Shape Selection in Hyperbolic Non-Euclidean Plates

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1. Differential growth has been replicated in the laboratory using gels that locally shrink or expand upon activation.
   ▪ Y. Klein, E. Efrati, and E. Sharon, Science 315, 1116.

2. This growth has been modeled using linear elasticity with a Riemannian manifold \((D, g)\) as the reference configuration.

3. The equilibrium configuration is modeled as a minimum of a Föppl - von Kármán type functional.

4. The full model can be simplified using the small slopes approximation.
The stress free configuration is modeled as a two dimensional Riemannian manifold \( D \) with a specified "target metric" \( g \).

The \textbf{equilibrium configuration} \( x(u, v) : W^{2,2}(D) \rightarrow \mathbb{R}^3 \) is a minimum of the following elastic energy functional:

\[
E[x] = \int_{D} \left\| \nabla x^T \cdot \nabla x - g \right\|^2 dA + h^2 \int_{D} \left( 4 \overline{H}^2 - \overline{K} \right) dA.
\]

where \( \overline{H} \) and \( \overline{K} \) are the mean and Gaussian curvature of \( x \) and \( h \) is the thickness of the sheet.

In this talk we will focus on the disk of radius \( R \) when \( g \) generates constant negative Gaussian curvature \( K \).

In polar coordinates \((\rho, \theta)\):

\[
g = d\rho^2 + \frac{1}{-K} \sinh^2(\sqrt{-K}\rho) d\theta^2.
\]
Crocheting the Hyperbolic Plane

D. Henderson, D. Tamina, Mathematical Intelligencer, 2002
Gel Disks

In the small slopes approximation we assume that $\epsilon = \sqrt{-KR} \ll 1$ and use the dimensionless variables

$$Rr = \rho, \; Ru = x \text{ and } Rv = y.$$ 

The target metric takes the form

$$g = R^2 dr^2 + R^2 \left( r^2 + \epsilon^2 \frac{r^4}{3} \right) d\theta^2$$

$$= R^2 \left( 1 + \frac{v^2}{3} \epsilon^2 \right) du^2 - R^2 \frac{2uv}{3} \epsilon^2 dudv + R^2 \left( 1 + \frac{u^2}{3} \epsilon^2 \right) dv^2.$$ 

To match the target metric to lowest order we assume that

$$x(u, v) = R \left( u + \epsilon^2 \chi(u, v), v + \epsilon^2 \xi(u, v), \epsilon \eta(u, v) \right).$$

A necessary condition that the surface is an isometric immersion is that

$$[\eta, \eta] = \eta_{xx} \eta_{yy} - \eta^2_{xy} = -1.$$
Assuming the dimensionless thickness $t = \frac{h}{R} \ll 1$ satisfies $\epsilon^2 \ll t \ll \epsilon$ then in terms of the small parameter $\tau = \frac{t}{\epsilon}$ we have that

$$E[x] = \int_D \left( (\gamma_{11} + \gamma_{22})^2 + \gamma_{11}^2 + 2\gamma_{12}^2 + \gamma_{22}^2 \right) dudv$$

$$+ \tau^2 \int_D \left( (\Delta \eta)^2 - [\eta, \eta] \right) dudv.$$  

The elastic energy of an isometric immersion is simply

$$E[x] = \tau^2 \int_D \left( (\Delta \eta)^2 + 1 \right) dudv.$$  

The minimizer of the energy over the class of immersions is

$$\chi = -\frac{xy^2}{3} \quad \text{and} \quad \xi = -\frac{x^2y}{3} \quad \text{and} \quad \eta = xy.$$  

This gives us the upper bound

$$\inf_{x \in W^{2,2}(D)} E[x] \leq \tau^2 \pi.$$
Convergence to Saddle

Scaled Energy vs Thickness

\[ \frac{E}{l^2} \]

\[ t \]

\[ 0.05 \quad 0.1 \quad 0.15 \quad 0.2 \quad 0.25 \quad 0.3 \quad 0.35 \quad 0.4 \quad 0.45 \quad 0.5 \]
A one parameter of isometric immersions are of the form

$$\eta_a = \frac{1}{2} \left( ax^2 - \frac{1}{a} y^2 \right).$$

By letting $a = \tan(\pi/2n)$ we can construct $n$-wave isometric immersions through odd periodic extensions.

This gives us the upper bound

$$E[x] \leq \pi \tau^2 \left( 4 \cot^2(\pi/n) + 1 \right).$$
Energy of Periodic Configurations

Elastic Energy of n Wave Configurations

- $n=2$
- $n=2$ Upper Bound
- $n=3$
- $n=3$ Upper Bound
- $n=4$
- $n=4$ Upper Bound
- $n=5$
- $n=5$ Upper Bound
**Definition**

\( \mathcal{A}_n \) is the set of configurations \( x = Id + \epsilon(0, 0, \eta) + \epsilon^2(\chi, \xi, 0) \) such that \( \eta \) is periodic in \( \theta \) with period \( \frac{2\pi}{n} \).

**Lemma**

For a fixed \( \tau > 0 \) and \( E_0 > 0 \) if there exists \( x \in \mathcal{A}_n \) such that \( E[x] \leq \tau^2 E_0 \) then there exists constants \( C_1, C_2 > 0 \) independent of \( \eta \) and \( n \) such that

\[
E_s[x] \geq C_1 - C_2 \frac{E_0^2}{n^2}.
\]

**Theorem**

There exists constant \( C_1, C_2 > 0 \) such that

\[
C_1 n \tau^2 \leq \inf_{y \in \mathcal{A}_n} E[y] \leq C_2 \tau^2 \left( 4 \cot^2\left(\frac{\pi}{n}\right) + 1 \right).
\]
Ideas of Proof:

- Sobolev/Poincare estimates gives us that

\[ \int_D \eta_x^4 \, dx \, dy \leq C \frac{E_0^2}{n^2} \]

- For \( a, b \in \mathbb{R} \) the following inequality is true \((a + b)^2 \geq \frac{a^2}{2} - 2b^2\). Therefore,

\[ \int_D \left( 2\chi_x + \eta_x^2 - \frac{y^2}{3} \right)^2 \, dudv \geq \frac{1}{2} \int_D \left( 2\chi_x - \frac{y^2}{3} \right)^2 \, dudv - C \frac{E_0^2}{n^2} \]

\[ \geq C_1 - C_2 \frac{E_0^2}{n^2}. \]

- Letting \( E_0 = \frac{E[x]}{tau^2} \) we get the inequality

\[ E_b \geq \sqrt{\frac{n^2 \tau^4}{C_2}} (C_1 - E_s) - E_s. \]

Minimizing \( E_b + E_s \) over this constraint gives us the lower bound.
A Chebychev Net is a configuration $x(u, v)$ with metric

$$
g = du^2 + \cos(\phi(u, v))dudv + dv^2. $$

The principal curvatures are given by

$$k_1^2 = \tan^2(\phi/2) \text{ and } k_2^2 = \cot^2(\phi/2).$$

$\phi$ satisfies the sine-Gordon equation

$$\frac{\partial^2 \phi}{\partial u \partial v} = -K \sin(\phi).$$

The bending energy can be put into the equivalent form

$$\mathcal{B}[\phi] \sim \int_D \left( \tan^2(\phi/2) + \cot^2(\phi/2) \right) dA.$$
Amsler Surfaces

A surface of constant negative Gaussian curvature $K$ and two straight asymptotic curves is called an Amsler Surface.

The similarity solution $\varphi(\lambda) = \phi(u, v)$, $\lambda = 2\sqrt{uv}$ satisfies the following Painleve III equation in trigonometric form:

$$\left\{ \begin{array}{l}
\varphi''(\lambda) + \frac{1}{\lambda} \varphi'(\lambda) - \sin(\varphi(\lambda)) = 0 \\
\varphi(0) = \frac{\pi}{n} \text{ and } \frac{d\varphi}{d\lambda}(0) = 0.
\end{array} \right.$$ 

A key property of these surfaces is that the surface has two asymptotic lines that intersect at the origin at an angle $\varphi(0) = \frac{\pi}{n}$. 
PERIODIC AMSLER SURFACES

Solutions to Painleve Equation

\[ \pi \]

Maximum Radius vs Number of Waves

Maximum Radius

\[ \sqrt{-K\lambda} \]
Periodic Amsler Surfaces
Elastic Energy of Amsler Surfaces
Elastic Energy of Amsler Surfaces

Scaled Bending Energy vs. Radius for Periodic Amsler Surfaces
1. We have found the lowest energy isometric immersion in the small slopes approximation.

2. We can construct low energy periodic isometric immersions.

3. Boundary layers are important when “adding thickness back” but do not change the qualitative shape of the sheet.

4. We can show an ansatz free lower bound for the growth of energy by adding periodic rippling.

5. Extensions of the periodic isometric immersions in the small slopes theory exist.

6. When considering the full geometric problem the radius of the disk is important. This is a consequence of geometry.

7. **Speculation:** Dynamic effects may select periodic shapes as local minimum.