Coloring the Integers with Rainbow Arithmetic Progressions

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A *k*-term arithmetic progression is a finite sequence of *k* terms of the form \( \{a, a + d, a + 2d, \ldots, a + (k - 1)d\} \), where *k*, *a*, and *d* are nonnegative integers.
A \textit{k-term arithmetic progression} is a finite sequence of \( k \) terms of the form \( \{a, a + d, a + 2d, \ldots, a + (k - 1)d\} \), where \( k \), \( a \), and \( d \) are nonnegative integers.

\[3, 10, 17, 24, 31\]
An \textit{r-coloring} of a set $S$ is a function $c : S \to C$, such that $|C| = r$. 
An $r$-coloring of a set $S$ is a function $c : S \to C$, such that $|C| = r$.

$3, 10, 17, 24, 31$
An $r$-coloring of a set $S$ is a function $c : S \rightarrow C$, such that $|C| = r$.

3, 10, 17, 24, 31

A set $S$ is monochromatic under an $r$-coloring $c$, if $c(s_1) = c(s_2)$, for each $s_1, s_2 \in S$. 
Van der Waerden's Theorem

Let $k$ and $r$ be positive integers. Then there exists a positive integer $w$ such that every $r$-coloring of $[w]$ contains a monochromatic $k$-term arithmetic progression.

The smallest integer $w(r, k)$ satisfying the theorem is called the van der Waerden number.
Van der Waerden’s Theorem

Let $k$ and $r$ be positive integers. Then there exists a positive integer $w$ such that every $r$-coloring of $[w]$ contains a monochromatic $k$-term arithmetic progression.

The smallest integer $w(r, k)$ satisfying the theorem is called the van der Waerden number.
Proving van der Waerden numbers.

To show \( w(r, k) = w \), the following two statements must be proven:

- There exists an \( r \)-coloring of \( [w - 1] \) with no monochromatic \( k \)-term APs.
- Every \( r \)-coloring of \( [w] \) has a monochromatic \( k \)-term AP.
$w(2, 3) \leq 9$

Case I:

1 2 3 4 5 6 7 8 9
$w(2, 3) \leq 9$

Case I:

\begin{align*}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{align*}
$w(2, 3) \leq 9$

Case I:

1 2 3 4 5 6 7 8 9
\( w(2, 3) \leq 9 \)

Case I:

1 2 3 4 5 6 7 8 9
$w(2, 3) \leq 9$

Case I:

\[
1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9
\]
$w(2, 3) \leq 9$

Case II:

1 2 3 4 5 6 7 8 9
$w(2, 3) \leq 9$

Case II:

\[1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9\]
$w(2, 3) \leq 9$

Case II:

1 2 3 4 5 6 7 8 9
\( w(2, 3) \leq 9 \)

Case II:

\[ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \]
$w(2, 3) \leq 9$

Case II:

1 2 3 4 5 6 7 8 9
Case II: 

\[ w(2, 3) \leq 9 \]
$w(2, 3) \leq 9$

Case II:

1 2 3 4 5 6 7 8 9
Case II:

\[ w(2, 3) \leq 9 \]
$w(2, 3) \geq 9$

Extremal Coloring:

$1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8$
Known van der Waerden numbers.

- $w(r, 1) = 1$. 

That's it!
Known van der Waerden numbers.

- $w(r, 1) = 1$.
- $w(1, k) = k$. 

That’s it!
Known van der Waerden numbers.

- $w(r, 1) = 1$.
- $w(1, k) = k$.
- $w(r, 2) = r + 1$. 
Known van der Waerden numbers.

- $w(r, 1) = 1$.
- $w(1, k) = k$.
- $w(r, 2) = r + 1$.
- $w(2, 3) = 9$, $w(3, 3) = 27$, $w(4, 3) = 76$. 
Known van der Waerden numbers.

- \( w(r, 1) = 1 \).
- \( w(1, k) = k \).
- \( w(r, 2) = r + 1 \).
- \( w(2, 3) = 9, w(3, 3) = 27, w(4, 3) = 76 \).
- \( w(2, 4) = 35, w(2, 5) = 178, w(2, 6) = 1132 \).
Known van der Waerden numbers.

- $w(r, 1) = 1.$
- $w(1, k) = k.$
- $w(r, 2) = r + 1.$
- $w(2, 3) = 9, w(3, 3) = 27, w(4, 3) = 76.$
- $w(2, 4) = 35, w(2, 5) = 178, w(2, 6) = 1132.$
- That's it!
Known van der Waerden numbers.

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anti-van der Waerden numbers

An exact $r$-coloring of a set $S$ is a surjective function $c : S \to C$, such that $|C| = r$.

3, 10, 17, 24, 31
An exact $r$-coloring of a set $S$ is a surjective function $c : S \rightarrow C$, such that $|C| = r$.

$$3, 10, 17, 24, 31$$

A set $S$ is rainbow under an $r$-coloring $c$, if $c(s_1) \neq c(s_2)$, for each distinct $s_1, s_2 \in S$. 
Given positive integers $n$ and $k$ with $k \leq n$, the anti-van der Waerden number, denoted by $aw(n, k)$, is the least positive integer $r$ such that every exact $r$-coloring of $[n]$ contains a rainbow $k$-term AP.
To show $aw(n, k) = r$, the following two statements must be proven:

- There exists an $(r - 1)$-coloring of $[n]$ with no rainbow $k$-term APs.
- Every $r$-coloring of $[n]$ has a rainbow $k$-term AP.
$aw(8, 3) \leq 5$
\( aw(8, 3) \leq 5 \)
\(aw(8, 3) \leq 5\)
$aw(8, 3) \leq 5$
aw(8, 3) \geq 5
Properties of $aw(n, k)$

- $k \leq aw(n, k) \leq n$. 
Properties of $aw(n, k)$

- $k \leq aw(n, k) \leq n$.
- $aw(n, k) = n$ if and only if $k \geq \frac{n}{2} + 1$. 
## Small anti-van der Waerden numbers

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$aw(n, k), \ k = 3$

$a, a + d, a + 2d$
aw(n, k), k = 3

\[ a, a + d, a + 2d \]

- The first and last term have the same parity.
$aw(n, k), \ k = 3$

\[ a, a + d, a + 2d \]

- The first and last term have the same parity.
- The terms are all the same or all different modulo 3.
Lowerbound for $aw(n, 3)$

Theorem

$aw(\frac{n}{3}, 3) + 1 \leq aw(n, 3)$.

1 2 3 4 5 6 7 8 9....
Lowerbound for $aw(n, 3)$

**Theorem**

$$aw\left(\frac{n}{3}, 3\right) + 1 \leq aw(n, 3).$$

**Corollary**

$$\log_3(n) + 2 \leq aw(n, 3).$$
Upperbound for $aw(n, 3)$

**Theorem**

\[ aw(n, 3) \leq aw\left(\frac{n}{2}, 3\right) + 1. \]
Upperbound for $aw(n, 3)$

**Theorem**

$aw(n, 3) \leq aw\left(\frac{n}{2}, 3\right) + 1.$

**Corollary**

$aw(n, 3) \leq \log_2(n) + 1.$
Upperbound for $aw(n, 3)$

<table>
<thead>
<tr>
<th>Theorem</th>
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<td>$aw(n, 3) \leq aw\left(\frac{n}{2}, 3\right) + 1$.</td>
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<td>$aw(n, 3) \leq \log_2(n) + 1$.</td>
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<td>$aw(n, 3) \leq \log_3(n) + 4$.</td>
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<td>$aw(3^m, 3) \leq m + 2$.</td>
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Thank You!