy measurements

\[ \|F^\delta - F\| \leq \delta \|F^\delta\| \quad \text{and} \quad \|B^\delta - B\| \leq \delta \|B^\delta\| \]

for some $\delta > 0$.

Remark:

\[ |\langle Bg, g \rangle| \leq |\langle B^\delta g, g \rangle| + \delta \|B^\delta\| \|g\|^2 \]
Main theorem of GLSM for noisy operators

$B^\delta$ and $F^\delta$ compact operators corresponding with noisy measurements

\[ \| F^\delta - F \| \leq \delta \| F^\delta \| \quad \text{and} \quad \| B^\delta - B \| \leq \delta \| B^\delta \| \]

for some $\delta > 0$.

**Remark:**

\[ |\langle Bg, g \rangle| \leq |\langle B^\delta g, g \rangle| + \delta \| B^\delta \| \| g \|^2 \]

$\Rightarrow$ We consider the functional:

\[ J^\delta_\alpha(\phi; g) := \alpha(|\langle B^\delta g, g \rangle| + \delta \| B^\delta \| \| g \|^2) + \| F^\delta g - \phi \|^2 \quad \forall g \in X, \]
Main theorem of GLSM for noisy operators

$B^\delta$ and $F^\delta$ compact operators corresponding with noisy measurements

$$\|F^\delta - F\| \leq \delta \|F^\delta\| \quad \text{and} \quad \|B^\delta - B\| \leq \delta \|B^\delta\|$$

for some $\delta > 0$.

$$J^\delta_\alpha(\phi; g) := \alpha(|\langle B^\delta g, g \rangle| + \delta \|B^\delta\| \|g\|^2) + \|F^\delta g - \phi\|^2 \quad \forall g \in X,$$

**Theorem:** Let $g^\delta_\alpha$ be the minimizer of $J^\delta_\alpha(\phi; \cdot)$ for $\alpha > 0$, $\delta > 0$ and $\phi \in X^*$. Then

\[\begin{align*}
\bigstar & \quad \phi \in \mathcal{R}(G) \Rightarrow \limsup_{\alpha \to 0} \limsup_{\delta \to 0} \left( |\langle B^\delta g^\delta_\alpha, g^\delta_\alpha \rangle| + \delta \|B^\delta\| \|g^\delta_\alpha\|^2 \right) < \infty \\
\bigstar & \quad \phi \notin \mathcal{R}(G) \Rightarrow \lim_{\alpha \to 0} \liminf_{\delta \to 0} \left( |\langle B^\delta g^\delta_\alpha, g^\delta_\alpha \rangle| + \delta \|B^\delta\| \|g^\delta_\alpha\|^2 \right) = \infty
\end{align*}\]
Main theorem of GLSM for noisy operators

$B^\delta$ and $F^\delta$ compact operators corresponding with noisy measurements

\[ \| F^\delta - F \| \leq \delta \| F^\delta \| \quad \text{and} \quad \| B^\delta - B \| \leq \delta \| B^\delta \| \]

for some $\delta > 0$.

**Theorem:** Let $g^\delta_\alpha$ be the minimizer of $J^\delta_\alpha(\phi; \cdot)$ for $\alpha > 0$, $\delta > 0$ and $\phi \in X^*$. Then

\[ \phi \in \mathcal{R}(G) \Rightarrow \lim_{\alpha \to 0} \limsup_{\delta \to 0} \left( |\langle B^\delta g^\delta_\alpha, g^\delta_\alpha \rangle| + \delta \| B^\delta \| \| g^\delta_\alpha \|^2 \right) < \infty \]

\[ \phi \notin \mathcal{R}(G) \Rightarrow \lim_{\alpha \to 0} \liminf_{\delta \to 0} \left( |\langle B^\delta g^\delta_\alpha, g^\delta_\alpha \rangle| + \delta \| B^\delta \| \| g^\delta_\alpha \|^2 \right) = \infty \]

⇒ From the numerical perspective this theorem indicates that a criterion to localize the object would be

\[ 1/ \left( |\langle B^\delta g^\delta_\alpha, g^\delta_\alpha \rangle| + \delta \| B^\delta \| \| g^\delta_\alpha \|^2 \right) \]

for small values of $\alpha$. 
On the numerical implementation

\[ J_{\alpha}(\phi; g) := \alpha(|\langle B^\delta g, g \rangle| + \delta \|B^\delta\| \|g\|^2) + \|F^\delta g - \phi\|^2 \]

For each \( z \) in the sampling grid, compute

\[ g_z = \arg\min J_{\alpha}(\phi_z^\infty; g), \]

then plot:

\[ z \mapsto 1/\left( |\langle B^\delta g_z, g_z \rangle| + \delta \|B^\delta\| \|g_z\|^2 \right) \]

**Initialization**: we use the Tikhonov-Morozov regularized solution

\[ (\eta(\delta) + (F^\delta)^* F^\delta)g_z^0 = (F^\delta)^* \phi_z^\infty \]

We choose: \( \alpha = \eta(\delta)/(\|F^\delta\| + \delta) \).
Numerical results

Without optimization, Noise $\delta = 0$

\[
\frac{1}{|\langle B^0 g_z^0, g_z^0 \rangle|} \quad \quad \quad \quad \frac{1}{||g_z^0||^2}
\]

\[
\frac{1}{\left( |\langle B^0 g_z^0, g_z^0 \rangle| + \delta \|B^0\| ||g_z^0||^2 \right)} \quad \quad \quad \quad F_\# \text{ method}
\]
Numerical results

Without optimization, Noise $\delta = 1\%$

\[
\frac{1}{|\langle B^\delta g_z^0, g_z^0 \rangle|} \quad \frac{1}{\|g_z^0\|^2}
\]

\[
\frac{1}{\left( |\langle B^\delta g_z^0, g_z^0 \rangle| + \delta \|B^\delta\| \|g_z^0\|^2 \right)}
\]

$F_\# \text{ method}$
Numerical results

Without optimization, Noise \( \delta = 5\% \)

\[
1 / \left| \langle B^\delta g^0_z, g^0_z \rangle \right| 
\]

\[
1 / \| g^0_z \| ^2 
\]

\[
1 / \left( \left| \langle B^\delta g^0_z, g^0_z \rangle \right| + \delta \| B^\delta \| \| g^0_z \|^2 \right) 
\]

\( F_\# \) method
Numerical results Optim GLSM

GLSM without optim, $\delta = 1\%$

GLSM without optim, $\delta = 5\%$

GLSM with optim, $\delta = 1\%$

GLSM with optim, $\delta = 5\%$
Towards applications to differential measurements

**Main ingredient:** exploit the link between GLSM and ITP.
Towards applications to differential measurements

**Main ingredient:** exploit the link between GLSM and ITP.

\[ F : X \rightarrow X^* \text{ and } B : X \rightarrow X^* \]

\[ F = GH \quad \text{and} \quad B = H^* TH \]

\[ J_\alpha(\phi; g) := \alpha |\langle Bg, g \rangle| + \|Fg - \phi\|^2 \quad \forall g \in X. \quad (5) \]

**Theorem:** We assume in addition that

\[ \varphi \mapsto |\langle T \varphi, \varphi \rangle| \quad \text{is uniformly convex} \]

\[ J_\alpha(\phi; g_\alpha) \leq j_\alpha(\phi) + p(\alpha) \quad \text{with} \quad \frac{p(\alpha)}{\alpha} \rightarrow 0 \quad \text{as} \quad \alpha \rightarrow 0. \quad (6) \]

If \( \phi \in \mathcal{R}(G) \) then \( Hg_\alpha \rightarrow \varphi \) such that \( G(\varphi) = \phi. \)

This is a consequence of Tikhonov applied to \( G(\varphi) = \phi. \)
Towards application to differential measurements

**Main ingredient:** exploit the link between GLSM and ITP.

**Corollary:** with $F = F$ and $B = F_{\#}$

$$J_\alpha(\phi_z^{\infty}; g_\alpha^{\infty}) \leq j_\alpha(\phi_z^{\infty}) + p(\alpha) \text{ with } \frac{p(\alpha)}{\alpha} \to 0 \text{ as } \alpha \to 0.$$ 

If $z \in D$ then $Hg_\alpha \to v_z$ strongly in $L^2(D)$ where $v_z$ is such that there exists $u_z \in L^2(D)$ for which $(u_z, v_z)$ is a solution of ITP with $(f, g) = (\Phi(z, \cdot), \partial_\nu \Phi(z, \cdot))$

**Notation** for ITP($D, n, f, g$): $(u, v) \in L^2(D) \times L^2(D)$ such that $u - v \in H^2(D)$ and

$$\begin{cases} 
\Delta u + k^2 nu = 0 \quad \text{in } D, \\
\Delta v + k^2 v = 0 \quad \text{in } D, \\
(u - v) = f \quad \text{on } \partial D, \\
\frac{\partial}{\partial \nu}(u - v) = g \quad \text{on } \partial D. 
\end{cases}$$
Application to differential measurements

Assumptions on the geometry:

\[ D_0 \subset D \quad \overline{D} = \overline{\Omega} \cup \overline{D_0} \quad \text{and } n = n_0 \text{ in } D_0 \setminus \Omega \]

\[ D_0 = \bigcup_{i} \tilde{D}_{0,i} \cup \bigcup_{i} D_{0,i}. \]

\( D_{0,i}, \ i = 1, \ldots, M \) the components of \( D_0 \) that intersect with \( \Omega \).
Application to differential measurements

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\( D_0, i, i = 1, \ldots, M \) the components of \( D_0 \) that intersect with \( \Omega \).

Comparison of ITP solutions:

**Theorem:** Assume that \( \mathcal{R}(n) > \mathcal{R}(n_0) > 1 \) or \( \mathcal{R}(n) < \mathcal{R}(n_0) < 1 \) in \( \Omega \).

Let \( z \in D \) and consider \( (u, v) \in L^2(D) \times L^2(D) \) (resp. \( (u_0, v_0) \in L^2(D_0) \times L^2(D_0) \)) solutions of \( \text{ITP}(D, n, \Phi_z, \frac{\partial \Phi_z}{\partial \nu}) \) (resp. \( \text{ITP}(D_0, n_0, \Phi_z, \frac{\partial \Phi_z}{\partial \nu}) \)).

Then, except for a countable set of values of \( k \),

- If \( z \in \tilde{D}_{0,i} \) then \( v = v_0 \) in \( D_0 \).
- If \( z \in D_{0,i} \), then \( v \neq v_0 \) in \( D_{0,i} \) and \( v = v_0 \) in \( D_0 \setminus D_{0,i} \).
Application to differential measurements

Assumptions on the geometry:

\[ D_0 \subset D \quad \overline{D} = \overline{\Omega} \cup \overline{D_0} \quad \text{and} \quad n = n_0 \text{ in } D_0 \setminus \Omega \]

\[ D_0 = \bigcup \tilde{D}_{0,i} \cup \bigcup_i D_{0,i}. \]

\( D_{0,i}, \; i = 1, \ldots, M \) the components of \( D_0 \) that intersect with \( \Omega \).

We use this to obtain characterizations of \( \tilde{\Omega} \) and \( \Omega_0 := \Omega \cup_i D_{0,i} \).
Characterization of $\Omega_0$ in terms of $F$ and $F_0$

$F$ farfield associated with $D$ and $n$. $B = F_#$
$F_0$ farfield associated with $D_0$ and $n_0$. $B_0 = F_0, #$. 

We introduce

$$\mathcal{D}(g, g_0) := |\langle B_0(g - g_0), g - g_0 \rangle|.$$

**Corollary:** Under previous assumptions on $D$, $D_0$, $n$, $n_0$ and $k$ and for $g_z^\alpha$ and $g_{0,z}^\alpha$ the minimizing sequences associated resp with $(F, B)$ and $(F_0, B_0)$

- If $z \in \bigcup_i \tilde{D}_0, i$ then $\lim_{\alpha \to 0} \mathcal{D}(g_z^\alpha, g_{0,z}^\alpha) = 0$.

- If $z \in \tilde{\Omega}$ then $\lim_{\alpha \to 0} \mathcal{D}(g_z^\alpha, g_{0,z}^\alpha) = \infty$.

- If $z \in \bigcup_i D_0, i$ then $\lim_{\alpha \to 0} \mathcal{D}(g_z^\alpha, g_{0,z}^\alpha) < \infty$. 
Characterization of $\Omega_0$ in terms of $F$ and $F_0$

The noise free case:

$$I(g, g_0) := |\langle Bg, g \rangle| (1 + |\langle Bg, g \rangle|/|\langle B_0(g - g_0), g - g_0 \rangle|).$$

**Corollary:** Under previous assumptions on $D, D_0, n, n_0$ and $k$. For $g^\alpha_z$ and $g^\alpha_{0,z}$ the minimizing sequences associated resp with $(F, B)$ and $(F_0, B_0)$

- If $z \notin \Omega_0$ then $$\lim_{\alpha \to 0} I(g^\alpha_z, g^\alpha_{0,z}) = \infty.$$

- If $z \in \Omega_0$ then $$\lim_{\alpha \to 0} I(g^\alpha_z, g^\alpha_{0,z}) < \infty.$$

Therefore, the limit as $\alpha \to 0$ of

$$z \mapsto 1/I(g^\alpha_z, g^\alpha_{0,z})$$

is an indicator for $\Omega_0 = \tilde{\Omega} \cup \bigcup_i D_{0,i}$.
Characterization of $\Omega_0$ in terms of $F$ and $F_0$

The noisy case:

For a fixed parameter $\eta \in (0, 1)$, we define

$$g_{0, z}^{\alpha, \delta} = \arg \min_{g} \alpha \left( \langle B_0^\delta g, g \rangle + \alpha^{-\eta} \delta \|B_0^\delta \| \|g\|^2 \right) + \|F_0^\delta g - \phi_\infty^z\|^2$$

$$g_z^{\alpha, \delta} = \arg \min_{g} \alpha \left( \langle B^\delta g, g \rangle + \alpha^{-\eta} \delta \|B^\delta \| \|g\|^2 \right) + \|F^\delta g - \phi_\infty^z\|^2$$

We then consider

$$D(\alpha, z) = \liminf_{\delta \to 0} \langle B_0^\delta (g_{0, z}^{\alpha, \delta} - g_z^{\alpha, \delta}), g_{0, z}^{\alpha, \delta} - g_z^{\alpha, \delta} \rangle$$

$$A(\alpha, z) = \liminf_{\delta \to 0} \left( \langle B^\delta g_z^{\alpha, \delta}, g_z^{\alpha, \delta} \rangle + \alpha^{-\eta} \delta \|B^\delta \| \|g_z^{\alpha, \delta}\|^2 \right)$$

$$\mathcal{I}(\alpha, z) := A(\alpha, z) \left(1 + A(\alpha, z)/D(\alpha, z)\right).$$

**Theorem:** Under previous assumptions on $D$, $D_0$, $n$, $n_0$ and $k$, $z \in \Omega_0$ iff $\lim_{\alpha \to 0} \mathcal{I}(\alpha, z) < +\infty$. 
Some numerical results

Exact configuration

Differential GLSM

Background Reconstruction

Medium Reconstruction
Some numerical results

Exact configuration

Differential GLSM

Background Reconstruction

Medium Reconstruction
Some numerical results for scattered background

Two defects of type inhomogeneities
Some numerical results for scattered background

One defect of type internal crack
Some numerical results for scattered background

A medium size external crack

Differential GLSM

Reconstructed crack using the exact background

Reconstructed crack using a homogeneous background
Thank you