DPG in Brief

- Discontinuous: solution values can differ at element boundaries, with fluxes that correspond to the difference.
- Petrov-Galerkin: Test and trial spaces may differ. Our test space is computed on the fly.
- Formulate PDE as a first-order system (for stability).  
- Integrate by parts to move derivatives to test functions, with flux terms as additional unknowns.
- Optimal test functions are computed on the fly by solving the local problem
  \[
  \begin{align*}
  \text{For basis element } u \in U, \text{ find } \tilde{v} \in V \text{ such that:} \\
  (T_v u, v)_{L^2} - (u, v)_H = \text{rhs}.
  \end{align*}
  \]

- Key result: always delivers the best approximation error in the energy norm
  \[
  |||v|||_E = \text{mp} \cdot \text{inf} |||v|||_V.
  \]
- The resulting stiffness matrix is symmetric positive definite:  
  \[
  \left( \tilde{T}_v u, \tilde{v} \right) = (T_v u, \tilde{v})_{L^2} = (T_v u, T_v \tilde{v}) = (T_v u, \text{rhs}_v).
  \]
- Optimal test functions are not generally polynomials. In practice, we approximate the optimal test functions using an enriched polynomial space (for today’s results, \( p = 3 \)).
- Key choice: the inner product on \( V \) determines the energy norm, and thus the sense in which the method is optimal.

Stokes Formulation

We applied the DPG method to the Stokes problem in two dimensions. The strong form of the problem is

\[
\begin{align*}
\frac{1}{2\mu} \left( \sigma_{11} + p \right) - \nabla \psi_1 + \left( \begin{array}{c}
\nu \nabla \cdot \psi_1 \\
\nu \nabla 
\end{array} \right) &= 0 \quad \text{in } \Omega, \\
\frac{1}{2\lambda} \left( \sigma_{12} \right) - \nabla \psi_2 + \left( \begin{array}{c}
\nu \nabla \cdot \psi_2 \\
\nu \nabla 
\end{array} \right) &= 0 \quad \text{in } \Omega, \\
- \nabla \cdot \psi_1 &= f_1 \quad \text{in } \Omega, \\
- \nabla \cdot \psi_2 &= f_2 \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega, \\
u &= \text{g}_D \quad \text{on } \partial \Omega.
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^2 \), \( \nu \) is viscosity, \( \psi \) stress, \( \nu \) vorticity, \( p \) pressure, \( u \) velocity, and \( f \) a vector forcing function.

Multiplying the first two equations by vector test functions \( q_1 \) and the following three by scalar test functions \( v_1, \) and integrating by parts over an element \( \mathcal{K} \), we obtain

\[
\begin{align*}
\int_{\mathcal{K}} \left( \frac{1}{2\mu} \left( \sigma_{11} + p \right) + q_1 \nu \right) \psi_{1} : \psi_1 + \int_{\mathcal{K}} \left( \psi_{1} : \frac{1}{2\lambda} \sigma_{12} \right) v_1 &= 0 \quad \text{in } \mathbb{R}^2, \\
\int_{\mathcal{K}} \left( \frac{1}{2\lambda} \left( \sigma_{12} \right) + q_2 \nu \right) \psi_2 : \psi_2 &= 0 \quad \text{in } \mathbb{R}^2, \\
\int_{\mathcal{K}} -\nabla \cdot \psi_1 \cdot q_1 &= 0 \quad \text{in } \mathbb{R}^2, \\
\int_{\mathcal{K}} -\nabla \cdot \psi_2 \cdot q_2 &= 0 \quad \text{in } \mathbb{R}^2, \\
\int_{\mathcal{K}} -\nabla \cdot u \cdot q_1 &= 0 \quad \text{in } \mathbb{R}^2, \\
\int_{\partial \Omega} \nu \cdot q_1 &= 0 \quad \text{on } \partial \mathcal{K},
\end{align*}
\]

where the “hatted” variables (\( \hat{\psi} \), e.g.) are the fluxes introduced by relaxing the continuity requirement at element boundaries.

Manufactured Solution Choice

Following Cockburn et al. in their LDG work on Stokes [CKS03], we use

\[
u_1 = -\text{c}(\psi_1 \nu + \sin y) \]
\[
u_2 = -\text{c}(\psi_2 \nu + \sin y)
\]
\[
u = 2\text{c} \sin y
\]

as our manufactured solution, and solve on domain \((-1, 1) \times (-1, 1)) using a uniform quadrilateral mesh. This gives us a baseline for comparison, but it should be noted that Cockburn et al. use a triangular mesh. Their \( n \times m \) mesh could be obtained from ours by cutting our elements along a diagonal.

References


[CKS03] Bernardo Cockburn, Guoguang Kanschat, Dominik Schötzau, and Christoph Schwab.


[DG10b] L. Demkowicz, J. Gopalakrishnan, and A. Nobile.


Mathematician’s Norm

A simple choice for the test space norm: \( q_i \in L^2(\text{div}) \) and \( v_i \in L^2(\text{grad}) \), so use

\[
\begin{align*}
\left\| (q_1, q_2, v_1, v_2) \right\|_E &= \sqrt{\int_{\Omega} \left( \nabla \cdot q_1 + q_1 \cdot v_1 + \nabla \cdot v_1 + v_1 \cdot v_1 \right) + \int_{\Omega} \left( \nabla v_1 \cdot \nabla v_1 + v_1 \cdot v_1 \right)},
\end{align*}
\]

as the norm on the test space.

Optimal Test Norm

- We know from the analysis that DPG delivers the best solution in the energy norm, and that the choice of norm on the test space determines the energy norm.
- \( \mathcal{Q} \): Can we select the test space norm in such a way that the energy norm is exactly the norm of interest \( \left( L^2 \right) \), in our case?  
- \( \mathcal{A} \): Yes, this is given in the abstract setting by

\[
\left\| \psi \right\|_E = \text{mp} \cdot \left\| u_{\text{meta}} \right\|_V,
\]

where \( \left\| u \right\|_V \) is the norm of interest, \( \left\| u \right\|_E \) is then called the optimal test norm.

The true optimal test norm is not localizable, so we approximate the optimal test norm for our Stokes formulation by the quasi-optimal test norm, given by

\[
\begin{align*}
\left( \int_{\Omega} \frac{1}{2\mu} \left( \sigma_{11} + p \right) \psi_1 : \psi_1 + \int_{\Omega} \frac{1}{2\lambda} \left( \sigma_{12} \right) \psi_2 : \psi_2 + \int_{\Omega} -\nabla \cdot \psi_1 \cdot q_1 + \int_{\Omega} -\nabla \cdot \psi_2 \cdot q_2 \right)^{1/2}
\end{align*}
\]

where \( \beta \) is a parameter, which we set as \( \beta = 10^{-1} \).

Quasi-Optimal Norm, Results

- DPG with quasi-optimal test norm \( \beta = 10^{-1} \) versus LDG convergence. \( p = 3, p_{\text{max}} = 5, p_{\text{min}} = 5 \).

Mathematician’s Norm, Results

- Error in Stochastic Solution with Cubic Elements: DPG vs. LDG (mathematician’s norm)

Other Authors

Joint work with
Denis Redal, Pavel B. Bochev, Kara J. Peterson, Christopher M. Siefert at Sandia, and Lecszek D. Demkowicz at UT Austin.