Algorithm Class ARODE
Adaptive stochastic Galerkin Runge-Kutta methods for Random Ordinary Differential Equations
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Governing Equations and Stochastic Galerkin Method:

\[ \frac{\partial}{\partial t} U(t, \xi) = f(U(t, \xi), \xi) \]
with \( \xi \in L_2(0, T, P) \).

Finite Dimensional Approximation Space

\[ S = \text{span} \{ \Phi_n : n \leq \theta \} \]

Multi-Element Approach

- Partition of \( \Omega \) and restriction of stochastic weak form to elements of the partition.

Convergence Properties:

Local Error

\[ \| \frac{\partial}{\partial t} U(t, \xi) - \sigma_{\tilde{U}} (U(t, \xi)) \|_{L^2} \leq O(\Delta^k h^p) \]

Global Error

\[ \| U(T, \xi) - \sigma (U(T, \xi)) \|_{L^2} \leq O(\Delta^{k+1}) + O(h^p) \]

- RK4 of order \( p \) and stepsize \( h \)
- \( k \) is the order of the approximation within \( S \)
- \( \sigma \) denotes the projection \( L_2 \rightarrow S \)
- \( U \) denotes the numerical approximation of \( U \)
- \( \tilde{U} \) is the order of the approximation within \( S \)

Numerical Results:

Random Van-Deer-Pol Oscillator

<table>
<thead>
<tr>
<th>Parameter Setting for ARODE</th>
<th>Numerical Results: (without/with (left/middle) refinement of ( \Omega ) and number of elements (right))</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial^2}{\partial t^2} U(t, \xi) = \lambda (1 - U(t, \xi)) \frac{\partial}{\partial t} U(t, \xi) - U(t, \xi) )</td>
<td>( \lambda = 1 ), ( \Delta t = 1 ), ( \xi = \text{uniform} )</td>
</tr>
<tr>
<td>Order of RK4</td>
<td>( p = 3 )</td>
</tr>
<tr>
<td>Degree of approximation</td>
<td>( \theta = 3 )</td>
</tr>
<tr>
<td>Inferences</td>
<td>( \lambda = 10^{-1}, \xi = 10^{-1} )</td>
</tr>
<tr>
<td>Inferences (elements)</td>
<td>( \lambda = 10^{-2}, \xi = 10^{-2} )</td>
</tr>
<tr>
<td>Refinement threshold</td>
<td>( \mu_1 = 1 )</td>
</tr>
<tr>
<td>Refinement threshold</td>
<td>( \mu_2 = 6 )</td>
</tr>
<tr>
<td>( 2 ) ( 4 ) ( 8 ) ( 16 ) ( 32 ) ( 64 ) ( 128 ) ( 256 ) ( 512 ) ( 1024 )</td>
<td>( \xi = \text{uniform} )</td>
</tr>
</tbody>
</table>

Statistics

- Rejected coarsenings (\( \rho_2 = 6 \)): 31
- Rejected coarsenings (\( \rho_2 = 10 \)): 0
- Rejected timestep 184
- Moments computed by MCS (square markers)
  - \( \times 10^3 \) samples
  - \( \times 10^{-5} \) error below plotting accuracy (\( 10^{-7} \) to \( 10^{-3} \))

Literature: