Multiscale Methods for Complex Systems

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Approximate Continuation of Harmonic Functions – Gabriela Jager

Geodesy: (upward) continuation of the potential field on bounded domains

Find $u : \Omega \rightarrow \mathbb{R}$, $\Delta u = 0$ from $(x_i, z_i)_{i=1}^{N}$ on $\Gamma \subset \Omega$ (ill-posed)

Local representation: $u_\lambda(x) = \sum_{\nu=1}^{M} \alpha_{\nu,\lambda} \psi_{\nu}(x), \lambda \in A$

Weighted least-squares regularization with: $J_\lambda(u_\lambda) := \sum_{\nu=1}^{M} \left( u_\lambda(x) - z_i \right)^2 + \eta \int_{\Omega} \left| \Delta u_\lambda(x) \right|^2 dx \rightarrow \min$

Solve $M + \eta G = b$, $A = (A_{\lambda\nu})_{\lambda=1}^{\nu=1}, A_{\lambda\nu} = \psi_\nu(x), M = A^T A$

$G = (G_{\lambda\nu})_{\lambda=1}^{\nu=1}$, $G_{\lambda\nu} = \int_{\Omega} \Delta \psi_\nu(x) \Delta \psi_\lambda(x) dx, b = A^T z$

Estimators for $\eta$: $\eta_{\text{cond}} = \left\{ \eta \text{ cond}(M + \eta G) = \min \right\}, \eta_{\text{opt}} = \frac{\eta_{\text{cond}}}{\min(\eta \text{ cond}(M + \eta G))}$

Adapted development: iterative, coarse-to-fine strategy based on hierarchical tensor products of B-splines/Wavelets and complete or sparse basis refinements

Solution of (2) with Monotone Multigrid (MMG)

Goal: Computation of fair price $V$ of American option with stochastic volatility and higher order derivatives (‘greeks’) of $V$

Solves free boundary value problem

\[
\begin{align*}
LV(\mathcal{V} - \mathcal{H}) = 0, \\
EV \leq 0, \mathcal{V} - \mathcal{H} \geq 0
\end{align*}
\]

in $G \subset \mathbb{R}^N \times (0, \infty)^T$ with differential operator $L$, obstacle (‘payoff’) function $H$ and boundary conditions.

Problem: no analytical solution to (1)

Finite difference/element discretization in time/space

Discrete linear complementary problem

Find $u \in \mathbb{R}^N$ satisfying

\[
\begin{align*}
u^T (Lu - f) = 0, \\
\mathcal{H} u - f \geq 0, u \geq 0, \end{align*}
\]

with $G \in \mathbb{R}^N \times \mathbb{R}^N$ and $f \in \mathbb{R}^N$

Option Prizing with B-Spline-based MMG – Katharina Wiechers

Approximate Absorption Spectra using Wavelets – Christian Mollet

Quantum wire:

- Higher potential in the neighborhood of the wire → electron can only move in $1-D$.
- Roughness at the interface → disorder potential.

Schrödinger eigenvalue equation:

\[ H \psi(x) = E \psi(x), \quad H = H_0 + H_{\text{def}} + H_{\text{dis}}, \]

with $H_0 = \frac{\hbar^2}{2m} \frac{d^2}{dx^2}, H_{\text{def}} = \sum_{k=1}^{N} R_{\text{def}}(x_k) \delta(x-x_k)$,

\[ R_{\text{def}}(x) = \frac{\hbar^2}{2m} \left( \frac{d^2}{dx^2} \right)^{1/2}, \quad \sigma(x) = \text{conductivity} \]

Idea: Weak formulation on fictitious domain

Potential distribution in a homogeneous fictitious body when current ($\mathbf{J}$) is applied to electrodes $l = 3, 4, j = 6, \zeta = 3$

Alternative: saddle point formulation with

\[ b(v, V, q) := \int_{\Omega} \nabla \cdot \mathbf{q} \implies q \text{ Lagrange multiplier} \]

Forward Model in EIT:

find potential distribution $u$ and potentials on electrodes $U_1, U_2, U_3, U_4$

\[ \nabla \cdot \sigma \nabla u = 0 \text{ in } \Omega \subset \mathbb{R}^3 \]

\[ u = \sigma \nabla u \cdot n - U_i \text{ on } \Gamma_i \]

\[ f_{\mathbf{r}} = \sigma \nabla u \cdot n = 0 \text{ on } \Gamma \]

boundary ($\delta \Omega$ = $\Gamma$)

Idea: Weak formulation on fictitious domain

Fictitious Domain Formulation for EIT – Panagiotis Kantartzis

Parabolic Wavelet Problem:

\[ u(t) + a(t, u, \cdot) = g(t), \quad u(0) = u_0 \in L_2(\Omega) \]

with $V := H_0^2(\Omega)$, $t \in I = [0, T]$

\[ \rightarrow \text{ new approach (Schwab, Stevenson '08)} \]

\[ \text{adaptive wavelet method based on weak}\]

\[ \text{space-time formulation} \]

\[ \text{place DOFs simultaneously in time and space}\]

\[ \text{wavelet preconditioning enables solvers of}\]

\[ \text{optimal complexity}\]

\[ \rightarrow \text{ own homogenization reformulation} : \]

\[ X = L_2(\mathbf{V}) \rightarrow I \subseteq H^1(\mathbf{V}), \quad Y = L_2(\mathbf{V}), \quad \mathbf{V} = L_2(\mathbf{V}). \]

Find $u^* \in \mathcal{X} : b(u^*, v^*) = (f, v), \quad \forall v \in \mathcal{Y}, f(x) = f_0^T (g_x(t), v(t)) dt, b(u, v) = f_1 (u(t), v(t)) + a(t, u(t), v(t)) dt$

Idea: For implementation investigate and use ingredients

\[ \text{methods based on tree-like index sets} \]

\[ \text{concept from nonlinear problems}\]

\[ \text{B-spline wavelets of order } d = 2, d = 4, \quad \text{i.e. primal scaling system consists of hat functions}\]

\[ \text{implementation based on AWM-Toolbox}\]

Adaptive Wavelets for Parabolic Evolution Problems – Florian Stapel

Adaptive Wavelet Methods – Roland Pabel

Nonlinear PDE:

\[ -\Delta g^\tau + g = f \quad \text{in } \Omega = (0, 1)^d \]

\[ \eta = g \quad \text{on } \partial \Omega \]

\[ \rightarrow \text{ uniform grids unsuitable for efficient}\]

\[ \rightarrow \text{ adaptive refinement of domain}\]

\[ \text{distributes DOFs only where needed, e.g. at singularities}\]

Solution of (2) with Monotone Multigrid (MMG)

Idea: For implementing wavelet decomposition and reconstruction

\[ \text{choose } \mathcal{X} = \text{Kronecker product of } L_2(\mathbf{V}) \text{ and } L_2(\mathbf{V}) \]

Idea: For implementing wavelet decomposition and reconstruction