Elastic Building Blocks in a Wrinkle Cascade

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Curtain Problem

Consider elastic sheet subject to stretching 
(U_{\text{stretch}} \sim Et \equiv Y) + bending (U_{\text{bend}} \sim Yt^2) energies

- Sheet subject to uniform uniaxial confinement in \( \hat{y} \) direction
- Compressive stress relieved by buckles with wavelength \( \lambda_b \)
- At \( x = 0 \), impose shape with \( \lambda_e < \lambda_b \)

Transitioning between wavelengths requires \( \hat{x} \) curvature
\( \Rightarrow \) Gaussian curvature \( \Rightarrow \) In-plane strain

System gets to choose how much strain to accommodate
Wrinkle Cascade

Thin polystyrene sheet floating on water
⇒ Surface tension favors small amplitude at edge
Huang et al, PRL 105, 038302 (2010)

Transition to optimal wavelength happens through cascade

What is basic shape of the unit cell? Is it smooth or sharp?

Belgacem et al, J. Nonlinear Sci. 10, 661 (2000);
Jin and Sternberg, J. Math. Phys. 42, 192 (2001);
Das et al, PRL 98, 014301 (2007); Pomeau,
Outline

- Curtain Problem
- Types of Building Blocks
- Smooth & Smooth Together
- Phase Space of Wrinkling
We wish to characterize the **building blocks** that make up the shapes of thin elastic sheets.

**Sharp** — Stress focused to corners and ridges.
Witten, RMP **79**, 643 (2007)

**Smooth** — Stress does not focus.
Cerda and Mahadevan, PRL **91**, 074302 (2003)
Sharp Building Blocks

Features where curvatures diverge as $t \to 0$
⇒ Stress is **focused** into vanishingly small areas

Shape reflects **geometric** principle: “Mostly developable” configuration to avoid Gaussian curvature

**Working Definition:** Let $A_S$ be the area of the sheet with significant elastic energy density. A feature is sharp if

$$\frac{A_S}{A_{Tot}} \to 0 \text{ as } t \to 0$$

Examples:
- *d*-cones: $A_S \sim t^{2/3}$
- minimal ridge: $A_S \sim t^{1/3}$
Smooth Building Blocks

Both the curvature and stress are **diffuse** throughout the sheet.

A **mechanical** property reigns: Compressive stresses vanish with $t$ (relaxed energy / membrane limit).

*Working definition:* A feature is smooth if:

$$\frac{A_S}{A_{Tot}} \to 0 \text{ as } t \to 0$$

Examples:
- Mahadevan-Cerda wrinkles from tension
- Lamé geometry — annulus under tension
Simplest Curtain Problem

Model a **single** generation in the cascade

1. Confine an elastic sheet
   \[ \Rightarrow \text{Euler buckle} \]

2. Force one edge into **3-buckle** shape

3. Make sheet long enough to achieve single buckle

Simulate with the Surface Evolver

- Program by Brakke to minimize energies over a mesh
- Built-in elastic and bending energies
- Use conjugate gradient and Hessian searches for local minima

http://www.susqu.edu/brakke/evolver/evolver.html
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Two Prominent Features

Plotting Gaussian curvature
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Terminates in crescent with large Gaussian curvature
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Long, apparently smooth transition region
Grows as thickness decreases, confinement increases

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Two Prominent Features

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- Grows as thickness decreases, confinement increases
- Terminates in crescent with large Gaussian curvature
- Plotting Gaussian curvature
Transition Region Scales with $t$

**Compliance** given by

$$ \epsilon \equiv \frac{t^2}{W^2} \Delta $$

Centerlines collapse under scaling $x \to x \epsilon^{1/4}$

Recalls scaling argument:

**Bending** $\sim$ **Stretching**

$$ B_K^2 \sim Y t^2 \left( \frac{\sqrt{\Delta}}{W} \right)^2 \sim Y u_{xx}^2 \sim Y \left( \sqrt{\Delta} \frac{W}{L_t} \right)^4 $$

$$ \Rightarrow L_t \sim \frac{W}{\epsilon^{1/4}} $$

Mahadevan, Vaziri, and Das, EPL 77, 40003 (2007)

Recent experiments: Vandeparre *et al*, arXiv:1012.4325
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Diffuse Stress in Transition

Argument suggests stretching energy is not focused

\[ \bar{x} = x \epsilon^{1/4} / W \]

This \textit{diffuse stress} region does not shrink with \( t \)

Suggests scaling solution in \( \bar{x} \)
Collapse of Compressive Stress

Airy (stress) potential has same scaling solution:

\[ \chi = \epsilon^{1/2} \Delta Y W^2 g(\bar{x}, y/W) \]

\[ \sigma_{xx} \sim Y \Delta \epsilon^{1/2} \]

\[ \sigma_{yy} \sim Y \Delta \epsilon \]

\[ \Rightarrow \frac{\sigma_{yy}}{\sigma_{xx}} \sim \epsilon^{1/2} \xrightarrow{t\to0} 0 \]

Suggests \textbf{mechanical} principle for diffuse-stress areas: compressive stress vanishes relative to tensile stress

Stein and Hedgepeth, NASA TN D-813 (1961)
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Plotting Gaussian curvature
There is a concentration of Gaussian curvature near $x^*$.

Associated length scales vanish with $\epsilon$.

Energy negligible in thin limit:

$$U_{\text{foc}} \sim Yt^{5/3} \ll U_{\text{dif}} \sim Yt^{3/2}$$
Focused Stress

There is a concentration of Gaussian curvature near $x^*$.

Associated length scales vanish with $\epsilon$.

Energy negligible in thin limit:

$$U_{foc} \sim Yt^{5/3} \ll U_{dif} \sim Yt^{3/2}$$
Focused and diffuse stress zones can coexist, despite differing constraints.

Open questions:

- How to match geometric and mechanical constraints at junction of diffuse and focused stress zones?
- What is role of focused structure in minimizing elastic energy?
- How does focused structure compare to \( d \)-cone?
Boundary Conditions Matter

Results up to now: Imposed shape must be planar

Rotation of plane has little impact

Releasing planarity constraint

⇒ Removes diffuse-stress region

Only focused structure; $L_t \sim W$
Phase Diagram for Curtains?

Buckling Transition

Sharp features

Thickness provides cut-off length

Thickness (\(\varepsilon\))
Phase Diagram for Curtains?

- Tension \( (T/\sigma) \)
- Thickness \( (\varepsilon) \)

- Buckling Transition
- Sharp features
- Thickness provides cut-off length
Apply tension $T \hat{x}$ (⊥ to confinement)

For $T < \text{compressive stress } \sigma$, no effect

For $T > \sigma$, focused structure "melts"

Shape well-described by two Fourier modes

Davidovitch, PRE 80, 025202 (2009).
Phase Diagram for Curtains?

- **Buckling Transition**
  - Tension irons out shape
  - Thickness provides cut-off length

- Sharp features

**Variables:**
- **Thickness** ($\varepsilon$)
- **Tension** ($T/\sigma$)
Phase Diagram for Curtains?

- Tension irons out shape
- Sharp features
- Thickness provides cut-off length

Buckling Transition

Thickness (\(\varepsilon\))

Tension (\(T/\sigma\))
Conclusions

- Smooth and sharp bulding blocks reflect different principles
  - Smooth features reflect mechanical property: compressive stress vanishes
  - Sharp features reflect geometric property: focus Gaussian curvature
- Model curtain shows coexistence of diffuse stress (smooth) and focused stress (sharp) regions
- Thickness and tension control degree of focusing
⇒ Beginning of phase diagram for wrinkling
Directions

Where else will we see coexistence of smooth and sharp features?

What is influence of boundary conditions?

How analogous is focused structure to $d$-cone?

Can we get insight into core size from “melting” under tension?

See PRL 106, 074301 (2011)

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Focused Scaling

\[ U \sim E t^3 \]

\[ U \sim E t^{8/3} \]

\[ U \sim E t^{5/2} \]
Boundary Conditions

a) $z$

b) $W$

c) $\Delta W$

d) $z$

2011 IMA Workshop – p. 23
Elastic Energies

Surface Evolver models sheet as triangulated 2D plane

Two edges of each triangle define a local “metric”

Strain measured as difference between current and target metrics

Stretching energy: integrated quadratics of strain

Bending energy calculated per vertex
- Mean curvature from gradient of volume
- Gaussian curvature from excess angle