Morphogenesis of growing tissues: elastic models, scaling laws and reduced theories of prestrained thin films

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This eight-hour course will be concerned with the analysis of thin elastic films which exhibit residual stress at free equilibria. Examples of such structures include, in particular, growing tissues such as leaves, flowers or marine invertebrates, as well as specifically engineered gels. There, it is conjectured that the growth process results in the formation of non-Euclidean target metrics, leading to complicated morphogenesis of the tissue which attains an orientation-preserving configuration closest possible to an isometric immersion of the metric.

This phenomenon can be studied through a variational model, pertaining to the non-Euclidean version of the nonlinear elasticity. For metrics with non-zero Riemann curvature, the infimum of the energy turns out to be positive at free equilibria. Further analysis of scaling of the energy minimizers in terms of the vanishing reference plate's thickness leads to the rigorous derivation of the corresponding limiting theories. These theories are differentiated by the embeddability properties of the target metrics - in the same spirit as different scalings of external forces lead to a hierarchy of nonlinear plate theories in classical elasticity whose rigorous (ansatz-free) derivation has been given by Friesecke, James and Muller.

The course will be self-contained and at a level suitable for PhD students having some familiarity with Mathematical Analysis. No prior experience with the research topics under discussion will be expected. A tentative outlay is as follows:

1. The notion of Γ-convergence and its fundamental properties.
2. Introduction to nonlinear elasticity, linear elasticity and non-Euclidean elasticity.
3. The Kirchhoff-like model of non-Euclidean plates as an example of dimension reduction in non-Euclidean elasticity. Statement of the theorem.
4. The fundamental role of Korn's inequality and the Friesecke-James-Muller geometric rigidity estimate. Introduction, discussion and outline (or more than outline) of the proofs.
5. Proof of the theorem in point 3.
6. Other energy scalings and the derivation of limiting theories in nonlinear elasticity of thin shells. Results on matching properties and density of infinitesimal isometries. Heuristic derivation of the hierarchy of theories in the Euclidean and non-Euclidean case. Discussion and open problems.