Mathematical Modeling in Industry Workshop XVI
Final Progress Report
Team 7: Valuation of Over-the-Counter Derivatives with Collateralization

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Abstract

In this report we try to derive formulas for prices of financial derivatives with collateralization. We first review approaches in [3] and [2]. Then we verify that the formula in Piterbarg [1] with single currency and single assets. Later we use the same framework to generalize the results to cross currency and multi-assets models. The report is a result of Team 7 in the Mathematical Modeling in Industry Workshop XVI held at the University of Calgary in June 2012.

1 Introduction

Collateralized OTC (over-the-counter) derivatives have grown rapidly over the last decade. According to the ISDA (International Swaps and Derivatives Association) Margin Survey [4], at the end of 2009 70% of the trade volumn of OTC derivatives were collateralized, in comparison with 30% in 2003. The percentage is expected to grow even faster due to the impact of the financial crisis and the growing attention to credit risk. The collateralization is usually done by a so-called credit support annex (CSA), which defines all detailed terms of the transaction.

Collateralization significantly reduces the credit risk for the party with negative derivative value. But in the framework of this report we don't consider credit risk when determine the derivative price. We only consider the difference of the prices of collateralized derivative and non-collateralized derivative brought by cost associated with collateralization. We assume the collateral account $C(t)$ is continuously adjusted. Thus the cost associated with collateralization is unknown unless $C(t)$ is known. Typically $C(t)$ is proportional to the value $V(t)$ of the derivative. So once $V(t)$ is known, the dynamics of $C(t)$ can be found.

The framework we are using in the report are adopted from Piterbarg [1], which considers constructing a replicating portfolio including the underlying, a collateral account and a cash account. We try to extend the results to cross currency and multi-assets models.

Although there is a trend to use standardized CSA agreements, due to the OTC nature of the derivatives we are considering in the paper, the specific terms of collateralization may vary from case to case. The report intends to cover a number of cases for how collateral accounts are maintained by the two counter parties or possibly a third party.

The report is organized as follows: Section 2 gives an overview of Literature. Subsection 2.1 gives an comparison of the three paper Piterbarg [1], Castagana

[3], and Fujii et al [2] (Written by Tong). Subsection 2.2 review the discrete approach in Castagna [3] (written by Henrike). Subsection 2.3 review the martingale approach in Fujii et al [2] (written by Henrike). Section 3 presents our approach under Piterbarg's framework. Subsection 3.1 review Piterbarg [1] and verifies results in the paper (written by Binbin). Subsection 3.2 uses the same framework to obtain formulas for cross currency cases (written by Binbin). Subsection 3.3 deals with multi-assets & single currency case (written by Shi), and Subsection 3.4 deals with multi-assets & cross currency case (written by Shi). Section 4 gives another idea of pricing collateralized derivatives (written by Tong).

2 Overview of literature

2.1 Comparison of literature

There are three major approaches for the valuation model of collateralized derivative: Vladimir Piterbarg used replication of portfolio to derive the most general solution; Antonio Castagna made the assumption for partial collateralized situation and gave the dynamic model for collateral account; Massaki Fujii considered the self-financing fact and include reinvestment factor into his collateral dynamic account. All of the above papers coincide into several important results, for example, the equation relating V_t^C, V_t^{NC} and LVA–Liquidity Value Adjustment:

$$V_t^C = V_t^{NC} + LVA \tag{1}$$

where

$$LVA = E\left[\int_t^T e^{-\int_t^s r_u du}(r_s - c_s)C_s ds\right] \tag{2}$$

here $r_\cdot$ is the risk free market rate, and $c_\cdot$ is the collateral return interest, which means that the derivative buyer(collateral poster) will get periodic interest return from derivative seller(collateral holder) and the return rate is defined as $c_\cdot$.
Even through the three authors shared a lot of great ideas, there are still some distinctions among their specific work. We can summarize them as follows:

- Piterbarg paper is the most general paper. It contains barely no assumption, which is favorable because of the solid cash flow replication:

  $$d\alpha(t) = [r_C(t)C(t) + r_F(t)(V(t) - C(t)) - r_R(t)\Delta(t)S(t) + r_D(t)\Delta(t)S(t)]dt$$

- Castagna paper is popular in financial market because of the nice dynamic equation for collateral account. But it is not flawless, when applied to partial collateral situations it is difficult to explain the inconsistency between partial collateral constraint and the differential equation of collateral dynamic:

$$\begin{aligned} dC_t &= c_t C_t dt + \alpha dV_t \\ C_t &= \alpha V_t \end{aligned}$$

- Fujii et al paper is also a widely discussed paper, it clarifies the cash flow within the collateral account, especially by indicating that the net return rate for collateral holder is equal to the difference of reinvestment rate(risk free market rate $r.$) and collateral interest return $c.$. But the drawback for Fujii et al paper is that he uses a different collateral dynamic from Castagna paper but leaves a gap between the dynamic and the conclusion.

$$\begin{aligned} dC_t &= (r_t - c_t) C_t dt + a(t) dV_t \\ a(t) &= e^{\int_t^s (r_u - c_u) du} \end{aligned}$$

Later on in our research work, we completed the assumption of partial collateral and applied to Piterbarg and Fujii et al papers. The final conclusions coincide into Castagna result.

$$V_t^C = E\left[d^{-\int_t^T [r_s(1-\alpha)+c_s\alpha]} ds V_T\right]$$

2.2 Discrete Approach of Pricing OTC with Collateralization

Antonio Castagna presents in [**?**] an approach to price derivatives with collateralization using binomial tree models. For this approach we introduce the following parameters:

- S is the underlying asset at time 0. It either goes up to Su or goes down to Sd (note that $u > 1$ and $d < 1$).
- V^C is the price of the collateralized contingent. If the underlying asset goes up we obtain the value V_u^C for the price of the collateral contingent and similarly if it goes down we obtain the value V_d^C.

- C is the value of the cash collateral account.
- The collateral rate is given by c (the interest one earns on the collateral account).
- The collateral factor is given by γ. Note if $\gamma = 1$ the derivative is fully collateralized.
- B is the value in the Bank account which earns at each period the risk-free interest r.

In order to find the arbitrage free collateralized pricing formula we will use the following replication strategy:

$$V_u^C - C(1+c) = \alpha Su + \beta B(1+r) \tag{3}$$
$$V_d^C - C(1+c) = \alpha Sd + \beta B(1+r). \tag{4}$$

With the help of Equation (3) and (4) we obtain the following representation for the coefficients α and β:

$$\alpha = \frac{V_u^C - V_d^C}{S(u-d)} \tag{5}$$
$$\beta = \frac{uV_d^C - dV_u^C - (1+c)C(u-d)}{(u-d)B(1+r)} \tag{6}$$

If we are able to replicate the pay-off of the collateralized contingent, then at time $t = 0$ we are receiving the arbitrage-free price of V^C. Hence:

$$V^C - C = \alpha S + \beta B = \frac{V_u^C - V_d^C}{(u-d)} + \frac{uV_d^C - dV_u^C - (1+c)C(u-d)}{(u-d)(1+r)}. \tag{7}$$

For the next step we assume that we allow partial collateralization, thus with $V^C = \gamma C$, we can rewrite Equation (7) as follows:

$$V^C - \gamma V^C = \frac{(1+r)(V_u^C - V_d^C) + uV_d^C - dV_u^C - (1+c)\gamma V^C(u-d)}{(u-d)(1+r)}$$
$$(1-\gamma)V^C + \frac{1+c}{1+r}\gamma V^C = \frac{(1+r)(V_u^C - V_d^C) + uV_d^C - dV_u^C}{(u-d)(1+r)}$$
$$\left(\frac{(1+r)(1-\gamma) + (1+c)\gamma}{1+r}\right) V^C = \frac{1}{1+r}\left(\underbrace{\frac{u-1-r}{u-d}}_{=p} V_d^C + \underbrace{\frac{1+r-d}{u-d}}_{=1-p} V_u^C\right)$$

Hence with some simplification we obtain the following pricing formula for collateral contingents:

$$V^C = \frac{1}{(1+r)(1-\gamma)+(1+c)\gamma}\left(pV_u^C + (1-p)V_d^C\right). \tag{8}$$

The pricing formula of a non-collateralized agreement is given by:

$$V^C\left(\frac{(1+r)(1-\gamma)+(1+c)\gamma}{1+r}\right) = \frac{1}{1+r}\left(pV_u^C + (1-p)V_d^C\right). \tag{9}$$

Hence by simplifying the left hand side of Equation (9) we have:

$$V^{NC} = V^C\left(\frac{(1+r)(1-\gamma)+(1+c)\gamma}{1+r}\right) = V^C - \gamma\frac{r-c}{1+r}V^C.$$

We define the last term

$$LVA = \gamma\frac{r-c}{1+r}V^C$$

as Liquidity Value Adjustment. Therefore the collateralized contingent minus the Liquidity Value Adjustment gives us the price of the non-collateralized contingent. Note that the price of the non-collateral contingent is the expected return under the risk neutral measure of an uncollateralized derivative.

2.3 Pricing Collateralized Derivatives by Fujii et al

Single Currency Case

In this section we are reviewing two approaches by Fujii et al [2] for fully collateralized derivatives. In the first approach a new dynamic to get the collateralized pricing formula is introduced, in the second approach the collateralized price is derived from the well known LVA-formula.

Fujii et al [2] introduces the following dynamics on the cash collateral account $C(t)$ with a self-financing strategy:

$$dC(s) = y(s)C(s)ds + a(s)dV(s), \tag{10}$$

where $y(s) = r(s) - c(s)$ is the difference of the risk-free interest rate and the collateral rate and $V(s)$ denotes the value of a derivative that matures at T with cash flow $V(T)$ at time s. This dynamic represents the perspective of the borrower and the lender. The borrower of the derivatives provides the cash collateral to the lender, the lender receives the risk-free interest r but has to pay the collateral

rate c to the borrower. Hence the lender receives $y(s) := r(s) - c(s)$. By the self-financing strategy the lender invests the gain from the interest rates to provide the buyer with $a(s)$ new derivatives. Equation (10) can be solved by the following integration:

$$\begin{aligned}
& dC(s) - y(s)C(s)ds = a(s)dV(s) \\
\Rightarrow \quad & \left(e^{-\int_0^s y(u)du}\right)(dC(s) - y(s)C(s)ds) = \left(e^{-\int_0^s y(u)du}\right)a(s)dV(s) \\
\Rightarrow \quad & d\left(e^{-\int_0^s y(u)du}C(s)\right) = \left(e^{-\int_0^s y(u)du}\right)a(s)dV(s) \\
\Rightarrow \quad & e^{-\int_0^T y(u)du}C(T) - e^{-\int_0^t y(u)du}C(t) = \int_t^T e^{-\int_0^s y(u)du}a(s)dV(s) \\
\Rightarrow \quad & C(T) = e^{\int_0^T y(u)du}\left(e^{-\int_0^t y(u)du}C(t) + \int_t^T e^{-\int_0^s y(u)du}a(s)dV(s)\right) \\
\Rightarrow \quad & C(T) = e^{\int_t^T y(u)du}C(t) + \int_t^T e^{\int_s^T y(u)du}a(s)dV(s).
\end{aligned}$$

We are considering the following trading strategy:

$$C(t) = V(t) \tag{11}$$

$$a(s) = e^{\int_t^s y(u)du}. \tag{12}$$

To model the amount of additional derivatives the borrower receives by its lender we use Equation (12). By substituting Equations (11) and (12) in

$$C(T) = e^{\int_t^T y(u)du}C(t) + \int_t^T e^{\int_s^T y(u)du}a(s)dV(s)$$

we have:

$$C(T) = e^{\int_t^T y(u)du}V(t) + \int_t^T e^{\int_s^T y(u)du}e^{\int_t^s y(u)du}dV(s).$$

This Equation can be simplified by integration methods as follows:

$$\begin{aligned}
C(T) &= e^{\int_t^T y(u)du}V(t) + \int_t^T e^{\int_t^T y(u)du}dV(s) \\
&= e^{\int_t^T y(u)du}\left(V(t) + \int_t^T dV(s)\right) \\
&= e^{\int_t^T y(u)du}\left(V(t) + V(T) - V(t)\right) \\
&= e^{\int_t^T y(u)du}V(T).
\end{aligned}$$

In order to obtain the risk-neutral collateralized pricing formula the equation above is discounted by the risk-free rate r:

$$\begin{aligned} V(t) &= E_t^Q\left[e^{-\int_t^T r(u)du} e^{\int_t^T y(u)du} V(T)\right] \\ &= E_t^Q\left[e^{-\int_t^T (r(s)-y(s))ds} V(T)\right] \\ &= E_t^Q\left[e^{-\int_t^T c(s)ds} V(T)\right]. \end{aligned}$$

Note that in [2] $E^Q[]$ denotes the expectation under the money-market account as a numeraire. Fujii et al [2] derived with a different approach the same result as stated above. This derivation involves the LVA-Formula as stated in [**?**]:

$$V^C = V^{NC} + LVA. \tag{13}$$

From arbitrage theory we know that the risk-neutral pricing formula for a non-collateral agreement is presented by the following formula:

$$V^{NC}(t) = E_t^Q\left[e^{-\int_t^T r(s)ds} V(T)\right].$$

The following LVA-formula for the continuous case was derived in [**?**]:

$$LVA = E_t^Q\left[\int_t^T e^{-\int_t^s r(u)du} y(s)V(s)ds\right].$$

Combining these two results we have:

$$\begin{aligned} V(t) &= E_t^Q\left[e^{-\int_t^T r(s)ds} V(T)\right] + E_t^Q\left[\int_t^T e^{-\int_t^s r(u)du} y(s)V(s)ds\right] \\ &= E_t^Q\left[e^{-\int_t^T r(s)ds} V(T) + \int_t^T e^{-\int_t^s r(u)du} y(s)V(s)ds\right]. \end{aligned}$$

Define the following process:

$$X(t) = e^{-\int_0^t r(s)ds} V(t) + \int_0^t e^{-\int_0^s r(u)du} y(s)V(s)ds. \tag{14}$$

In order to show that the process $X(t)$ is a Q-martingale it should hold that for $0 \leq \tau \leq t$

$$E^Q[X(t)|F_\tau] = X(\tau).$$

With the calculation below we obtain the following result:

$$
\begin{aligned}
E^Q[X(t)|F_\tau] =& E^Q\left[e^{-\int_0^t r(s)ds}V(t) + \int_0^t e^{-\int_0^s r(u)du}y(s)V(s)ds|F_\tau\right] \\
=& E^Q\left[e^{-\int_0^t r(s)ds}\left(E_t^Q\left[e^{-\int_t^T r(s)ds}V(T)\right] + E_t^Q\left[\int_t^T e^{-\int_t^s r(u)du}y(s)V(s)ds\right]\right)|F_\tau\right] \\
&+ E^Q\left[\int_0^t e^{-\int_0^s r(u)du}y(s)V(s)ds|F_\tau\right] \\
=& E^Q\left[e^{-\int_0^t r(s)ds}\left(e^{-\int_t^T r(s)ds}V(T) + \int_t^T e^{-\int_t^s r(u)du}y(s)V(s)ds\right)|F_\tau\right] \\
&+ E^Q\left[\int_0^t e^{-\int_0^s r(u)du}y(s)V(s)ds|F_\tau\right] \\
=& E^Q\left[\left(e^{-\int_0^T r(s)ds}V(T) + \int_t^T e^{-\int_0^s r(u)du}y(s)V(s)ds\right) + \int_0^t e^{-\int_0^s r(u)du}y(s)V(s)ds|F_\tau\right] \\
=& E^Q\left[e^{-\int_0^T r(s)ds}V(T) + \int_0^T e^{-\int_0^s r(u)du}y(s)V(s)ds|F_\tau\right] \\
=& E^Q[X(T)|F_\tau].
\end{aligned}
$$

Similarly we can show that

$$E^Q[X(T)|F_\tau] = E^Q[X(\tau)|F_\tau].$$

Hence in the end we obtain that $X(t)$ is a martingale as:

$$E^Q[X(t)|F_\tau] = E^Q[X(T)|F_\tau] = E^Q[X(\tau)|F_\tau] = X(\tau).$$

By differentiation of $X(t)$ with the help of Itô's Lemma we have:

$$
\begin{aligned}
dX(t) &= e^{-\int_0^t r(s)ds}\left(-r(t)V(t)dt + dV(t)\right) + e^{-\int_0^t r(s)ds}y(t)V(t)dt \\
dX(t) &= e^{-\int_0^t r(s)ds}\left(y(t) - r(t)\right)V(t)dt + e^{-\int_0^t r(s)ds}dV(t)
\end{aligned}
$$

With rearrangements of the factors we have:

$$dV(t) = \left(r(t) - y(t)\right)V(t)dt + \underbrace{e^{-\int_0^t r(s)ds}dX(t)}_{:=dM(t)},$$

where $dM(t)$ is a martingale. Therefore the dynamics of the collateralized derivative value can be expressed as a stochastic differential equation with martingale $dM(t)$:

$$dV(t) = (r(t) - y(t))\, V(t)dt + dM(t).$$

To get the collateralized pricing formula we use the following derivation:

$$\begin{aligned}
& dV(s) - c(s)V(s)ds = dM(s) \\
\Rightarrow \quad & \left(e^{-\int_0^s c(u)du}\right)(dV(s) - c(s)V(s)ds) = \left(e^{-\int_0^s c(u)du}\right) dM(s) \\
\Rightarrow \quad & d\left(e^{-\int_0^s c(u)du} V(s)\right) = \left(e^{-\int_0^s c(u)du}\right) dM(s) \\
\Rightarrow \quad & e^{-\int_0^T c(u)du} V(T) - e^{-\int_0^t c(u)du} V(t) = \int_t^T e^{-\int_0^s c(u)du} dM(s) \\
\Rightarrow \quad & V(t) = e^{\int_0^t c(u)du}\left(e^{-\int_0^T c(u)du} V(T) - \int_t^T e^{-\int_0^s c(u)du} dM(s)\right) \\
\Rightarrow \quad & V(t) = e^{-\int_t^T c(u)du} V(t) - \int_t^T e^{-\int_t^s c(u)du} dM(s).
\end{aligned}$$

Applying the conditional expectation to the equation above we get the final collateralized pricing formula:

$$\begin{aligned}
\Rightarrow \quad & E_t[V(t)] = E_t\left[e^{-\int_t^T c(u)du} V(t) - \int_t^T e^{-\int_t^s c(u)du} dM(s)\right] \\
\Rightarrow \quad & V(t) = E_t\left[e^{-\int_t^T c(u)du} V(t)\right].
\end{aligned}$$

Extension to Partially Collateralized Derivatives

We will know extend Fujii et al [2] approach to partial collateralized derivatives. We are therefore considering the well known LVA-formula in [**?**] that allows partial collateralization (note: $0 < \gamma < 1$):

$$V_t = E\left[e^{-\int_t^T r(u)du} V(T) + \gamma \int_t^T e^{-\int_t^s r(v)dv} \underbrace{(r(s) - c(s))}_{:=y(s)} V(s)ds\right].$$

Similarly as in the fully collateral case, we define the following process:

$$X(t) = e^{-\int_0^t r(s)ds} V(t) + \gamma \int_0^t e^{-\int_0^s r(u)du} y(s)V(s)ds,$$

where we show by the same argumentation that $X(t)$ is a martingale. By differentiation of $X(t)$ using Itô's Lemma we obtain:

$$dX(t) = -e^{-\int_0^t r(s)ds} r(t)V(t)dt + e^{-\int_0^t r(s)ds} dV(t) + \gamma e^{-\int_0^t r(s)ds} y(t)V(t)dt.$$

Defining $dM(t) = e^{\int_0^t r(s)ds} dX(t)$ we derive the following stochastic process:

$$\begin{aligned} dM(t) &= -r(t)V(t)dt + dV(t) + \gamma(r(t) - c(t))V(t)dt \\ dV(t) &= ((1-\gamma)r(t) + \gamma c(t))\, V(t)dt + dM(t) \end{aligned}$$

with the martingale $dM(t)$. Integrating this stochastic process and taking the conditional expectation on t we derive the result:

$$V(t) = E\left[e^{-\int_t^T [(1-\gamma)r(s)+\gamma c(s)]ds} V_T\right]$$

which coincides with the results presented in [**?**].

Cross-Currency Case

Suppose the underlying and the derivative are in domestic currency i but the collateral is posted in the foreign currency j. Also suppose $f_x^{(i,j)}(t)$ is the foreign exchange rate at time t representing the price of the unit amount of currency j in terms of currency i. Then the collateralized derivative price with collateral $C^{(i)}(t) = V^{(i)}(t)$ is given by [2]

$$\begin{aligned} V^{(i)}(t) =& E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds} V^{(i)}(T)\right] \\ &+ f_x^{(i,j)}(t) E_t^{Q_j}\left[\int_t^T e^{-\int_t^s r^{(j)}(u)du} y^{(j)}(s)\left(\frac{C^{(i)}(t)}{f_x^{(i,j)}(t)}\right) ds\right], \end{aligned}$$

where $E_t^{Q_i}$ is the conditional expectation under risk neutral measure of domestic currency i and $E_t^{Q_j}$ is the conditional expectation under risk neutral measure of foreign currency j. Change all the measure to Q_i, Fujii et al [2] obtained

$$V^{(i)}(t) = E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds} V^{(i)}(T) + \int_t^T e^{-\int_t^s r^{(i)}(u)du} y^{(j)}(s) V^{(i)}(s)ds\right].$$

Define

$$X(t) = e^{-\int_0^t r^{(i)}(s)ds} V^{(i)}(t) + \int_0^t e^{-\int_0^s r^{(i)}(u)du} y^{(j)}(s) V^{(i)}(s)ds.$$

Using the argument similar as above single currency case, we can show $X(t)$ is a Q_i martingale. Let $0 < \tau \leq t$, then

$$\begin{aligned}
E^{Q_i}[X(t)|F_\tau] =& E^{Q}\left[e^{-\int_0^t r^{(i)}(s)ds}V^{(i)}(t) + \int_0^t e^{-\int_0^s r^{(i)}(u)du}y^{(j)}(s)C^{(i)}(s)ds|F_\tau\right] \\
=& E^{Q_i}\left[e^{-\int_0^t r^{(i)}(s)ds}\left(E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds}V^{(i)}(T) + \int_t^T e^{-\int_t^s r^{(i)}(u)du}y^{(j)}(s)V^{(i)}(s)ds\right]\right)|F_\tau\right] \\
&+ E^{Q_i}\left[\int_0^t e^{-\int_0^s r^{(i)}(u)du}y^{(j)}(s)V^{(i)}(s)ds|F_\tau\right] \\
=& E^{Q_i}\left[e^{-\int_0^T r^{(i)}(s)ds}V^{(i)}(T) + \int_t^T e^{-\int_0^s r^{(i)}(u)du}y^{(j)}(s)V^{(i)}(s)ds\right] \\
&+ E^{Q_i}\left[\int_0^t e^{-\int_0^s r^{(i)}(u)du}y^{(j)}(s)V^{(i)}(s)ds|F_\tau\right] \\
=& E^{Q_i}\left[e^{-\int_0^T r^{(i)}(s)ds}V^{(i)}(T) + \int_0^T e^{-\int_0^s r^{(i)}(u)du}y^{(j)}(s)V^{(i)}(s)ds|F_\tau\right] \\
=& E^{Q_i}[X(T)|F_\tau].
\end{aligned}$$

Thus, it is obvious that $E^{Q_i}[X(t)|F_\tau] = E^{Q_i}[X(\tau)|F_\tau] = X(\tau)$ and therefore $X(t)$ is a Q_i martingale. By Ito's lemma,

$$dX(t) = e^{-\int_0^t r^{(i)}(s)ds}\left[(y^{(j)}(t) - r^{(i)}(t))V^{(i)}(t)dt + dV^{(i)}(t)\right].$$

Let $dM(t) = e^{\int_0^t r^{(i)}(s)ds}dX(t)$, then $M(t)$ is also a Q_i martingale and it follows that

$$dV(t) = (r^{(i)}(t) - y^{(j)}(t))V^{(i)}(t)dt + dM(t).$$

Therefore, $e^{-\int_s^t (r^{(i)}(u)-y^{(j)}(u))du}V(t)$ is also a Q_i martingale. Hence,

$$e^{-\int_s^t (r^{(i)}(u)-y^{(j)}(u))du}V^{(i)}(t) = E_t^{Q_i}\left[e^{-\int_s^T (r^{(i)}(u)-y^{(j)}(u))du}V^{(i)}(T)\right]$$

Then,

$$V^{(i)}(t) = E_t^{Q_i}\left[e^{-\int_t^T r^{(i)}(s)ds}\left(e^{\int_t^T y^{(j)}(s)ds}V^{(i)}(T)\right)\right].$$

3 Our approach under Piterbarg's framework

3.1 Single currency & Single asset

In this section we review Piterbarg's [1] general setting for solving the pricing problem. The main results are verified with detailed proof. The same setting are

used in the cross currency and multi-assets cases in Section 3.2, Section 3.3, and Section 3.4.

Let $V(t) = V(t, S_t)$ be the time t price of a collateralized derivative written on an asset with price dynamics

$$dS_t/S_t = \mu_t dt + \sigma_t dW_t$$

in the real world probability measure. Then by Ito's formula, the dynamics of $V(t)$ is

$$dV(t) = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_t^2 S_t \frac{\partial^2 V}{\partial S^2}\right] dt + \Delta(t) dS_t \tag{15}$$

where

$$\Delta(t) = \frac{\partial V}{\partial S}$$

To replicate the derivative we consider a self-financing portfolio with a collateral account $C(t)$, $\Delta(t)$ units of the underlying asset, and a cash account. Let $\gamma(t)$ be the sum of the collateral account and the cash account, then the growth of $\gamma(t)$ are determined by each of following rates:

- the collateral rate $r_C(t)$ for the collateral account $C(t)$;
- the repo rate $r_R(t)$ used to borrow/lend $\Delta(t)S_t$ to purchase $\Delta(t)$ units of the underlying;
- the interest rate $r_F(t)$ used to borrow/lend the rest of the cash $V(t) - C(t)$;
- the stock pays dividend at a rate $r_D(t)$

Remark In the view point of a bank, say bank A, if the other party are required to post collateral to the bank, then the bank must pay interest at rate $r_C^A(t)$ to the collateral account but can also invest the collateral $C(t)$ to get an investment rate $r_v^A(t)$ (or other discounting rate the bank use for internal purpose). Hence the interest rate $r_C(t)$ need to be replaced by $r_C^A(t) - r_v^A(t)$. If the bank has to post collateral to an account under his own bank, the story in the same. But if the other party is another bank, say bank B, then the rate we should use is $r_C^B(t) - r_v^A(t)$. However we don't consider such complication in this report.

Hence the growth of $\gamma(t)$ is

$$d\gamma(t) = [r_C(t)C(t) - r_R(t)\Delta(t)S_t + r_F(t)(V(t) - C(t)) + r_D(t)\Delta(t)S_t]dt \tag{16}$$

The value of the portfolio ia

$$X(t) = \Delta(t)S_t + \gamma(t),$$

by the self-financing assumption we have

$$dX(t) = \Delta(t)dS_t + d\gamma(t),$$

hence by Equation (15) we have

$$d\gamma(t) = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_t^2 S_t \frac{\partial^2 V}{\partial S^2}\right] dt \tag{17}$$

Comparing equation (16) and (17) we get

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_t^2 S_t \frac{\partial^2 V}{\partial S^2} = r_C(t)C(t) + r_F(t)(V(t) - C(t)) + (r_D(t) - r_R(t))\frac{\partial V}{\partial S}S_t$$

which can be rearranged as

$$\frac{\partial V}{\partial t} + (r_R(t) - r_D(t))\frac{\partial V}{\partial S}S + \frac{1}{2}\sigma_t^2 S_t \frac{\partial^2 V}{\partial S^2} = r_F(t)V(t) - (r_F(t) - r_C(t))C(t) \tag{18}$$

This partial deferential can be solved using the Feynman-Kac theorem. To remind the reader we state the Feynman-Kac theorem.

Theorem 3.1 (Feynman-Kac Theorem) *Consider the PDE*

$$\frac{\partial u}{\partial t} + \mu(x,t)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 u}{\partial x^2} - V(x,t)u + f(x,t) = 0$$

defined for all real x and t in the interval $[0,T]$, subject to the terminal condition $u(x,T) = \psi(x)$, where μ, σ, ψ, V are known functions, T is a parameter and $u : \mathbb{R} \times [0,T] \to \mathbb{R}$ is the unknown. Then the solution can be written as an conditional expectation as follows:

$$u(x,t) = E\left[\int_t^T e^{-\int_t^s V(X_\tau)\,d\tau} f(X_s,s)ds + e^{-\int_t^T V(X_\tau)\,d\tau}\psi(X_T) \mid X_t = x\right]$$

where X is an Itô process driven by the equation

$$dX = \mu(X,t)\,dt + \sigma(X,t)\,dW,$$

where $W(t)$ is a Brownian motion and the initial condition for $X(t)$ is $X(0) = x$.

Therefore by applying the Feynman-Kac Theorem we get the solution of the PDE (18):

$$V(t) = E_t^Q \left[e^{-\int_t^T r_F(u)du} V_T + \int_t^T e^{-\int_t^u r_F(v)dv} (r_F(u) - r_C(u)) C(u) du \right] \tag{19}$$

where Q is the measure under which the underlying price dynamics is

$$dS_t/S_t = (r_R(t) - r_D(t))dt + \sigma_t d\widetilde{W}_t$$

By rearrange the terms in the PDE (18) one can get the solution in a different form:

$$V(t) = E_t^Q \left[e^{-\int_t^T r_C(u)du} V_T + \int_t^T e^{-\int_t^u r_F(v)dv} (r_F(u) - r_C(u))(V(u) - C(u)) du \right] \tag{20}$$

Now setting $C = 0$ in equation (19) we get the classic result

$$V(t) = E_t^Q \left[e^{-\int_t^T r_F(u)du} V_T \right].$$

with no collateral. Setting $V = C$ in equation (20) we get the formula for the derivative price with full collateral:

$$V(t) = E_t^Q \left[e^{-\int_t^T r_C(u)du} V_T \right].$$

While $C(t)$ usually depends on $V(t)$, equation (19) and equation (20) are not of practical use when in the imperfect collateral cases. But if we assume $C(t) = \alpha V(t)$ we can write the PDE (18) as

$$\frac{\partial V}{\partial t} + (r_R(t) - r_D(t)) \frac{\partial V}{\partial S} S + \frac{1}{2} \sigma_t^2 S_t \frac{\partial^2 V}{\partial S^2} = \left[r_F(t) - \alpha (r_F(t) - r_C(t)) \right] V(t)$$

and again by the Feynman-Kac theorem we obtain an important formula

$$V(t) = E_t^Q \left[e^{-\int_t^T [(1-\alpha) r_F(u) + \alpha r_C(u)] du} V_T \right] \tag{21}$$

which gives the derivative price for all type of collateralization with $C(t) = \alpha V(t)$ for $0 \leq \alpha \leq 1$. It is consistent with the formulas for the no collateral case and full collateral case.

3.2 Cross currency & Single asset

In this section we consider that the two counter parties are in different countries hence use different currency. This complicates the problem but it still can be solved using the same framework in Section 3.1.

Assuming the two currencies are the domestic currency D and foreign currency F, and party A (domestic) and B (foreign) are involved in the derivative trading. The exchange rate is given by r_{df}, i.e. M_f in foreign currency is worth $M_d = r_{df} M_f$ in domestic currency. The underlying is dominated in the domestic currency. We consider the following two cases:

- Case 1: the collateral are posted in domestic currency D
- Case 2: the collateral are posted in foreign currency F
- Case 3: the collateral are posted in in domestic currency D when $V_t < 0$ and in foreign currency F when $V_t > 0$

For Case 1, standing on the view point of A, it is exactly the same case as a single currency case in Section 3.1, hence the same formulas in Section 3.1 apply.

For Case 2, let $C_f(t)$ be the collateral account. Then the collateral account is worth $C(t) = r_{df}(t)C_f(t)$ in domestic currency. The growth of the cash accounts are slightly different than the single currency case. It is determined by the following:

- the collateral rate $r_C^f(t)$ for the collateral account $C_f(t)$ in foreign currency, which is $C(t)$ in domestic currency;
- the interest rate $r_F^f(t)$ used to create a foreign cash account $-C_f(t)$;
- the rest of cash is $V(t) - \Delta t S_t$, but the portion $-\Delta t S_t$ can be borrowed/lend at a secured repo rate $r_R(t)$, and $V(t)$ is borrowed at a unsecured rate $r_F(t)$
- the stock pays dividend at a rate $r_D(t)$

The growth of $\gamma(t)$ is given by

$$d\gamma(t) = [(r_C^f(t) - r_F^f(t))C(t) - r_R(t)\Delta(t)S_t + r_F(t)V(t) + r_D(t)\Delta(t)S_t]dt \quad (22)$$

Note that the $C(t)$ is actually $r_{df}(t)C_f(t)$.

The difference to the single currency case that that the coefficients for $C(t)$ are replaced by the foreign counterpart as it is in foreign currency.

Then we can use (22) to get the PDE for $V(t)$ and apply Feynman Kac theorem to get the following solution:

$$V(t) = E_t^Q \left[e^{-\int_t^T r_F(u)du} V_T + \int_t^T e^{-\int_t^u r_F(v)dv} (r_F^f(u) - r_C^f(u)) C(u) du \right] \tag{23}$$

The measure Q is the same measure in Section 3.1. Again if we assume $C(t) = \alpha V(t)$, we get a formula similar to equation 21:

$$V(t) = E_t^Q \left[e^{-\int_t^T [\alpha(r_C^f(u) - r_F^f(u)) + r_F(u)]du} V_T \right] \tag{24}$$

For Case 3 we can get similar formulas like equation (23) and (24) but the $r_C^f(t)$ has to be replaced by $r_C(t)1_{V_t<0} + r_C^f(t)1_{V_t>0}$. But since the sign of V_t is unknown, these formulas are not of practical use.

3.3 Multi-asset Model for single currency

Let $\mathbf{S_t} = (S_t^{(1)}, \ldots, S_t^{(d)})^T$ be the d-dimensional vector of underlyings, where the ith underlying dynamic is

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = \mu_t^{(i)} dt + \sum_{j=1}^{d} \sigma_t^{(i,j)} dW_t^{(j)}, \quad i = 1, 2, \ldots, d$$

where the measure Q such that $W_t^{(i)}$ is Wiener process and the corresponding drift $\mu_t^{(i)}$ will be specified later for all $i = 1, \ldots, d$. Denote by V_t a derivative with underlying $\mathbf{S_t}$. Then by Ito's lemma,

$$dV_t = (\mathcal{L}V_t)dt + \sum_{i=1}^{d} \Delta_t^{(i)} dS_t^{(i)}.$$

where

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_t^{(i,j)} S_t^{(i)} S_t^{(j)} \frac{\partial^2}{\partial S_t^{(i)} \partial S_t^{(j)}}.$$

To replicate the derivative, at time t we hold $\Delta_t^{(i)}$ units of the i-th underlying and γ_t amount of cash. The value of the replication portfolio $\Pi(t)$ is then

$$V(t) = \Pi(t) = \sum_{i=1}^{d} \Delta_t^{(i)} S_t^{(i)} + \gamma_t$$

The cash amount γ_t is split among a number of accounts: [adds in]

The growth of all cash accounts is given by:

$$\begin{aligned} d\gamma_t = &[r_C(t)C(t) + r_F(t)(V(t) - C(t)) \\ &- \sum_{i=1}^{d} r_R(t)^{(i)} \Delta_t^{(i)} S_t^{(i)} + \sum_{i=1}^{d} r_D(t)^{(i)} \Delta_t^{(i)} S_t^{(i)} \Bigg] dt \end{aligned}$$

By the self-financing condition,

$$d\gamma_t = dV(t) - \sum_{i=1}^{d} \Delta_t^{(i)} dS_t^{(i)} = (\mathcal{L}V(t))dt$$

Thus we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \sum_{i=1}^{d} \left(r_R(t)^{(i)} - r_D(t)^{(i)} \right) S_t^{(i)} \frac{\partial}{\partial S_t^{(i)}} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_t^{(i,j)} S_t^{(i)} S_t^{(j)} \frac{\partial^2}{\partial S_t^{(i)} \partial S_t^{(j)}} \right) V(t) \\ = r_F(t)V(t) + (r_C(t) - r_F(t))C(t) \end{aligned}$$

The solution is obtained by Feynman-Kac formula as

$$V(t) = E_t^Q \left[e^{-\int_t^T r_F(u)du} V(T) + \int_t^T e^{-\int_t^u r_F(v)dv} (r_F(u) - r_C(u))C(u)du \right]$$

in the measure Q such that the ith underlying asset has a drift $\mu_t^{(i)} = r_R(t)^{(i)} - r_D(t)^{(i)}$ and furthermore, the ith underlying asset has a dynamic in Q as:

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = \left(r_R(t)^{(i)} - r_D(t)^{(i)} \right) dt + \sum_{j=1}^{d} \sigma_t^{(i,j)} dW_t^{(j)}, \quad i = 1, 2, \ldots, d.$$

Rewrite the above PDE in the form of

$$\begin{aligned} \left(\frac{\partial}{\partial t} + \sum_{i=1}^{d} \left(r_R(t)^{(i)} - r_D(t)^{(i)} \right) S_t^{(i)} \frac{\partial}{\partial S_t^{(i)}} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_t^{(i,j)} S_t^{(i)} S_t^{(j)} \frac{\partial^2}{\partial S_t^{(i)} \partial S_t^{(j)}} \right) V(t) \\ = r_C(t)C(t) + r_F(t)(V(t) - C(t)) \end{aligned}$$

and use the proof of Feynmac-Kac theorem, we can obtain another useful formula

$$\begin{aligned} V(t) = &E_t^Q \left[e^{-\int_t^T r_C(u)du} V(T) \right] \\ &- E^Q \left[\int_t^T e^{-\int_t^u r_C(v)dv} (r_F(u) - r_C(u))(V(u) - C(u))du \right] \end{aligned}$$

Let $C(t) = \alpha V(t)$, we obtain a practical formula:

$$V(t) = E_t^Q \left[e^{-\int_t^T [r_F(u)(1-\alpha)+r_C(u)\alpha]du} V(T) \right].$$

3.4 Multi-asset Model for cross currency

Let $\mathbf{S_t} = (S_t^{(1)}, \ldots, S_t^{(d)})^T$ be the d-dimensional vector of underlyings, where the ith underlying dynamic is

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = \mu_t^{(i)} dt + \sum_{j=1}^{d} \sigma_t^{(i,j)} dW_t^{(j)}, \quad i = 1, 2, \ldots, d$$

where the measure Q such that $W_t^{(i)}$ is Wiener process and the corresponding drift $\mu_t^{(i)}$ will be specified later for all $i = 1, \ldots, d$. Denote by V_t a derivative with underlying $\mathbf{S_t}$. Then by Ito's lemma,

$$dV_t = (\mathcal{L}V_t)dt + \sum_{i=1}^{d} \Delta_t^{(i)} dS_t^{(i)}.$$

where

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \sigma_t^{(i,j)} S_t^{(i)} S_t^{(j)} \frac{\partial^2}{\partial S_t^{(i)} \partial S_t^{(j)}}.$$

To replicate the derivative, at time t we hold $\Delta_t^{(i)}$ units of the i-th underlying and γ_t amount of cash. The value of the replication portfolio $\Pi(t)$ is then

$$V(t) = \Pi(t) = \sum_{i=1}^{d} \Delta_t^{(i)} S_t^{(i)} + \gamma_t$$

The cash amount γ_t is split among a number of accounts: [adds in]

The growth of all cash accounts is given by:

$$\begin{aligned} d\gamma_t = \Big[& r_C^f(t)C(t) + r_F(t)V(t) - r_F^f(t)C(t) \\ & - \sum_{i=1}^{d} (r_R(t))^{(i)} \Delta_t^{(i)} S_t^{(i)} + \sum_{i=1}^{d} (r_D(t))^{(i)} \Delta_t^{(i)} S_t^{(i)} \Big] dt \end{aligned}$$

By the self-financing condition,

$$d\gamma_t = dV(t) - \sum_{i=1}^{d} \Delta_t^{(i)} dS_t^{(i)} = (\mathcal{L}V(t))dt$$

Thus we have

$$\left(\frac{\partial}{\partial t} + \sum_{i=1}^{d} \left((r_R(t))^{(i)} - (r_D(t))^{(i)}\right) S_t^{(i)} \frac{\partial}{\partial S_t^{(i)}} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} \sigma_t^{(i,j)} S_t^{(i)} S_t^{(j)} \frac{\partial^2}{\partial S_t^{(i)} \partial S_t^{(j)}}\right) V(t)$$
$$= r_C^f(t)C(t) + r_F(t)V(t) - r_F^f(t)C(t)$$

The solution is obtained by Feynman-Kac formula as

$$V(t) = E_t^{Q_i}\left[e^{-\int_t^T r_F(u)du} + \int_t^T e^{-\int_t^u r_F(v)dv}(r_F^f(u) - r_C^f(t))C(u)du\right]$$

in the measure Q such that the ith underlying asset has a drift $\mu_t^{(i)} = r_R(t)^{(i)} - r_D(t)^{(i)}$ and furthermore, the ith underlying asset has a dynamic in Q as:

$$\frac{dS_t^{(i)}}{S_t^{(i)}} = \left(r_R(t)^{(i)} - r_D(t)^{(i)}\right) dt + \sum_{j=1}^{d} \sigma_t^{(i,j)} dW_t^{(j)}, \quad i = 1, 2, \ldots, d.$$

Let $C(t) = \alpha V(t)$, we obtain a practical formula:

$$V(t) = E_t^Q\left[e^{-\int_t^T [r_F(u) - (r_F^f(u) - r_C^f(u))\alpha]du} V(T)\right].$$

4 Yet another idea

Here we provide another approach to the final valuation model. To clarify the dynamic and eliminate the possibility of introducing any confliction, we use the portfolio replication framework and only one collateral dynamic modified from Fujii et al's paper. For convenience we will not consider the underlying dividend in this case.

$$dS(t) = r_t S(t)dt + \sigma S(t)dz \tag{25}$$

$$dV_t = \mathcal{L}^r V_t dt + \sigma S(t)\frac{\partial V_t}{\partial S(t)}dz \tag{26}$$

where

$$\mathcal{L}^{\dot{a}}V = \frac{\partial V}{\partial t} + \dot{a}S\frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V}{\partial S^2}$$

and on the other hand, for the replicated portfolio:

$$dC_t = \alpha dV_t + (r_t - c_t)C_t dt \tag{27}$$
$$dB_t = rB_t dt \tag{28}$$

set

$$V_t - C_t = \alpha S(t) + B_t \tag{29}$$

hence

$$dV_t - dC_t = \alpha dS(t) + dB_t \tag{30}$$

specifically,

$$RHS = \alpha r_t S(t)dt + \alpha\sigma S(t)dz + r_t B_t dt$$

$$\begin{aligned} LHS &= (1-\alpha)dV_t - (r_t - c_t)C_t dt \\ &= (1-\alpha)\mathcal{L}^r V_t dt + (1-\alpha)\sigma S(t)\frac{\partial V}{\partial S}(t)dz - (r_t - c_t)C_t dt \end{aligned}$$

To eliminate dz, set

$$\alpha = (1-\alpha)\frac{\partial V}{\partial S}(t) \tag{31}$$
$$B_t = V_t - C_t - \alpha S(t) \tag{32}$$

the the PDE turns out to be, after simplification

$$\mathcal{L}^r V + \frac{c_t}{1-\alpha}C_t - \frac{r_t}{1-\alpha}V_t = 0$$

By Feynman-Kac formula,

$$V_t^C = E\left[\int_t^T e^{-\int_t^s \frac{r_u}{1-\alpha}du}\frac{c_s}{1-\alpha}C_s ds + e^{-\int_t^T \frac{r_u}{1-\alpha}du}V_T\right] \tag{33}$$

If we define the discounted return:

- $\tilde{r} = \frac{r}{1-\alpha}$

- $\tilde{c} = \frac{x}{1-\alpha}$

then the above result coincide Paper one result

$$V_t^C = E\left[\int_t^T e^{-\int_t^s \tilde{r_u} du} \tilde{c_s} C_s ds + e^{-\int_t^T \tilde{r_u} du} V_T\right] \tag{34}$$

On the other hand, we can achieve the expression of collateral from its dynamic.

By moving $(r-c)Cdt$ together with dC, we can achieve

$$dC_s - (r_s - c_s)C_s ds = \alpha dV_s$$

We are going to multiply the discount factor with above equation, then we will have

$$de^{\int_0^s (r_\tau - c_\tau) d\tau} C_s = e^{-\int_0^s (r_\tau - c_\tau) d\tau} \alpha dV_s$$

Then integrate both sides, firstly from t to T.

$$e^{-\int_0^T (r_\tau - c_\tau) d\tau} C_T - e^{-\int_0^t (r_\tau - c_\tau) d\tau} C_t = \int_t^T e^{-\int_0^s (r_\tau - c_\tau) d\tau} \alpha dV_s$$

$$C_t = e^{-\int_t^T (r_\tau - c_\tau) d\tau} C_T + \int_t^T e^{-\int_t^s (r_\tau - c_\tau) d\tau} \alpha dV_s$$

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