Dispersion and Dissipation in Shallow Water

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in collaboration with

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Outline

I. The Indian Ocean Tsunami
II. Experiments
III. Modeling
   a. Governing Equations
   b. Linearity
   c. Dispersion
   d. Dissipation
IV. Summary
2004 Indian Ocean Earthquake and Tsunami

- Occurred on 26 December 2004
- Indian Plate was subducted by the Burma Plate
- Magnitude of between 9.1 and 9.3
- Roughly 30 cubic kilometers of water were displaced
- Waves up to 30 meters high
- Over 230,000 people died
http://www.youtube.com/watch?v=1J8Feyr38Ss
Earthquakes

From the USGS Earthquake Hazard Program’s Sumatra-Andman Islands Earthquake webpage.
Initial Water Displacement

Figure 1. Map of the northern Indian Ocean, showing the shape and intensity of the initial tsunami in 2004. The coloured regions show where the water level was raised (red) and lowered (blue) by a sequence of earthquakes. This image is the first frame of a simulation of the entire tsunami by K. Satake. See http://staff.aist.go.jp/kenji.satake/animation.html for the full animation.

Mathematical models of water waves are discussed in the appendix. For this event, the important conclusions are easy to draw:

— Wave amplitudes were 1000 times smaller than the local ocean depth, so the waves that evolved from these initial data were small amplitude waves. Hence a linearized theory should describe the propagation of the waves with reasonable accuracy, at least for short times and away from shore. (See the appendix for proper definitions of 'short time' and 'long time'. Crudely, 'short time' includes the time required for the wave to propagate some reasonable number of wavelengths of the initial disturbance. For the tsunami of 2004, this would include the time to propagate across either the Bay of Bengal (less than $15 \times 10^3 km$) or the Andaman Sea (less than $6 \times 10^3 km$).

— Wavelengths in the east–west direction were about 30 times longer than the fluid depth for the wave propagating westward, and even more so for the wave propagating eastward. In deriving the proper mathematical model, it turns out that the square of this ratio is the relevant parameter, and the squared ratio is of the order of 1000. These were very long waves.

— For long waves of small amplitude, the speed of propagation is approximately $c = \sqrt{gd}$, where $g$ represents gravitational acceleration and $d$ is the fluid depth.
Photo from Mt. Pu

Photo was taken from the top of Mount Pu on the island of Koh Jum. Copyright Anders Grawin, 2006.
Photo was taken on Hat Ray Leah beach on the Krabi coast, Thailand. Copyright Scanpix, 2006.
Plate 1. The tsunami of 26 December 2004 approaching Hat Ray Leah beach on the Krabi coast, Thailand. (Copyright Scanpix, 2006.)

Plate 2. The tsunami of 26 December 2004 approaching Hat Ray Leah beach on the Krabi coast, Thailand. (Copyright Scanpix, 2006.)

in the pictures (detailed in Plates 3 and 4)—is entering a calm, undisturbed region of water, and the same undisturbed nature is evident well behind the wave. Indeed, we see here one of the hallmarks of a weak jump: a train of waves is formed behind the jump, enabling the excess energy to be transported away, energy which would otherwise be dissipated by turbulent action at the front (which is evident in...
Experiments
Experimental Facility

*Not the actual experimental facility where the experiments were conducted.
*Nor the actual experimentalist who conducted the experiments.
Experimental Setup

Figure not to scale!

Experiments conducted by Joe Hammack.
Experiments conducted by Joe Hammack.
\( z = 0 \) at the bottom

\( z = h_0 \) mean fluid level

\( h_0 \)

\( x \)

\( z = z, \) water depth

\[ \epsilon = \frac{H}{h_0} \] is a (dimensionless) measure of nonlinearity

\[ \delta = \frac{h_0}{\lambda} \] is a (dimensionless) measure of shallowness
Experimental Measurements: $A_0 = 0.5\text{cm}$

\begin{align*}
\epsilon &= \frac{0.5}{10} = 0.05 \\
\delta &= \frac{10}{2 \times 61} = 0.08
\end{align*}

Experimental Measurements: \( A_0 = 1.5 \text{cm} \)

\[
\epsilon = \frac{1.5}{10} = 0.15 \\
\delta = \frac{10}{2 \times 61} = 0.08
\]

Modeling

Our goal is to accurately model the experimental data.

Really, our goal is to accurately model real-world tsunamis.
Assume

- The fluid is incompressible
- The fluid is inviscid
- The motion of the fluid is irrotational
- There is zero surface tension
- The fluid rests on a horizontal impenetrable bed
Physicial System

\[ z = 0 \] at the bottom

\[ z = h_0 \] mean fluid level

\[ h_0 \] water depth

\[ z = \zeta, \text{ water depth} \]

\[ z = 0 \] at the bottom

\[ z = h_0 \] mean fluid level

\[ H \]

\[ \lambda \]

\[ h_0 \]
Let

- $g$ represent the acceleration due to gravity
- $h_0$ represent the undisturbed water depth
- $\zeta = \zeta(x, t)$ represent the local water depth
- $\phi = \phi(x, z, t)$ represent the velocity potential of the water
- $\eta = \eta(x, t)$ represent the free surface displacement
- Note: $\zeta(x, t) = h_0 + \eta(x, t)$

- $u(x, t)$ represents the horizontal fluid velocity
- $\bar{u}(x, t)$ represents the depth-averaged horizontal fluid velocity
The model equations are the Euler equations:

\[
\phi_{xx} + \phi_{zz} = 0 \quad \text{for} \quad 0 < z < \zeta(x, t)
\]

\[
\phi_z = 0 \quad \text{for} \quad z = 0
\]

\[
\zeta_t + \phi_x \zeta_x - \phi_z = 0 \quad \text{for} \quad z = \zeta(x, t)
\]

\[
\phi_t + g\zeta + \frac{1}{2}(\phi_x^2 + \phi_z^2) = 0 \quad \text{for} \quad z = \zeta(x, t)
\]
If we assume
\[ \zeta(x, t) = e^{ikx - i\omega t} + c.c. \]
we find that the linear phase speed of the Euler equations is
\[ c^2 = \left( \frac{\omega}{k} \right)^2 = \frac{g \tanh(kh_0)}{k} \]

- The Euler equations are dispersive.
- Waves can travel in both directions.

The Euler equations are complicated, so we use simplified models.
Initial Conditions for Numerics

Figure not to scale!

\[ h_0 = 10 \text{ cm} \]
\[ A_0 \]

\[ x = 61 \text{ cm} \]
\[ x = -61 \text{ cm} \]

\[ \eta(x, 0) = \begin{cases} 
0 & 0 \leq x < 7,625 \\
-A_0 + A_0 \text{sn}(0.0925x, 0.9999) & 7,625 \leq x \leq 7,869 \\
0 & 7,869 < x \leq 15,616 
\end{cases} \]
Linearized Euler Equations

The linearized Euler equations are obtained by assuming $\phi = \mathcal{O}(\epsilon)$ and $\zeta = \mathcal{O}(\epsilon)$ and truncating the Euler system at $\mathcal{O}(\epsilon)$.

\[
\phi_{xx} + \phi_{zz} = 0 \quad \text{for} \quad 0 < z < \zeta(x, t)
\]
\[
\phi_z = 0 \quad \text{for} \quad z = 0
\]
\[
\zeta_t - \phi_z = 0 \quad \text{for} \quad z = \zeta(x, t)
\]
\[
\phi_t + g\zeta = 0 \quad \text{for} \quad z = \zeta(x, t)
\]
Linearized Euler Expt #2: \( A_0 = 0.5 \text{cm}, \, \varepsilon = 0.05, \, \delta = 0.08 \)
Linearized Euler Expt #2: $A_0 = 0.5\text{cm}, \epsilon = 0.05, \delta = 0.08$
Linearized Euler Expt #3: $A_0 = 1.5\text{cm}, \epsilon = 0.15, \delta = 0.08$
The dimensional St. Venant (a.k.a. the classical shallow-water equations) are obtained by assuming $\delta \ll 1$ and truncating at $O(\delta)$ without making any assumptions on $\epsilon$. 
St. Venant Equations

The dimensional St. Venant equations are given by

\[
\begin{align*}
ht + (hu)_x &= 0 \\
(hu)_t + \left(\frac{1}{2}gh^2 + hu^2\right)_x &= 0
\end{align*}
\]

These equations are nondispersive and the phase speed is given by

\[
c^2 = \left(\frac{\omega}{k}\right)^2 = gh_0
\]

This is the first term in the small-\(k\) series expansion of the Euler phase speed.
Note that waves can travel in both directions.
St. Venant Expt #2: \( A_0 = 0.5 \text{cm}, \ \epsilon = 0.05, \ \delta = 0.08 \)

St. Venant simulations computed by David George.
St. Venant Expt #3: $A_0 = 1.5\text{cm}$, $\epsilon = 0.15$, $\delta = 0.08$

St. Venant simulations computed by David George.
The Korteweg-deVries (KdV) equation is derived by assuming $\delta^2 = \mathcal{O}(\epsilon)$ and truncating the Euler system at $\mathcal{O}(\epsilon^3)$. 

KdV
The dimensional KdV equation is given by

\[ \eta_t + \sqrt{gh_0} \eta_x + \frac{3}{2h_0} \sqrt{gh_0} \eta \eta_x + \frac{1}{6} h_0^2 \sqrt{gh_0} \eta_{xxx} = 0 \]

KdV is dispersive and its phase speed is given by

\[ c = \frac{\omega}{k} = \sqrt{gh_0} \left( 1 - \frac{1}{6} h_0^2 k^2 \right) \]

These are the first two terms in the small-\(k\) series expansion of the Euler phase speed.

Note that waves are supposed to only travel in one direction.
Dispersion in KdV

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KdV Expt #2: $A_0 = 0.5\text{cm}$, $\epsilon = 0.05$, $\delta = 0.08$
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KdV Expt #3: $A_0 = 1.5\text{cm}, \epsilon = 0.15, \delta = 0.08$
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The fifth-order KdV (KdV5) equation is derived by assuming $\delta^2 = \mathcal{O}(\epsilon)$ and truncating the Euler system at $\mathcal{O}(\epsilon^5)$. 
Fifth-Order KdV

The dimensional KdV5 equation is given by

\[ \eta_t + \sqrt{gh_0} \eta_x + \frac{3}{2h_0} \sqrt{gh_0} \eta \eta_x + \frac{1}{6} h_0^2 \sqrt{gh_0} \eta_{xxx} = \frac{19}{360} \sqrt{gh_0} h_0^4 \eta_{xxxx} + \frac{5}{12} \sqrt{gh_0} h_0 \eta \eta_{xxx} + \frac{5}{6} \sqrt{gh_0} h_0 \eta_x \eta_{xx} + \frac{15}{8h_0^2} \eta^2 \eta_x \]

KdV5 is dispersive and its phase speed is given by

\[ c = \frac{\omega}{k} = \sqrt{gh_0} \left( 1 - \frac{1}{6} h_0^2 k^2 + \frac{19}{360} h_0^4 k^4 \right) \]

These are the first three nonzero terms in the small-\( k \) series expansion of the Euler phase speed.

Note waves only travel in one direction.
Dispersion in KdV5
KdV5 Expt #2: $A_0 = 0.5\text{cm}$, $\epsilon = 0.05$, $\delta = 0.08$
KdV5 Expt #2: $A_0 = 0.5\text{cm}$, $\epsilon = 0.05$, $\delta = 0.08$.
The Serre (Green-Naghdi) equations are derived by depth averaging the horizontal velocity and truncating the Euler system at $O(\delta^4)$. No assumptions are made on $\epsilon$. 
The dimensional Serre equations are given by

\[ h_t + (h \tilde{u})_x = 0 \]
\[ \tilde{u}_t + \tilde{u}\tilde{u}_x + gh_x - \frac{1}{3h} \left( h^3 \left( \tilde{u}_{xt} + \tilde{u}\tilde{u}_{xx} - (\tilde{u}_x)^2 \right) \right)_x = 0 \]

The Serre equations are dispersive and their phase speed is

\[ c^2 = \left( \frac{\omega}{k} \right)^2 = \frac{3gh_0}{3 + k^2 h_0^2} \]

This is the (1, 3)-Padé Approximant of the Euler phase speed.

Note that waves can travel in both directions.
Dispersion in Serre

\[
\frac{\omega}{k c_0}
\]

\[
\begin{array}{c}
\text{Euler} \\
\text{St. Venant} \\
\text{KdV} \\
\text{Serre}
\end{array}
\]

\( k h_0 \)

\( w_k \)
Serre Expt #2: $A_0 = 0.5\text{cm}, \epsilon = 0.05, \delta = 0.08$
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Serre versus KdV Expt #2

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Serre Expt #3: $A_0 = 1.5\text{cm}, \epsilon = 0.15, \delta = 0.08$
The dimensional Whitham equation is

$$
\eta_t + \frac{3}{2h_0} \sqrt{gh_0} \eta \eta_x + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sqrt{gk \tanh(kh_0)}}{2\pi k} e^{ik(x-\xi)} \eta_\xi \, dk \, d\xi = 0
$$

The Whitham equation is dispersive and its phase speed is

$$
c = \frac{\omega}{k} = \sqrt{\frac{g \tanh(kh_0)}{k}}
$$

Note that waves can travel only in one direction.
Dispersion and Dissipation in Shallow Water

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Whitham Expt #2: \( A_0 = 0.5 \text{cm}, \epsilon = 0.05, \delta = 0.08 \)
Whitham Expt #2, Larger $t$ Interval
Whitham Expt #3: $A_0 = 1.5\text{cm}, \epsilon = 0.15, \delta = 0.08$
So far, we haven’t said anything about dissipation.

We modify KdV because of its familiarity/simplicity.
\[ \eta_t + \sqrt{gh_0} \eta_x + \frac{3}{2h_0} \sqrt{gh_0} \eta \eta_x + \frac{1}{6} h_0^2 \sqrt{gh_0} \eta_{xxx} + \gamma \eta = 0 \]
Dissipative KdV #2

\[ \eta_t + \sqrt{gh_0} \eta_x + \frac{3}{2h_0} \sqrt{gh_0} \eta \eta_x + \frac{1}{6h_0^2} \sqrt{gh_0} \eta_{xxx} - \gamma \eta_{xx} = 0 \]

(a) \hspace{5cm} (b) \hspace{5cm} (c) \hspace{5cm} (d) \hspace{5cm} (e)
\[ \eta_t + \sqrt{gh_0} \eta_x + \frac{3}{2h_0} \sqrt{gh_0} \eta \eta_x + \frac{1}{6} h_0^2 \sqrt{gh_0} \eta_{xxx} + \gamma \eta_{xxxx} = 0 \]
\[ \eta_t + \sqrt{gh_0} \eta_x + \frac{3}{2h_0} \sqrt{gh_0} \eta \eta_x + \frac{1}{6} h_0^2 \sqrt{gh_0} \eta_{xxx} + \gamma \eta_{\frac{1}{2}x} = 0 \]

where

\[
\eta_{\frac{1}{2}x} = \mathcal{F}^{-1} \left( (ik)^\frac{1}{2} \mathcal{F}(\eta) \right)
\]
Dissipative KdV #4

\[
\eta_t + \sqrt{gh_0} \eta_x + \frac{3}{2h_0} \sqrt{gh_0} \eta \eta_x + \frac{1}{6} h_0^2 \sqrt{gh_0} \eta_{xxx} + \gamma \eta^{\frac{1}{2}} = 0
\]
A promising model (Keulegan 1948, Liu & Orfila 2004):
A promising model:

\[
\eta_t + \sqrt{gh_0} \eta_x + \frac{3}{2h_0} \sqrt{gh_0} \eta \eta_x + \frac{1}{6} h_0^2 \sqrt{gh_0} \eta_{xxx} + \gamma \int_0^t \frac{\eta_x(\tau)}{\sqrt{t-\tau}} \, d\tau = 0
\]
Summary

1. Nonlinear effects are important in shallow water
2. Dispersion can be important in shallow water
3. “Any old form” of dispersion is not necessarily sufficient
4. Dissipation likely plays an important role, but that role is unclear
5. None of the examined models get the amplitudes correct
6. Bathymetry likely plays an important role, but that role is unclear