Nucleation of Localised Ferro-patterns

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• Motivation

• Energy & Hamiltonian Formulation

• 1D Centre-Manifold reduction

• 2D radial spots

• Conclusion and Open problems

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Motivation

**Ferrofluid:** colloidal liquids made of nanoscale ferromagnetic, particles suspended in a carrier fluid

**Experiment:** uniform vertical magnetic field

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**Solitons**

**Hexagons**

[Richter & Barashenkov, 2005]

Q: Is the system (spatially) Hamiltonian?
Q: Can one establish the existence of radial spots?
Ferrofluid Equations

Maxwell’s equations:
\[ \nabla \times H = 0 \quad \text{in } \Omega_A, \quad \nabla \cdot B = 0 \quad \text{in } \Omega_F \]

Magnetic field:
\[ H = \begin{cases} \nabla \chi & \text{in } \Omega_A, \\ -\nabla \phi & \text{in } \Omega_F \end{cases} \]

Magnetic induction:
\[ B = \begin{cases} \mu_0 H & \text{in } \Omega_A, \\ \mu_0 (H + M) & \text{in } \Omega_F \end{cases} \]

Magnetisation Law:
\[ M = M(H) \]

Magnetic permeability (\( \mu \)) > 1:
\[ \mu = \begin{cases} 1 & \text{in } \Omega_A, \\ 1 + |M|/|H| & \text{in } \Omega_F \end{cases} \]

Dynamic bc:
\[ -\rho g \xi + \mu_0 \int_0^{|H|} |M| d|H| + \frac{\mu_0}{2} (M \cdot n)^2 + C = \tau \nabla \cdot \left[ \frac{\nabla \xi}{\sqrt{1 + |\nabla \xi|^2}} \right] \]

Kinematic boundary conditions on \( z = \xi \):
\[ \phi = \chi \quad \nabla (\chi - \mu \phi) \cdot n = 0 \]
Ferrofluid eqns (cont.)

- **Base state:**
  \[ \chi = H_0 z, \quad \text{in } \Omega_A \quad \phi = H_F z, \quad \text{in } \Omega_F \]

  \( H_0 \) - applied magnetic field strength

  \( H_F \) - magnetic field strength in fluid: \( \mu(H_F)H_F = H_0 \)

- **Boundary conditions on top and bottom of domain:**
  \[ \phi_z(x,-D) = H_F \quad \chi_z(x,-D) = H_0 \]
  \[ \chi_z|_{z=D} = H_F \mu(H_F) \quad \nabla^2(\phi) \phi_z|_{z=D} = H_F \]

- **Magnetisation Law:**
  
  **Linear:** \( M = M_s H \)

  **Langevin:** \( M = M_s \left( \coth(\gamma|H|) - \frac{1}{\gamma|H|} \right) \hat{H} \)

  Linear error \( \sim 5\% \), Fitted Langevin \( \sim \text{exp. error} \)
Previous work

Linear Magnetisation Law
Concentrate on doubly periodic patterns

[Silber & Knobloch, 1988], [Friedrichs & Engel, 2003], [Bohlius, Brand & Pleiner, 2008]

Methods:
1. Solve Laplace’s equations for the magnetic potentials
2. Plug into interface conditions/energy functional to yield expansions for the free-surface

Closest approach: Lyapunov-Schmidt reduction [Twombly & Thomas, 1983]

Methods fail to capture localised patterns...

Water-waves:
Centre-Manifold reduction [Buffoni, Groves & Toland, 1996] [Mielke, 1991]


**Generalised Energy Formulation**

**Previous Energy functionals:**  [Gailitis, 1977]  [Friedrichs & Engel, 2003]

\[
\mathcal{E} = \int \left\{ \frac{\mu_0}{2} \int_{-D}^{\xi} \Gamma(|\nabla \phi|^2)\,dz + \frac{\mu_0}{2} \int_{\xi}^{D} |\nabla \chi|^2\,dz + \frac{\rho \mu_0^2}{2} \right. \\
\left. + C \xi + \tau \left( \sqrt{1 + |\nabla \xi|^2} - 1 \right) \right\} \,dx \\
+ \mu_0 \int H_0 \phi \big|_{z=-D} \,dx - \mu_0 \int \mu(H_F) H_F \chi \big|_{z=D} \,dx
\]

**where:** \( \Gamma'(|\nabla \phi|^2) = \mu(|\nabla \phi|) \) \quad and \quad \( \phi \big|_{z=\xi} = \chi \big|_{z=\xi} \)

**Langevin Law:**

\[
\Gamma(|\nabla \phi|^2) = |\nabla \phi|^2 + 2 M_s \left[ \ln(\gamma |\nabla \phi|) + \ln(\coth(\gamma |\nabla \phi| - 1)) + \ln(\coth(\gamma |\nabla \phi| + 1)) \right]
\]
Energy Formulation

**Proposition**

Critical points of $\mathcal{E}$ subject to the condition $\phi|_{z=\xi} = \chi|_{z=\xi}$ satisfy the following equations

\[-\nabla^2 \chi = 0 \quad \text{on} \quad \xi < z < D \quad -\nabla \cdot (\mu(|\nabla \phi|) \nabla \phi) = 0 \quad \text{on} \quad -D < z < \xi\]

\[\text{on} \quad z = \xi\]

\[-\rho g \xi + \mu_0 \int_0^{\mathcal{F}|H|} |M|dH + \frac{\mu_0}{2} (M \cdot n)^2 + C = \tau \nabla \cdot \left( \frac{\nabla \xi}{\sqrt{1 + |\nabla \xi|^2}} \right)\]

with kinematic bc: $\nabla (\chi - \mu \phi) \cdot n = 0$

with bcs: $\chi_z|_{z=D} = H_F \mu(H_F)$ $\Gamma'(|\nabla \phi|^2) \phi_z|_{z=D} = H_F$
Legendre transformation

Coordinate transform -> Hamiltonian system

Flatten free-surface: [Twombly & Thomas, 1983]
\[ \tilde{z} = D(z - \xi)/(D - \xi), \quad \text{on } \xi < z < D, \]
\[ \tilde{z} = D(z - \xi)/(D + \xi), \quad \text{on } -D < z < \xi \]

Consider perturbation of base state
\[ \phi = H_F z + \tilde{\phi}, \]
\[ \chi = H_0 z + \tilde{\chi} \]

Compute generalised coordinates
\[ \alpha = \frac{\delta \mathcal{E}}{\delta \tilde{\phi}_x}, \quad \beta = \frac{\delta \mathcal{E}}{\delta \tilde{\chi}_x}, \quad \eta = \frac{\delta \mathcal{E}}{\delta \xi_x} \]

Solve (nonlinear) system for \( \tilde{\phi}_x, \tilde{\chi}_x, \xi_x \)

Hamiltonian: \[ H = \int_{-D}^{0} \alpha \tilde{\phi}_x dz + \int_{0}^{D} \beta \tilde{\chi}_x dz + \eta \xi_x - \mathcal{E} \]

Near onset - solve \( \tilde{\phi}_x, \tilde{\chi}_x, \xi_x \) using IFT -> Hamiltonian approximation

Linear Mag. law -> Generalised coordinates globally invertible

Monday, 3 June 13
Theorem: Linear Magnetisation Law

Coordinate transformation: [Twombly & Thomas, 1983]
\[ \tilde{z} = D(z - \xi)/(D - \xi), \quad \text{on} \quad \xi < z < D, \]
\[ \tilde{z} = D(z - \xi)/(D + \xi), \quad \text{on} \quad -D < z < \xi \]

Ferrofluid system has a Hamiltonian formulation \((M, \Omega, H)\)

where \( M = \{(\xi, \eta, \phi, \alpha, \chi, \beta) \in X, |W| < 1, \phi(x,0) = \chi(x,0)\} \)

\[ \Omega = d\eta \wedge d\xi + \int_{-D}^{0} d\alpha \wedge d\phi dz + \int_{0}^{D} d\beta \wedge d\chi dz \]

\[ H = \frac{1}{2} \int_{0}^{D} \left( K^+_2 \beta^2 - K^+_2 \chi^2 \right) dz + \frac{1}{2} \int_{-D}^{0} \left( \frac{K^-_2 \alpha^2}{\mu} - \mu K^-_2 \phi^2 \right) dz - \sqrt{1 - W^2} + 1 - \frac{1}{2} \xi^2 + C \xi \]

\[ -H \phi|_{z=-D} + H \chi|_{z=D} \]

and Hamilton’s equations solve the ferrofluid system

Bcs \( z=0 \)
\[ \phi = \chi \quad \mu K^-_2 \phi_z - K^+_2 \chi_z = \xi_x (K^-_2 \alpha - K^+_2 \beta) \]

Proof: [Groves & Toland, 1997] Non-dimensionalise
Linear Stability Analysis

Linearise Hamilton’s equations + bcs about

\[ \xi = \eta = \alpha = \beta = 0 \quad \phi = Hz \quad \chi = \mu Hz \]

\( H \) - magnetic field strength

Look for solns: \( e^{ikx} \)

Dispersion relation:

\[ -\frac{\mu(\mu - 1)^2 H^2}{(\mu + 1)} \tanh(kD)k + 1 + k^2 = 0 \]

\[ \mathcal{H} = \sqrt{\frac{\mu}{2(\mu + 1)(\mu - 1)}}H \]

\[ \mathcal{H}_{\text{crit}} = 1 \quad D \to \infty \]

Hamiltonian-Hopf Bif

\[ d \]

\[ k \]
Centre-Manifold Reduction

Apply Centre-Manifold Thm  [Mielke, 1991]

Removing the nonlinear boundary condition

\[ \Gamma(z) = \phi(-z) - \chi(z), \quad \Phi(z) = \mu K^-_2 \phi(-z) - K^+_2 \chi(z), \]
\[ \Phi(z) = \Theta(z) + \int_0^z \xi_x(K^-_2 \alpha(-z) - K^+_2 \beta(z)) \frac{(D-z)}{D} dz \]

Proposition: The spectrum of \( L \) consists of isolated, geometrically simple eigenvalues of finite algebraic multiplicity

Estimate:

\[ \| (L - ik)^{-1} \| \leq \frac{C}{1 + |k|} \]

Reduction:

\[ u_{xxxx} + Pu_{xx} + u + au^2 = 0 \]

Pulses exist if \( a < 0 \)  [Buffoni & Sere, 1996]
Spots

- Thm generalises - radially dependent Hamiltonian
- Hamilton's eqns solve the radial problem
- Radial centre-manifold reduction of Scheel (2003)?

\[ \Delta^2_{r} u + P \Delta_{r} u + u + au^2 = 0 \]

Existence of Centre-unstable manifold?
- Existence of a spot for \( a \neq 0 \) can then be established

[L. & Sandstede, 2009]
Numerics: 1D

- Flatten free-surface -> Computational domains are boxes
- Discretise with pseudo-spectral methods
- Phase condition -> $\chi, \phi$ only defined up to a constant

$\mu = 2$

$\mu = 6$

Check: Amplitude analysis  [Silber & Knobloch, 1988]
Numerics: 1D

No pulses

\( \mu \)

\( H \)

Lz=20
Lz=40
Lz=60
Lz=80
Lz=100

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Localised Hexagons

\[ \mu = 2.5 \]

Ferro-hexagon patches

[Gollwitzer, Rehberg & Richter]

Planar Hexagons

[L., Gollwitzer, et al.]

Patch bifurcation off spot

\[ \mathcal{H} = 0.995 \]
Conclusion

- Spatial Hamiltonian exists - Maxwell points?
- Centre-manifold reduction

Work to be done:
- 1D Normal form computations - Linear Magnetisation Law
- 1D Normal form computations - Nonlinear Magnetisation Law
- Establish existence of spots?

Numerical continuation of Ferrofluid equations

Outlook:
- Applied Field with x-component?
- Hexagon holes?
- Experiments?