T1 Theorem and local Tb Theorems for Square functions

Ana Grau de la Herrán – University of Missouri

(The work presented is a joint work with Steve Hofmann)

June 2, 2012. Minneapolis. MN.
Goal

\[ \int \int_{\mathbb{R}^{n+1}_+} |\theta_t f(x)|^2 \frac{dxdt}{t} \leq C\|f\|_{L^2(\mathbb{R}^n)}^2 \]

In the **Square function** case:

\[ \theta_t f(x) := \int_{\mathbb{R}^n} \psi_t(x, y)f(y)dy, \quad t > 0, \quad x \in \mathbb{R}^n \]

In the **Singular integral function** case:

\[ \theta f(x) := \int_{\mathbb{R}^{n+1}} \phi(x - y)f(y)dy, \quad x \in \mathbb{R}^{n+1} \]
A Calderón Zygmund operator is an operator whose kernel \( \{ \psi_t(x, y) \}_{t \in (0, \infty)} \) is a Littlewood-Paley family of operators, that means that for some exponent \( \alpha > 0 \) they satisfy:

(a) (size condition) \(|\psi_t(x, y)| \leq C \frac{t^\alpha}{(t+|x-y|)^{n+\alpha}}\) 

\[\text{(vs} \ |\phi(x, y)| \leq C \frac{1}{|x-y|^{n+1}} \text{)}\]

(b) (continuity condition) 
\[|\psi_t(x, y + h) - \psi_t(x, y)| + |\psi_t(x + h, y) - \psi_t(x, y)| \leq C \frac{|h|^\alpha}{(t+|x-y|)^{n+\alpha}}\]

whenever \(|h| \leq t/2\) 

\[\text{(vs cancellation} \ |\phi(x, y) - \phi(x', y)| + |\phi(y, x) - \phi(y, x')| \leq C \frac{|x-x'|^\alpha}{|x-y|^{n+1+\alpha}} \text{)}\]
T1 Theorem for Square functions [Christ-Journé]

Let \( \theta_t f(x) \equiv \int_{\mathbb{R}^n} \psi_t(x, y)f(y)dy \), and assume that \( \psi_t \) is a LP family. Suppose that we have the Carleson measure estimate

\[
\sup_Q \frac{1}{|Q|} \int_0^{\ell(Q)} \int_Q |\theta_t 1(x)|^2 \frac{dxdt}{t} \leq C
\]

Then we have the square function bound that we wanted.

Remark 1.- The converse direction is essentially due to Fefferman and Stein.

Remark 2.- Remember that the departing point in CZ theory is the \( L^2 \) boundedness.
Tb Theorem for Square functions [Semmes]

Let \( \theta_t f(x) \equiv \int_{\mathbb{R}^n} \psi_t(x, y)f(y)dy \), assume that \( \psi_t \) is a LP family and \( b \) an accretive function.

Suppose that we have the Carleson measure estimate

\[
\sup_Q \frac{1}{|Q|} \int_0^{\ell^4(\mathbb{Q})} \int_Q |\theta_t b(x)|^2 \frac{dxdt}{t} \leq C
\]

Then we have the square function bound that we wanted.

Remark.-
An accretive function \( b \) is an \( L^\infty \) function that satisfies \( \text{Re}(b) \geq c > 0 \).
Local Tb Theorem for Square functions [HMc, HLMc,AHLMcT]

(Case $p = 2$ [HMc, HLMc,AHLMCT], Case $1 < p < 2$ [H])

Let $\theta_t f(x) \equiv \int_{\mathbb{R}^n} \psi_t(x, y)f(y)dy$,

Assume that $\psi_t$ is a LP family (+ (b)) and $1 < p < \infty$

Suppose also that there exist $\delta > 0$, $C_0 < \infty$ such that for any dyadic cube $Q$, there exists a function $b_Q$ satisfying:

(i) $\int_{\mathbb{R}^n} |b_Q|^p \leq C_0 |Q|

(ii) $\int_Q \left( \int_{\mathbb{R}^n} |\theta_t b_Q(x)|^2 \frac{dt}{t} \right)^{\frac{p}{2}} dx \leq C_0 |Q|

(iii) $\delta \leq |\int_Q b_Q|

Then we have the square function bound that we wanted.
Local Tb Theorem for Square functions (matrix version)  
[G.,Hofmann]  
(Case $p = 2$ by [HMc, HLMc,AHLMCT])

Let $\Theta_t f(x) \equiv \int_{\mathbb{R}^n} \Psi_t(x, y)f(y)dy$,

Assume that $\psi_t$ is a $\mathbb{C}^N$-valued LP family (+ (b)) and $1 < p < \infty$

Suppose also that there exist $\delta > 0$, $C_0 < \infty$ such that for any dyadic cube $Q$, there exists a $N \times N$ (complex) matrix valued mapping $B_Q$ satisfying:

(i) $\int_{\mathbb{R}^n} |B_Q|^p \leq C_0 |Q|$

(ii) $\int_Q \left( \int_0^{\ell(Q)} |\Theta_t B_Q(x)|^2 \frac{dt}{t} \right)^{\frac{p}{2}} dx \leq C_0 |Q|$

(iii) $\delta \leq \text{Re} \xi \cdot (|Q|^{-1} \int_Q B_Q) \bar{\xi}$, for all unit vectors $\xi \in \mathbb{C}^N$

Then we have the square function bound that we wanted.
Goal is that with the conditions of our theorem gives us the **Carleson measure estimate** from the T(1) Theorem.

- We fix a cube $Q$ and construct a family of dyadic subcubes $\{Q_j\}$ of $Q$ satisfying:
  - $\sum_j |Q_j| \leq (1 - \eta)|Q|$, $\eta \in (0, 1)$.
  - $\int_E \left( \int_0^\ell(Q) |\theta_t 1(x)|^2 \frac{dt}{t} \right)^{\frac{p}{2}} dx \leq C|Q|$
  - $\exists N > 0$ and $\beta \in (0, 1)$ such that
    - $|\Omega_N| := |\{x \in Q : \int_0^\ell(Q) |\theta_t 1(x)|^2 \frac{dt}{t} > N\}| \leq (1 - \beta)|Q|$
Suppose that $\psi_t : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{C}^N$ and for $f : \mathbb{R}^n \rightarrow \mathbb{C}^N$ we define

$$\theta_t f(x) = \int \psi_t(x, y) f(y) dy$$

We assume that $\Theta_t$ satisfies

(a) (Uniform $L^2$ bounds) $\sup_{t > 0} \|\theta_t f\|_2 \leq C \|f\|_2$

and (off-diagonal decay in $L^2$)

$$\|\theta_t f_j\|_{L^2(Q)} \leq C 2^{-\frac{(n+2+\beta)}{2}} j \|f_j\|_{L^2(2^{i+1}Q \setminus 2^i Q)},$$

for $f_j = f 1_{2^{i+1}Q \setminus 2^i Q}$ where $\ell(Q) \leq |t| \leq 2\ell(Q)$ and $\beta > 0$
More conditions

(b) (Quasi-orthogonality in $L^2$)

$$\|\theta_t Q_s f\|_{L^2(\mathbb{R}^n)} \leq C \left(\frac{s}{t}\right)^\beta \|f\|_{L^2(\mathbb{R}^n)}, \beta > 0, s < t,$$

where $\{Q_s\}_{0 \leq s \leq \infty}$ is some family of operators with

$$\int_{\mathbb{R}^n} \int_0^\infty |Q_s f|^2 \frac{dsdx}{s} \leq C \|f\|_2^2, \|\nabla Q_s f\|_{L^2} \leq \frac{1}{s} \|f\|_2.$$
More conditions

(c) (“Hypercontractive” off-diagonal decay)

\[
\left( \int_{Q^*} |\theta_t (g^1 S_j(Q))(y)|^2 dy \right)^{\frac{1}{2}} \leq C 2^{-j \nu} t^{-n(\frac{1}{r} - \frac{1}{2})} \left( \int_{S_j(Q)} |g|^r \right)^{\frac{1}{r}}
\]

for some $1 < r < 2$, $\forall j$, $\nu > \frac{n}{r}$, ($\nu = \frac{n}{r} + \epsilon$), $\epsilon > 0$

$Q^* = 8Q$

$S_0(Q) = 16Q$

$S_j(Q) = 2^{j+4} Q \setminus 2^{j+3} Q$, $j \geq 1$

(d) (Improved integrability) For some $s > 2$

\[
\|\theta_t f\|_s \leq C \|f\|_s
\]
Introduction
Generalized problem (no-pointwise estimates)
Applications

Generalized T1 Theorem [G., Hofmann]

Let \( \theta_t f(x) \equiv \int_{\mathbb{R}^n} \psi_t(x, y) f(y) \, dy \), which satisfies the previous general conditions.

Suppose that we have the Carleson measure estimate

\[
\sup_Q \frac{1}{|Q|} \int_Q \int_0^{l(Q)} |\theta_t 1(x)|^2 \frac{dx \, dt}{t} \leq C
\]

Then we have the square function bound that we wanted

Remark.-

For the T1 Theorem we only need conditions (a) and (b) for \( \theta_t \)
Generalized Local Tb Theorem [G., Hofmann]

Let \( \theta_t f(x) \equiv \int_{\mathbb{R}^n} \psi_t(x, y) f(y) dy \), which satisfies the previous general conditions.

Suppose also that exists a system \( \{b_Q\} \) of function indexed by cubes \( Q \subseteq \mathbb{R}^n \), a system of Lipschitz functions \( \{\Phi_Q\} \) also indexed by cubes and some constants \( 0 < C_0 < \infty, 0 < C_1 < 1, p > r \) such that for each cube \( Q \),

(i) \( \int_{\mathbb{R}^n} |b_Q|^p \leq C_0 |Q| \)

(ii) \( \int_Q \left( \int_0^{\ell(Q)} \int_{|x-y|<t} |\theta_t b_Q(y)|^2 \frac{dt}{t^{n+1}} \right)^{\frac{p}{2}} dy \leq C_0 |Q| \),

(iii) \( \delta m_Q(Q) \leq |\int_Q b_Q(x) dm_Q(x)| \) where \( dm_Q(x) = \Phi_Q(x) dx \) and \( \Phi_Q(x) \) satisfies \( \|\nabla \Phi_Q\|_\infty \leq C_0 \ell(Q)^{-1} \), \( C_1 \leq \Phi_Q(x) \leq 1 \) on \( Q \).

Then we have the square function bound that we wanted
Generalized Local Tb Theorem (matrix version) [G., Hofmann]

Let \( \Theta_t f(x) \equiv \int_{\mathbb{R}^n} \psi_t(x, y) f(y) dy \), which satisfies the previous general conditions.

Suppose also that exists a system \( \{B_Q\} \) of functions indexed by dyadic cubes \( Q \subseteq \mathbb{R}^n \) and some constants \( C_0 > 0, 0 < C_1 < 1 \) and \( p > r \) such that for each cube \( Q \).

(i) \( \int_{\mathbb{R}^n} |B_Q|^p \leq C_0 |Q| \)

(ii) \( \int_Q \left( \int_0^{\ell(Q)} \int_{|x-y|<t} |\Theta_t B_Q(x)|^2 \frac{dt}{t} \right)^{\frac{p}{2}} \, dx \leq C_0 |Q| , \)

(iii) \( \delta |\xi|^2 m_Q(Q) \leq Re \xi (\int_Q B_Q(x) dm_Q(x)) \bar{\xi} , \forall \xi \in \mathbb{C}^N \) and \( dm_Q(x) = \Phi_Q(x) dx \) as in the previous version.
Reduction to work just with dyadic cones

It is possible to prove this results by constructing the family of \( \{b_Q\} \) as a family indexed only by dyadic cubes. In order to have that result we need to change slightly one condition satisfied by such a family.

The condition

\[
\int_Q \left( \int_0^{\ell(Q)} \int_{|x-y|<t} |\theta_t b_Q(y)|^2 \frac{dtdy}{t^{n+1}} \right)^{\frac{p}{2}} dx \leq C_0 |Q|
\]

is changed by condition

\[
\int_Q \left( \int_{\tilde{\Gamma}_Q(x)} \int_{|x-y|<t} |\theta_t b_Q(y)|^2 \frac{dtdy}{t^{n+1}} \right)^{\frac{p}{2}} dx \leq C_0 |Q|,
\]

where \( \tilde{\Gamma}_Q(x) = \bigcup_{Q' \ni x, Q' \subset Q} U_Q \) and \( U_Q = Q \times (\ell(Q)/2, \ell(Q)) \).
Kato estimate

\[ \| \sqrt{L} h \|_2 \leq C \| \nabla h \|_2, \quad \forall h \in L^2_1(\mathbb{R}^n) \]

where \( Lu = -\text{div}(A(x)\nabla u) \) and \( A \) has \( L^\infty \) coefficients and satisfy the ellipticity condition.

Such result is equivalent to prove the \( L^2 \) boundedness on \( L^2_1(\mathbb{R}^n) \) of the square function operator \( \Theta_t = te^{-t^2L} \text{div}A \)

And the local test functions are \( B_Q = \nabla(e^{\epsilon^2\ell(Q)^2L} \varphi_Q) \) with

\[ \varphi_Q(x) := (x - x_Q)\eta_Q, \quad \eta_Q \in C^\infty_0(5Q), \quad \eta_Q \equiv 1 \text{ on } 4Q. \]
Layer Potential

Let’s assume the solvability of the Dirichlet problem $\left( D \right)_{p'}$ and the Regularity problem $\left( R \right)_{q}$ for $L^*$ in the lower half-space with $p > \frac{2n}{n+2}$ and $q > 1$. Then

$$\int_{\mathbb{R}^n} \int_{0}^{\infty} \left| t \partial_t^2 S_t f(x) \right|^2 \frac{dt dx}{t} \leq C \| f \|_2^2.$$ 

$L = -\text{div}A \nabla$ defined in $\mathbb{R}^{n+1}$, $n \geq 2$.

$A = A(x)$ is a $t$-independent $(n + 1) \times (n + 1)$ matrix of complex-valued coefficients defined on $\mathbb{R}^n$

$\Gamma(X, Y)$ the fundamental solution for $L$

$(\Rightarrow \Gamma(x, t, y, s) = \Gamma(x, t - s, y, 0))$

$S_t f(x) := \int_{\mathbb{R}^{n+1}} \Gamma(x, t, y, s) f(y) dy$
Thank you!!!