We extend hybridizable discontinuous Galerkin (HDG) methods [3, 5, 6, 7, 8] to CFD applications. The HDG methods inherit the geometric flexibility and high-order accuracy of discontinuous Galerkin methods, and offer a significant reduction in the computational cost. In order to capture shocks, we employ an artificial viscosity model based on an extension of existing artificial viscosity methods [4, 9, 2, 1, 10]. In order to integrate the Spalart-Allmaras turbulence model [11] using high-order methods, some modification of the model is necessary. Mesh adaptation based on shock indicator is used to improve shock profiles. Several test cases are presented to illustrate the proposed approach.

Shock Capturing

The equations of compressible fluid dynamics with an artificial viscosity (AV) term in the physical domain $\Omega \subset \mathbb{R}^3$ are written in nondimensional conservation form as

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{F}(\mathbf{u}) + \epsilon \mathbf{G}(\mathbf{u})) = 0,$$

where $\mathbf{u}$ is the $m$-dimensional vector of conserved dimensionless quantities, $\mathbf{F}(\mathbf{u})$ and $\mathbf{G}(\mathbf{u})$ are the artificial fluxes of dimension $m \times d$ added to the original equations for the purpose of capturing shocks. Here $\epsilon$ is the artificial viscosity and $\mathbf{G}(\mathbf{u})$ is the same as $\epsilon$ except that the energy is replaced with the enthalpy $[1]$. Following the previous work [2, 10], we use the dilatation as a shock indicator. In particular, we define the artificial viscosity as

$$\epsilon = \alpha f(\sqrt{\mathbf{G}(\mathbf{u})}),$$

where $\alpha$ is a user-specified constant, $r$ is a characteristic length scale, $v$ is the velocity field, $c_\infty$ is the sound speed, and $f$ is an analytic function. Not wanting to add viscosity at the wall, we specify $f$ as

$$f = \min(0, \epsilon_0),$nh(\delta h)_0,$$

where $\delta_h$ is a representative size of the finite elements and $\epsilon_0$ is the distance from the closest wall. To complete our artificial viscosity model we define $\psi$ as

$$\psi(\alpha) = \psi_\epsilon(1 + \psi_\epsilon(\psi_\chi - 1)),$$

for $\alpha = 0, 1$ and $\beta = -0.5$. However, for hypersonic flows where the sound speed is very stiff and close to zero at the shock, we replace $\psi$ in (2) with

$$g(\psi) = \psi_\epsilon(1 + 0.5 \psi_\epsilon(1 + \psi_\epsilon(2\psi_\chi - 1)))^{1/2},$$

where $c_\infty = \sqrt{\gamma p / \rho_\infty}$ is the sound speed at far field.

Turbulence Modeling

The Spalart-Allmaras (SA) turbulence model [11] is used for RANS solution of high-Reynolds turbulent flow. A drawback of this model is that the eddy viscosity can be negative at the edge of the boundary layer if mesh resolution is not sufficient. This situation is even more severe when high-order methods are used to integrate the SA equation. We thus propose some modifications of the SA model: For the terms involving algebraic expressions of $\gamma$ and $\chi = \psi / \psi_0$ we replace $\psi_0$ with $\psi_0 = 0.05 \log(1 + \psi_\epsilon(2\psi_\chi - 1))$ where $\psi_\epsilon$ and $\psi_\chi$ are always positive even when $\psi$ is negative.

It is important to point out that our modification does not aim to alleviate the issue of negative eddy viscosity in the SA model. The eddy viscosity may still be negative if the boundary layer is under-resolved. However, the modification is necessary for high-order methods since it renders the resulting system easier to integrate in the presence of negative eddy viscosity.

Numerical Applications

Several examples for subsonic, transonic, supersonic, and hypersonic flows are presented below to demonstrate the performance of the proposed approach. In all test cases, polynomials of degree $k = 1$ are used to represent the approximate solution.

Inviscid Flows

Figure 1: Pressure and AV for transonic flow past a NACA 0012 foil at $M_\infty = 0.8$ and $\alpha = 1.5^\circ$.

Figure 2: Mach and AV for supersonic flow past NACA 0012 foil at $M_\infty = 2.0$ and $\alpha = 1.5^\circ$.

Figure 3: Pressure and AV for inviscid supersonic flow past a circular bump at $M_\infty = 1.4$.

Figure 4: Pressure and AV for inviscid hypersonic flow past a circular cylinder at $M_\infty = 7$.

Figure 5: Mach number and pressure coefficient distribution over airfoil surface for turbulent subsonic flow past NACA 0012 foil at $M_\infty = 0.1$, $\alpha = 0^\circ$, and $Re = 1.8 \times 10^6$.

Figure 6: Pressure and pressure coefficient distribution over airfoil surface for turbulent transonic flow past RAE 2822 foil at $M_\infty = 0.729$, $\alpha = 3^\circ$, and $Re = 6.5 \times 10^6$.

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