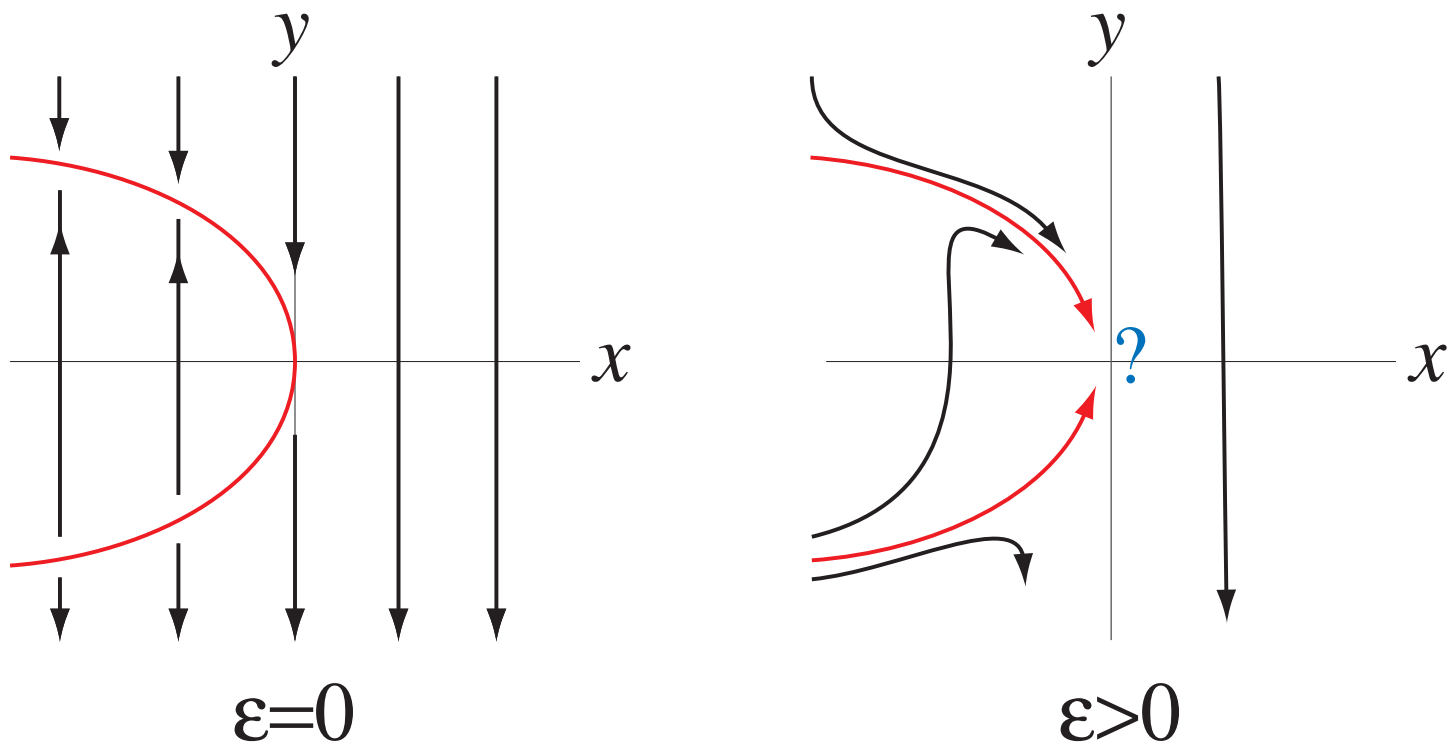


Loss of Normal Hyperbolicity



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“Happy families are all alike; every unhappy family is unhappy in its own way.”

Leo Tolstoy, *Anna Karenina*

Anna Karenina Principle: “By that sentence, Tolstoy meant that, in order to be happy, a marriage must succeed in many different respects: sexual attraction, agreement about money, child discipline, religion, in-laws, and other vital issues. Failure in any one of those essential respects can doom a marriage even if it has all the other ingredients needed for happiness.

“This principle can be extended to understanding much else about life besides marriage.”

Jared Diamond, *Guns, Germs, and Steel*

Principle of Fragility of Good Things: “Good things (e.g. stability) are more fragile than bad things. It seems that in good situations a number of requirements must hold simultaneously, while to call a situation bad even one failure suffices.”

Vladimir Arnold, *Catastrophe Theory*

Plan

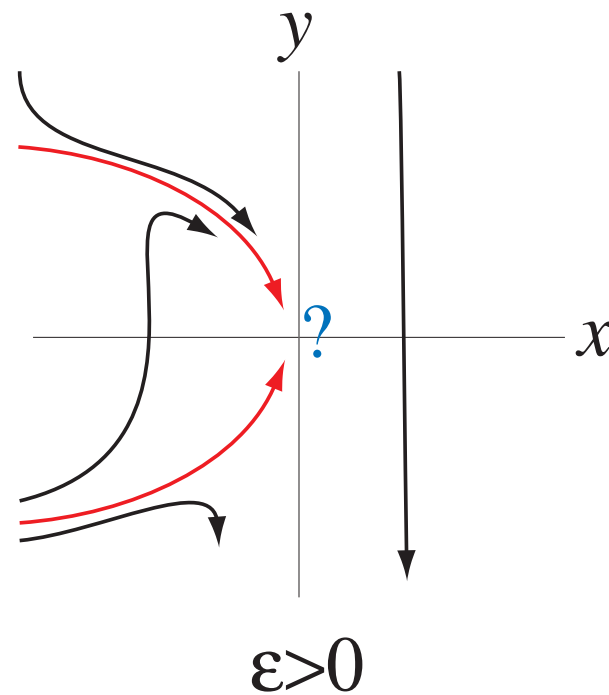
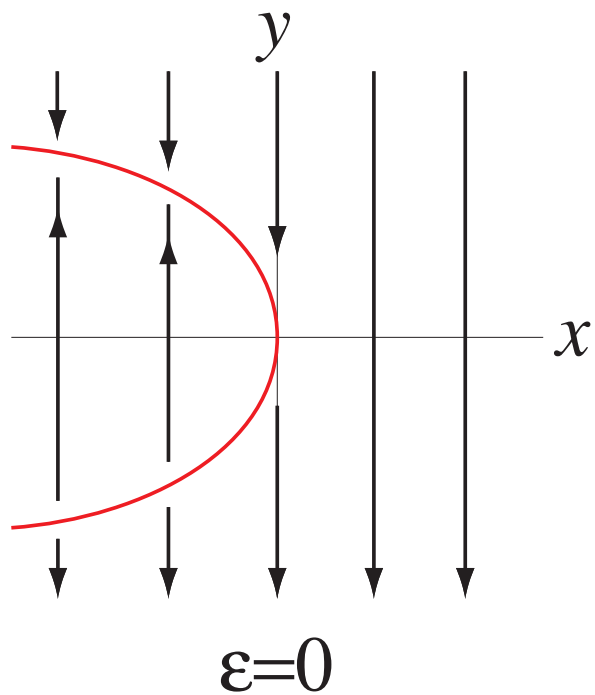
- (1) Saddle-node bifurcation in the fast equation
- (2) Rarefactions in the Dafermos regularization of a system of conservation laws
- (3) Crystalline interphase boundaries

I. Saddle-Node Bifurcation in the Fast Equation

M. Krupa and P. Szmolyan, 2001, expanding on ideas of F. Dumortier and R. Roussarie.

System:

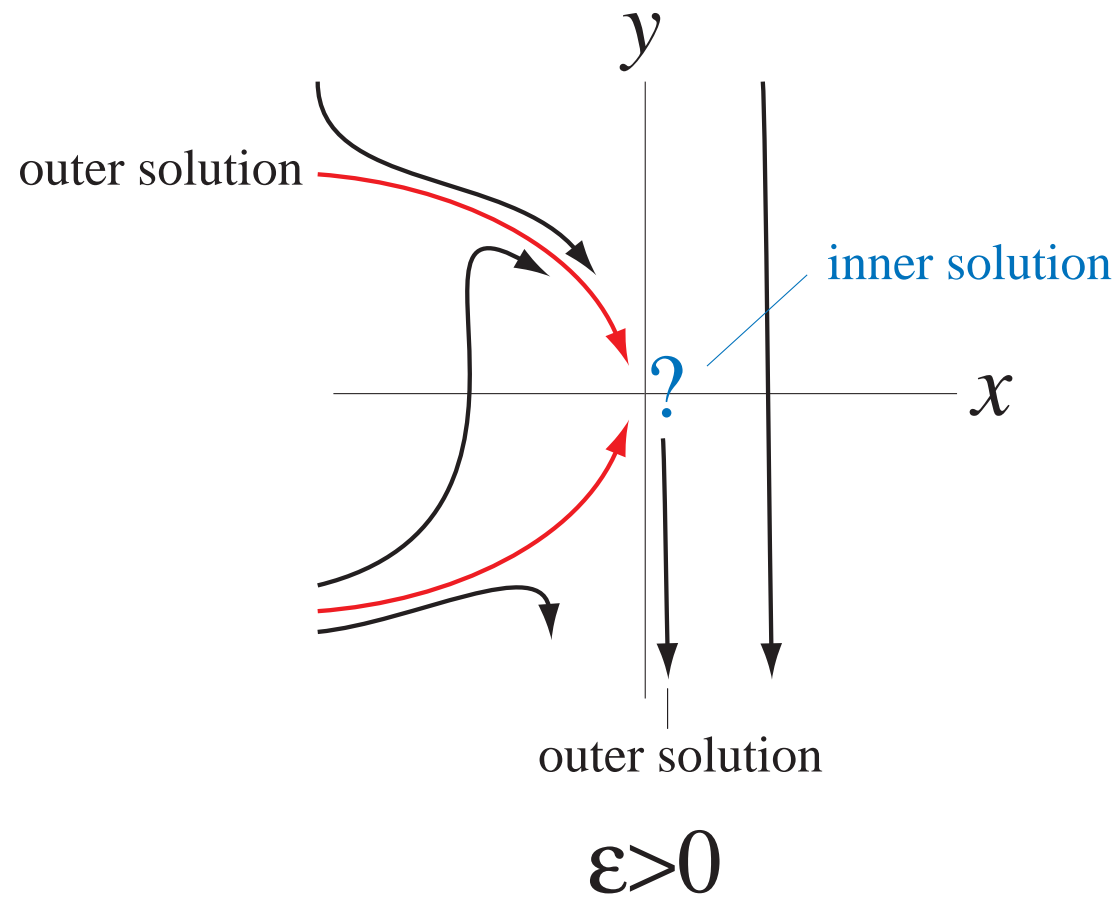
$$\begin{aligned}\dot{x} &= \varepsilon(1 + \dots) \\ \dot{y} &= -x - y^2 + \dots \quad (x^2, xy \text{ terms allowed})\end{aligned}$$



Question: For $\varepsilon > 0$, where does the normally attracting invariant curve go?

Classical analysis

$$\begin{aligned}\dot{x} &= \varepsilon(1 + \dots) \\ \dot{y} &= -x - y^2 + \dots \quad (x^2, xy \text{ terms allowed})\end{aligned}$$



To get inner solution, let $x = \varepsilon^{\frac{2}{3}}a$, $y = \varepsilon^{\frac{1}{3}}b$.

System:

$$\begin{aligned}\dot{x} &= \varepsilon(1 + \dots) \\ \dot{y} &= -x - y^2 + \dots \quad (x^2, xy \text{ terms allowed})\end{aligned}$$

Let $x = \varepsilon^{\frac{2}{3}}a$, $y = \varepsilon^{\frac{1}{3}}b$.

$$\begin{aligned}\varepsilon^{\frac{2}{3}}\dot{a} &= \varepsilon \left(1 + O(\varepsilon^{\frac{1}{3}})\right) \\ \varepsilon^{\frac{1}{3}}\dot{b} &= -\varepsilon^{\frac{2}{3}}a - \varepsilon^{\frac{2}{3}}b^2 + O(\varepsilon)\end{aligned}$$

Simplify.

$$\begin{aligned}\dot{a} &= \varepsilon^{\frac{1}{3}} \left(1 + O(\varepsilon^{\frac{1}{3}})\right) \\ \dot{b} &= -\varepsilon^{\frac{1}{3}}a - \varepsilon^{\frac{1}{3}}b^2 + O(\varepsilon)\end{aligned}$$

Rescale time (divide by $\varepsilon^{\frac{1}{3}}$).

$$\begin{aligned}a' &= 1 + O(\varepsilon^{\frac{1}{3}}) \\ b' &= -a - b^2 + O(\varepsilon^{\frac{2}{3}})\end{aligned}$$

Note three terms have order ε^0 . Set $\varepsilon = 0$.

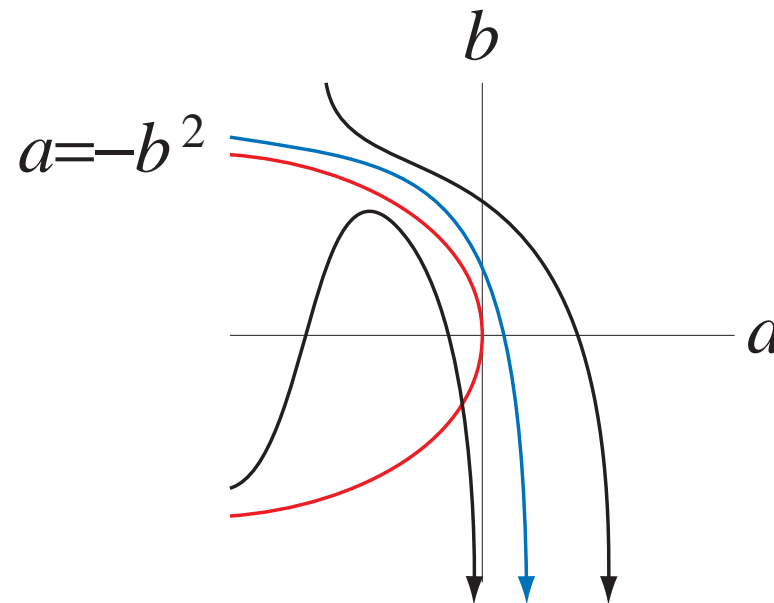
$$\frac{db}{da} = -a - b^2$$

$$\frac{db}{da} = -a - b^2$$

Convert this Riccati equation to a linear equation by the substitution $b = \frac{1}{c} \frac{dc}{da}$.

$$\frac{d^2c}{da^2} + ac = 0.$$

There is an explicit solution in terms of Airy functions. Convert back to get b in terms of a .



Use the blue one and try to match to the outer solutions using asymptotic expansions.

Solution is asymptotic to $a = k$ ($-k =$ first zero of Airy function), i.e., $x = k\epsilon^{\frac{2}{3}}$.

Blow-Up: a Geometric Approach to Matching

Extend the original system:

$$\begin{aligned}\dot{x} &= \varepsilon(1 + \dots) \\ \dot{y} &= -x - y^2 + \dots \\ \dot{\varepsilon} &= 0\end{aligned}$$

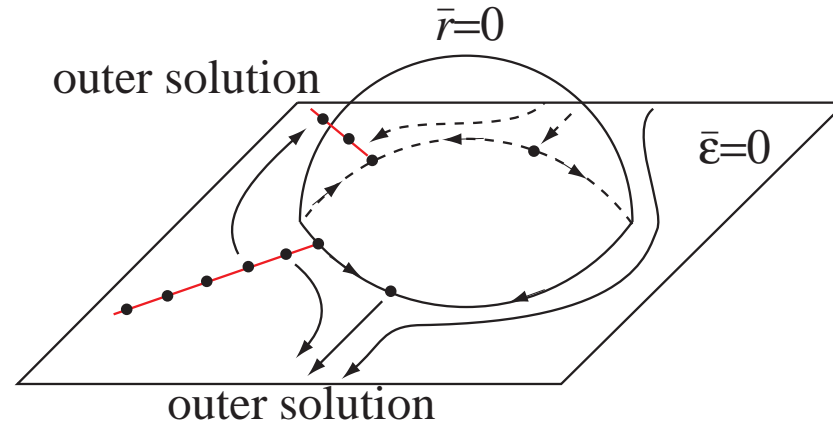
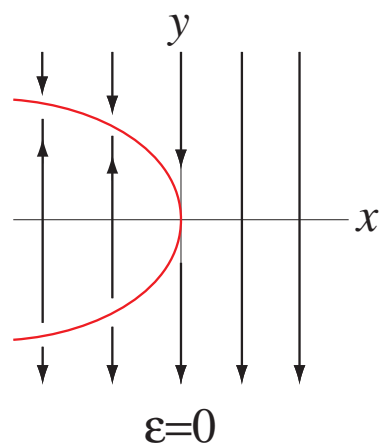
The blow-up transformation for this problem is a map from $S^2 \times [0, \infty)$ (blow-up space) to $xy\varepsilon$ -space. Let $((\bar{x}, \bar{y}, \bar{\varepsilon}), \bar{r})$ be a point of $S^2 \times [0, \infty)$, so $\bar{x}^2 + \bar{y}^2 + \bar{\varepsilon}^2 = 1$. Then

$$x = \bar{r}^2 \bar{x}, \quad y = \bar{r} \bar{y}, \quad \varepsilon = \bar{r}^3 \bar{\varepsilon}.$$

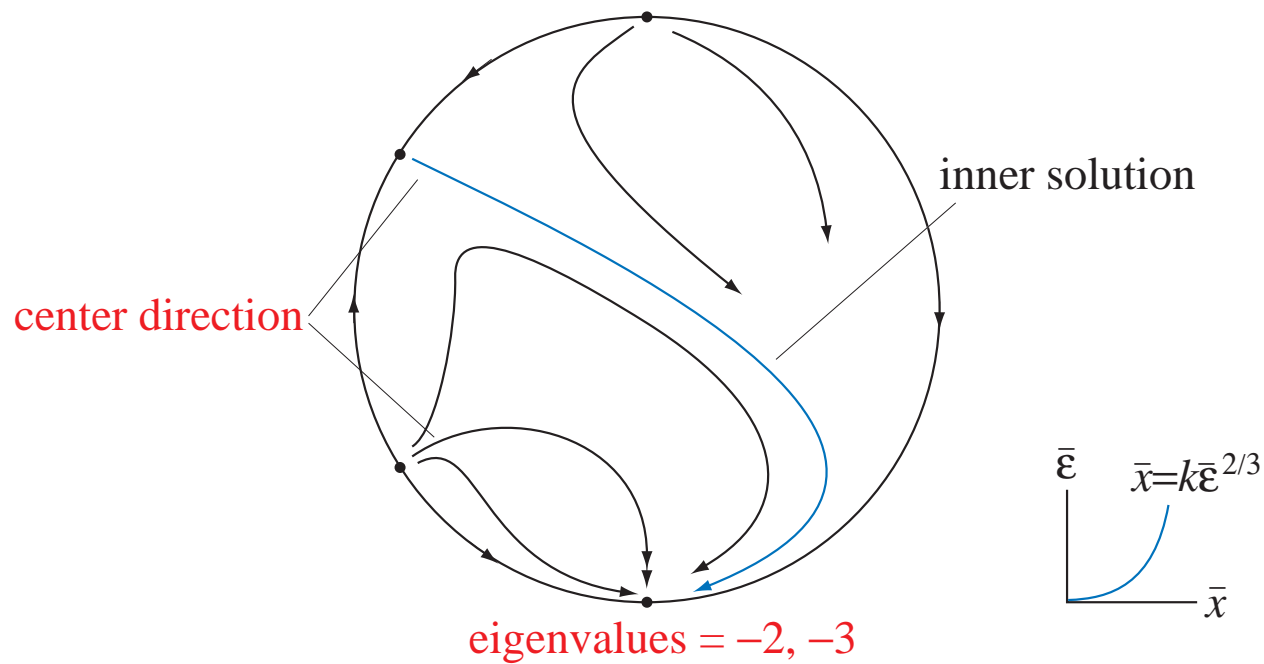
The origin has been “blown up” to a sphere (“quasi-homogeneous” spherical coordinates).

Under this transformation the system pulls back to a vector field X on $S^2 \times [0, \infty)$ for which the sphere $\bar{r} = 0$ consists entirely of equilibria. The vector field we shall study is $\tilde{X} = \bar{r}^{-1}X$. Division by \bar{r} desingularizes the vector field on the sphere $\bar{r} = 0$ but leaves it invariant. It is equivalent to rescaling time.

In blow-up space there is no loss of normal hyperbolicity.



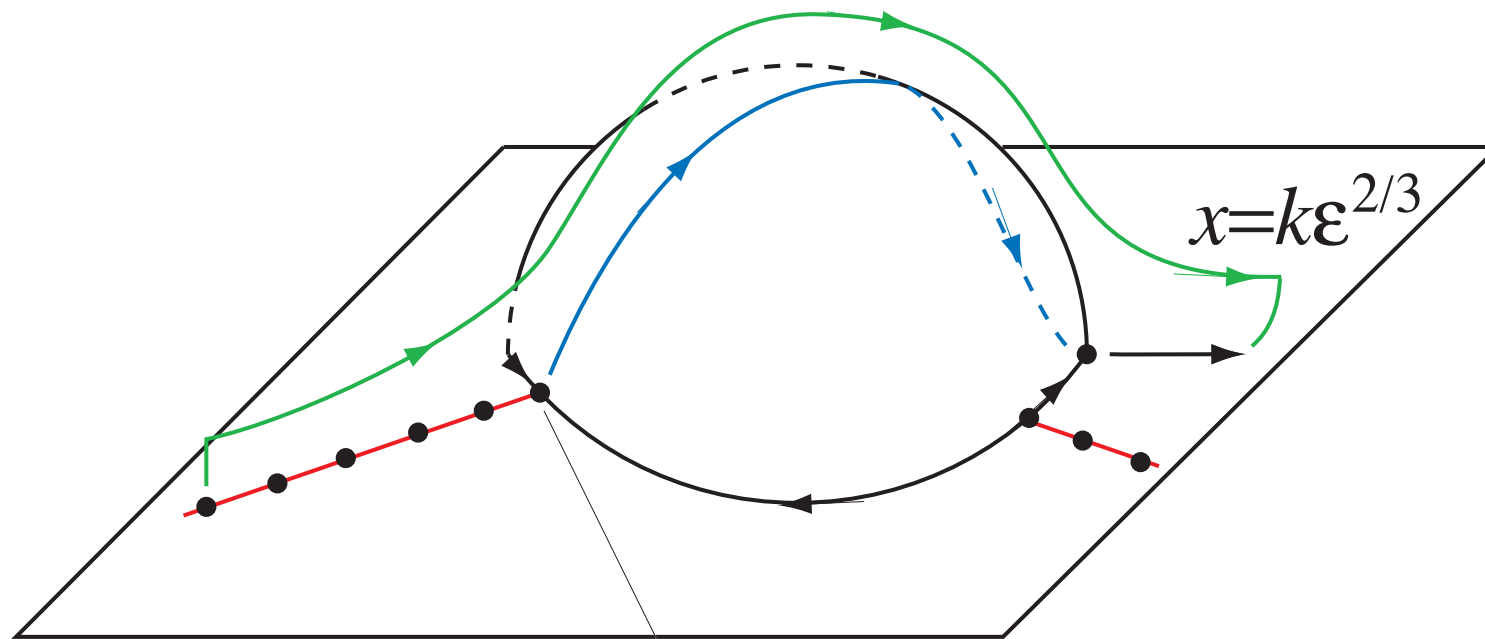
Top view of sphere:



Coordinate system on top of the sphere gives inner solution.

Coordinate systems on side and front of sphere allow geometric matching to outer solutions.

Flow past sphere:



Center manifold of this point is shown. It is the extension of the normally hyperbolic invariant manifold of outer solutions in $xy\epsilon$ -space.

Calculations

$$\begin{aligned}\dot{x} &= \varepsilon(1 + \dots) \\ \dot{y} &= -x - y^2 + \dots \\ \dot{\varepsilon} &= 0\end{aligned}$$

$$x = \bar{r}^2 \bar{x}, \quad y = \bar{r} \bar{y}, \quad \varepsilon = \bar{r}^3 \bar{\varepsilon}, \quad ((\bar{x}, \bar{y}, \bar{\varepsilon}), \bar{r}) \in S^2 \times [0, \infty)$$

Chart for $\bar{\varepsilon} > 0$

$$x = r^2 a, \quad y = r b, \quad \varepsilon = r^3, \quad a \in \mathbb{R}, b \in \mathbb{R}, r \geq 0 \quad (r = \bar{\varepsilon}^{\frac{1}{3}} \bar{r}, a = \bar{\varepsilon}^{-\frac{2}{3}} \bar{x}, b = \bar{\varepsilon}^{-\frac{1}{3}} \bar{y})$$

$$\begin{aligned}r^2 \dot{a} &= r^3(1 + O(r)) \\ r \dot{b} &= -r^2 a - r^2 b^2 + O(r^3) \\ \dot{r} &= 0\end{aligned}$$

Simplify and rescale time (divide by r).

$$\begin{aligned}a' &= 1 + O(r) \\ b' &= -a - b^2 + O(r^2) \\ r' &= 0\end{aligned}$$

We have seen this before: “rescaling chart.”

Side and front charts are used for geometric matching.

Chart for $\bar{x} < 0$

$$\begin{aligned}\dot{x} &= \varepsilon(1 + \dots) \\ \dot{y} &= -x - y^2 + \dots \\ \dot{\varepsilon} &= 0\end{aligned}$$

$$x = -r^2, \quad y = rb, \quad \varepsilon = r^3c, \quad b \in \mathbb{R}, \quad c \in \mathbb{R}, \quad r \geq 0.$$

$$\begin{aligned}-2r\dot{r} &= r^3c(1 + O(r)) \\ \dot{r}b + r\dot{b} &= r^2 - r^2b^2 + O(r^3) \\ 3r^2\dot{r}c + r^3\dot{c} &= 0\end{aligned}$$

Solve for \dot{r} , \dot{b} , \dot{c} .

$$\begin{aligned}\dot{r} &= -\frac{1}{2}r^2c(1 + O(r)) \\ \dot{b} &= r - rb^2 + \frac{1}{2}rbc + O(r^2) \\ \dot{c} &= \frac{3}{2}rc^2(1 + O(r))\end{aligned}$$

Rescale time (divide by r).

$$r' = -\frac{1}{2}rc + O(r^2)$$

$$b' = 1 - b^2 + \frac{1}{2}bc + O(r)$$

$$c' = \frac{3}{2}c^2(1 + O(r))$$

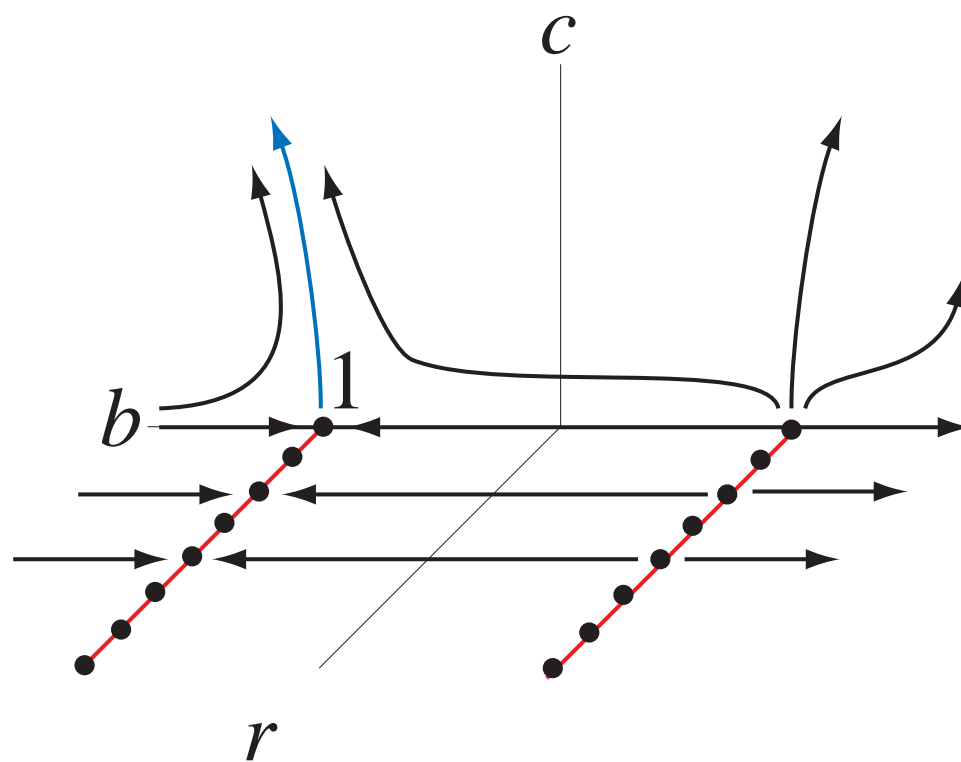


Chart for $\bar{y} < 0$

$$\begin{aligned}\dot{x} &= \varepsilon(1 + \dots) \\ \dot{y} &= -x - y^2 + \dots \\ \dot{\varepsilon} &= 0\end{aligned}$$

$$x = r^2 a, \quad y = -r, \quad \varepsilon = r^3 c, \quad a \in \mathbb{R}, \quad c \in \mathbb{R}, \quad r \geq 0.$$

$$\begin{aligned}2r\dot{r}a + r^2\dot{a} &= r^3c(1 + O(r)) \\ -\dot{r} &= -r^2a - r^2 + O(r^3) \\ 3r^2\dot{r}c + r^3\dot{c} &= 0\end{aligned}$$

Solve for \dot{a} , \dot{r} , \dot{c} .

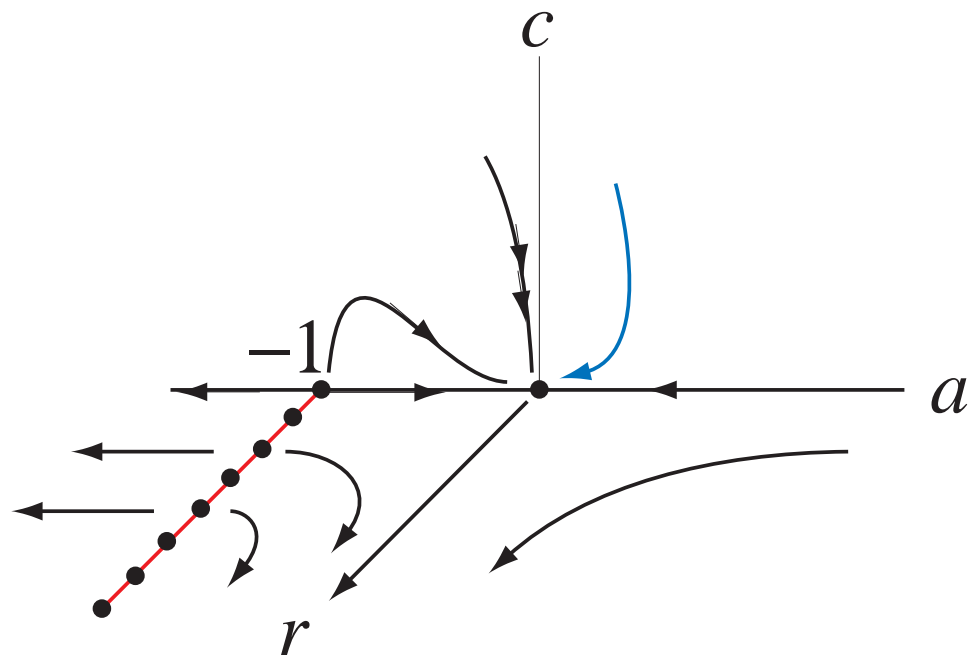
$$\begin{aligned}\dot{a} &= rc(1 + O(r)) - 2ra(a + 1 + O(r)) \\ \dot{r} &= r^2(a + 1 + O(r)) \\ \dot{c} &= -3cr(a + 1 + O(r))\end{aligned}$$

Rescale time (divide by r).

$$a' = c(1 + O(r)) - 2a(a + 1 + O(r))$$

$$r' = r(a + 1 + O(r))$$

$$c' = -3c(a + 1 + O(r))$$



$$a = kc^{\frac{2}{3}} \text{ so } x = r^2 a = r^2 kc^{\frac{2}{3}} = k(r^3 c)^{\frac{2}{3}} = k\varepsilon^{\frac{2}{3}}$$

To deal with loss of normal hyperbolicity in a manifold of equilibria for $\varepsilon = 0$:

- (1) Identify manifolds of possible outer solutions.
- (2) Extend the system by making ε into a variable.
- (3) Decide on blow-up coordinates. Expect ε to be among the variables that are blown up.
- (4) Use one chart to identify inner solution.
- (5) Use other charts to match.

II. Gain-of-Stability Turning Points (Rarefactions in the Dafermos Regularization)

Consider the system

$$\begin{aligned}\dot{u} &= v, \\ \dot{v} &= (A(u) - xI)v, \\ \dot{x} &= \varepsilon,\end{aligned}$$

with $(u, v, x) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and $A(u)$ an $n \times n$ matrix.

Let $n = k + l + 1$. Assume that on an open set U in \mathbb{R}^n :

- There are numbers $\lambda_1 < \lambda_2$ such that $A(u)$ has
 - k eigenvalues with real part less than λ_1 ,
 - l eigenvalues with real part greater than λ_2 ,
 - a simple real eigenvalue $\lambda(u)$ with $\lambda_1 < \lambda(u) < \lambda_2$.
- $A(u)$ has an eigenvector $r(u)$ for the eigenvalue $\lambda(u)$, and $D\lambda(u)r(u) > 0$.

Notice ux -space is invariant for every ε . For $\varepsilon = 0$ it consists of equilibria, but loses normal hyperbolicity along the surface $x = \lambda(u)$. (Not in standard form for a slow-fast system.)

Goal: For $\varepsilon > 0$, find a solution that connects $u = u^-$ to $u = u^+$ as x passes $\lambda(u)$.

Simplify by restricting to a normally hyperbolic invariant manifold.

$$\begin{aligned}\dot{u} &= v, \\ \dot{v} &= (A(u) - xI)v, \\ \dot{x} &= \varepsilon,\end{aligned}$$

Near $x = \lambda(u)$, there is a normally hyperbolic invariant manifold with coordinates (u, z_1, x, ε) with z_1 a coordinate along $r(u)$ in v -space.

For $\varepsilon = 0$, within the normally hyperbolic invariant manifold, the equilibria $z_1 = 0$ still lose normal hyperbolicity when $x = \lambda(u)$.

We therefore make the change of variables $x = \lambda(u) + \sigma$ and blow up the set $z_1 = \sigma = \varepsilon = 0$:

$$\begin{aligned}u &= u, \\ z_1 &= \bar{r}^2 \bar{z}_1, \\ \sigma &= \bar{r} \bar{\sigma}, \\ \varepsilon &= \bar{r}^2 \bar{\varepsilon},\end{aligned}$$

with $\bar{z}_1^2 + \bar{\sigma}^2 + \bar{\varepsilon}^2 = 1$ (quasi-homogeneous “spherical cylindrical” coordinates).

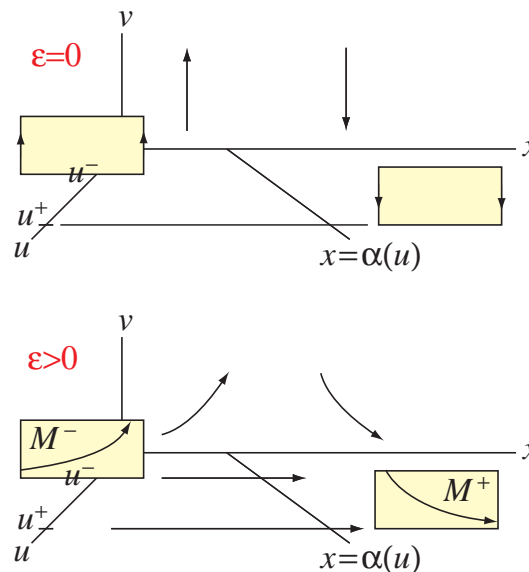
For the new system, the spherical cylinder $\bar{r} = 0$ consists entirely of equilibria. Divide by \bar{r} to desingularize.

System with $n = 1$, so $u \in \mathbb{R}$ and $v \in \mathbb{R}$:

$$\begin{aligned}\dot{u} &= v, \\ \dot{v} &= (\alpha(u) - x)v, \quad \alpha' > 0 \\ \dot{x} &= \varepsilon.\end{aligned}$$

ux -space is invariant for every ε . For $\varepsilon = 0$ it consists of equilibria, but loses normal hyperbolicity along the curve $x = \alpha(u)$.

Look for a solution with $u(-\infty) = u^-$, $u(\infty) = u^+$, $u^- < u^+$.



Second picture shows possible outer solutions.

Preliminary change of coordinates: $x = \alpha(u) + \sigma$. Also, extend system.

$$\begin{aligned}\dot{u} &= v, \\ \dot{v} &= -\sigma v, \\ \dot{\sigma} &= \varepsilon - \alpha'(u)v, \\ \dot{\varepsilon} &= 0.\end{aligned}$$

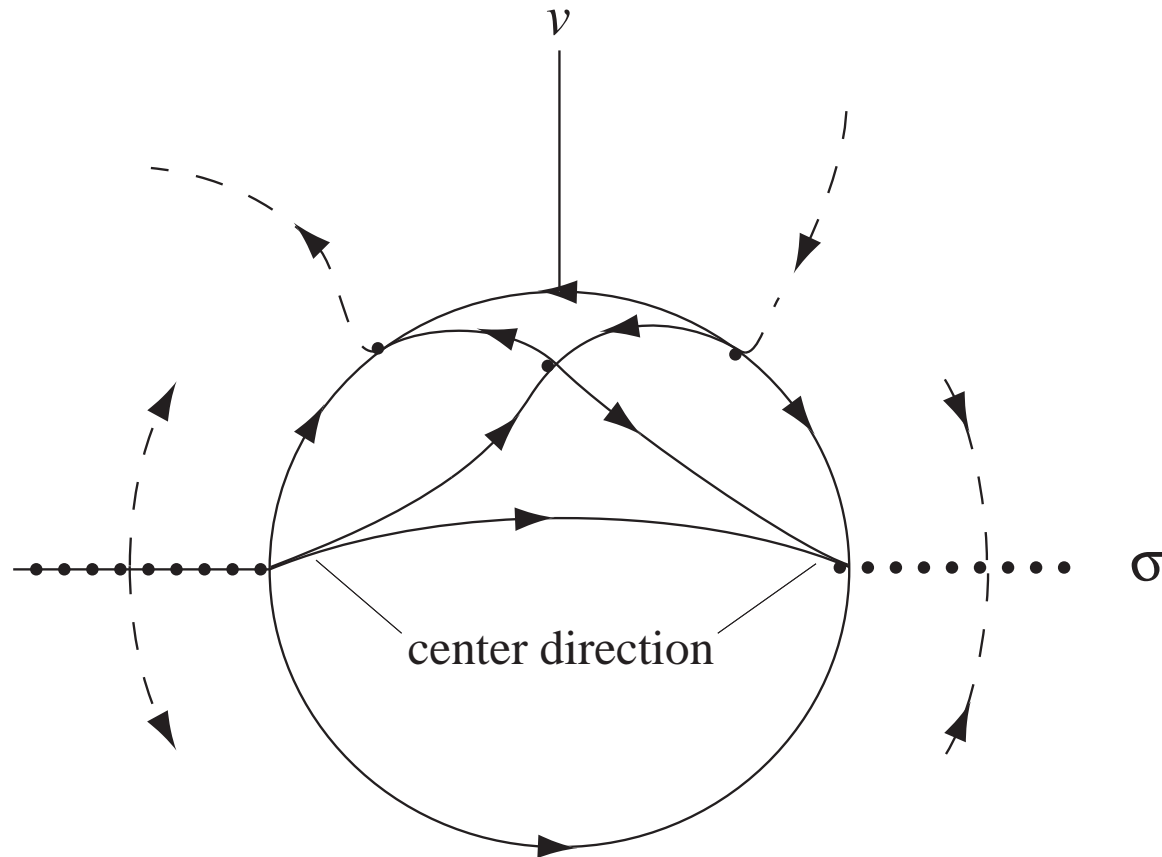
For $\varepsilon = 0$, normal hyperbolicity of $u\sigma$ -space is lost along the u -axis ($\sigma = 0$).

Blow-up:

$$\begin{aligned}u &= u, \\ v &= \bar{r}^2 \bar{v}, \\ \sigma &= \bar{r} \bar{\sigma}, \\ \varepsilon &= \bar{r}^2 \bar{\varepsilon},\end{aligned}$$

with $u \in \mathbb{R}$, $(\bar{v}, \bar{\sigma}, \bar{\varepsilon}) \in S^2$, $\bar{r} \geq 0$.

Divide vector field on blow-up space by \bar{r} to desingularize. The spherical cylinder $\bar{r} = 0$ remains invariant, and on it $\dot{u} = 0$.



Flow on blow-up space for fixed u . The ε -axis points toward you.

- No loss of normal hyperbolicity.
- Dashed curves do not have constant u .
- The “plane” $u = \text{constant}$, $\bar{v} = 0$ is invariant.

Use chart for $\bar{\varepsilon} > 0$ (rescaling chart) to find inner solution

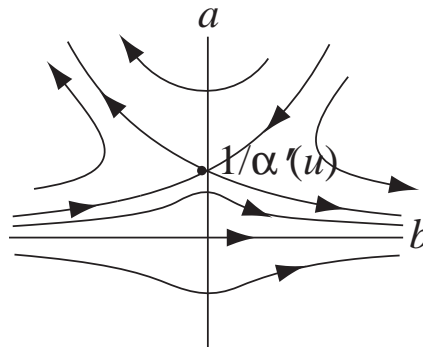
$$\begin{aligned}\dot{u} &= v, \\ \dot{v} &= -\sigma v, \\ \dot{\sigma} &= \varepsilon - \alpha'(u)v, \\ \dot{\varepsilon} &= 0.\end{aligned}$$

$$v = r^2 a, \quad \sigma = rb, \quad \varepsilon = r^2, \quad a \in \mathbb{R}, \quad b \in \mathbb{R}, \quad r \geq 0.$$

Substitute, solve for \dot{a} and \dot{b} , rescale time (divide by r):

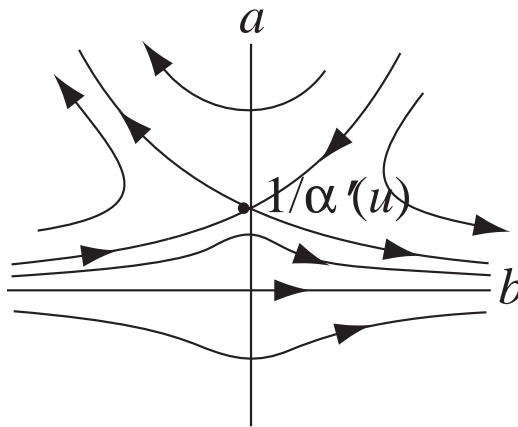
$$\begin{aligned}\dot{u} &= ra, \\ \dot{a} &= -ab, \\ \dot{b} &= 1 - \alpha'(u)a, \\ \dot{r} &= 0.\end{aligned}$$

Note three terms in \dot{a} and \dot{b} equations have order ε^0 . First three equations are a slow-fast system with small parameter r . For $r = 0$, flow with $u = \text{constant}$:



$$\begin{aligned}
 \dot{u} &= ra, \\
 \dot{a} &= -ab, \\
 \dot{b} &= 1 - \alpha'(u)a, \\
 \dot{r} &= 0.
 \end{aligned}$$

For $r = 0$, flow with $u = \text{constant}$:



For $r = 0$, $I_0 = \{(u, a, b) : a = \frac{1}{\alpha'(u)}, b = 0\}$ is a normally hyperbolic curve of equilibria.

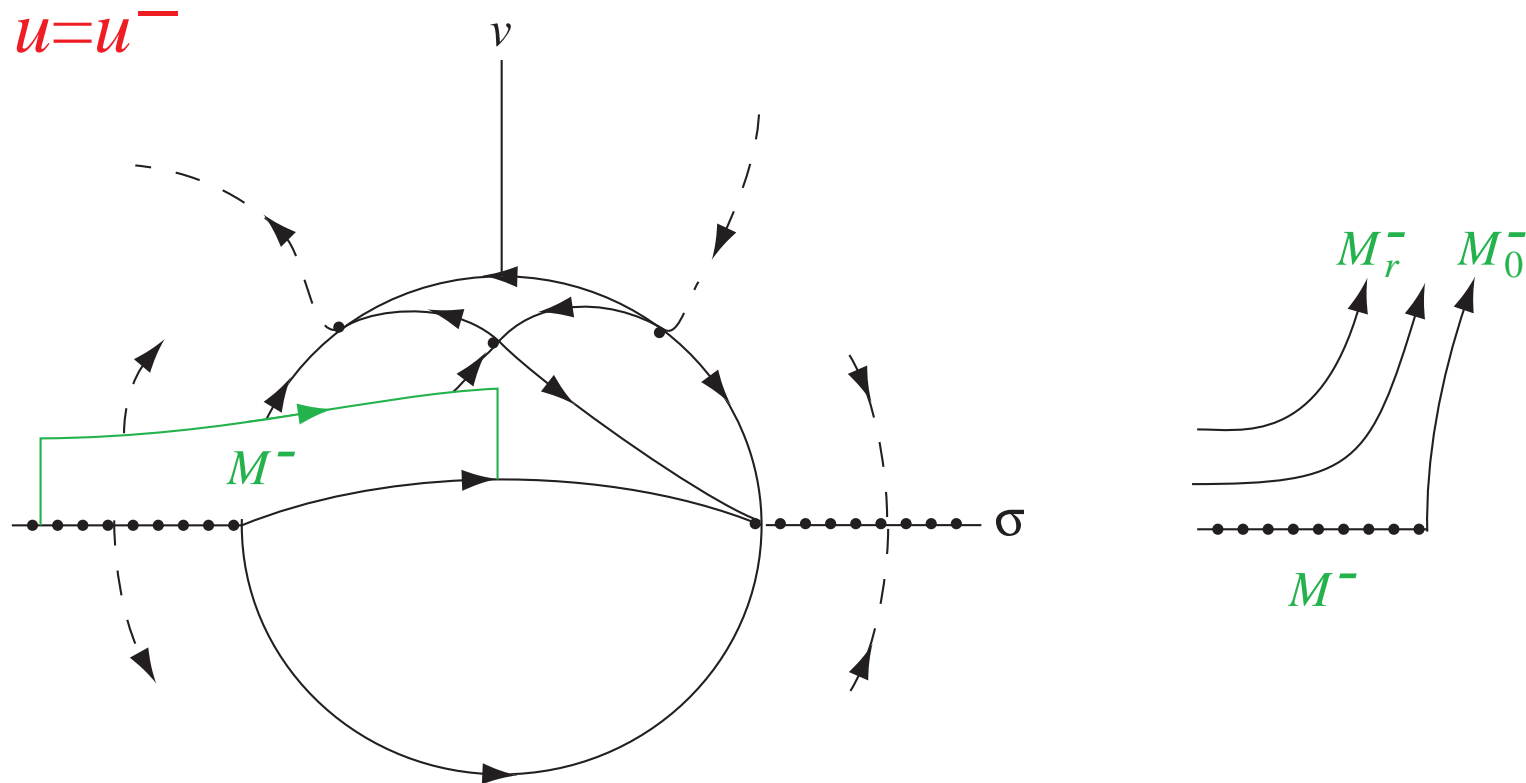
Therefore for $r > 0$ there is a nearby normally hyperbolic invariant curve I_r parameterized by u .

On I_r , to lowest order the differential equation (slow equation) is $\dot{u} = r \frac{1}{\alpha'(u)} > 0$: gives inner solution. The Exchange Lemma can help with matching.

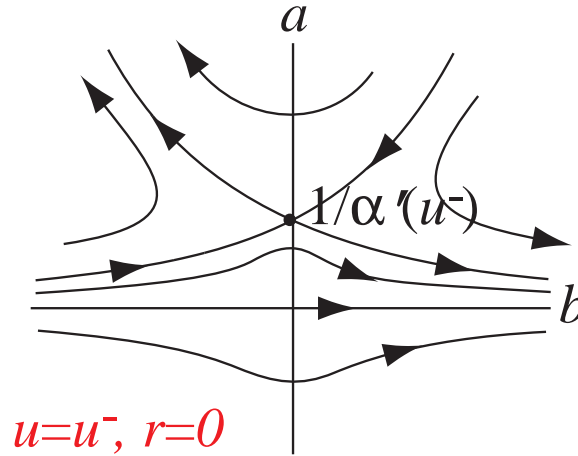
Matching: In $uv\sigma\varepsilon$ -space, $\{(u, v, \sigma, \varepsilon) : v = 0, \sigma < -\delta < 0\}$ is 3-dimensional normally repelling invariant manifold.

Let $M^- = \{(u, v, \sigma, \varepsilon) : u = u^-, v = 0, \sigma < -\delta < 0\}$ a 2-dimensional invariant manifold.

In blow-up space M^- extends as an invariant manifold to the sphere $\bar{r} = 0$: it's now $\{(u, (\bar{v}, \bar{\sigma}, \bar{\varepsilon}), \bar{r}) : u = u^-, \bar{v} = 0, \bar{\sigma} < 0\}$.



$W^u(M^-)$ (dimension = 3) includes an open subset of $\{u^-\} \times S^2 \times \{0\}$ that we call $W^u(M_0^-)$.



For $r = 0$, in uab -space, $W^u(M_0^-)$ (dimension = 2) is transverse to $W^s(I_0)$ (dimension = 2).

For small r , $W^u(M_r^-)$ is close to $W^u(M_0^-)$.

By the Exchange Lemma, for small $r > 0$, $W^u(M_r^-)$ is close to $W^u(I_r)$ when u reaches u^+ .

(Transversality to $W^s(I_r)$ is exchanged for closeness to $W^u(I_r)$.)

For $r = 0$, in uab -space, $W^u(I_0)$ is transverse to $W^s(M_0^+)$.

Therefore for small $r > 0$, $W^u(M_r^-)$ is transverse to $W^s(M_r^+)$.

This gives, for small $r > 0$, an intersection of $W^u(M_r^-)$ and $W^s(M_r^+)$

Heteroclinic orbits in a Hamiltonian system

Motivation

Sourdis and Fife, *Existence of heteroclinic orbits for a corner layer problem in anisotropic interfaces*, Advances in Differential Equations **12** (2007), 623–668:

The physical motivation comes from a multi-order-parameter phase field model, developed by Braun et al. for the description of crystalline interphase boundaries. The smallness of ε is related to large anisotropy. [The heteroclinic orbit represents a moving interface between ordered and disordered states.] The mathematical interest stems from the fact that the smoothness and normal hyperbolicity of the critical manifold fails at certain points. Thus the well-developed geometric singular perturbation theory does not apply. The existence of such a heteroclinic, and its dependence on ε , is proved via a functional analytic approach.

Second-order system

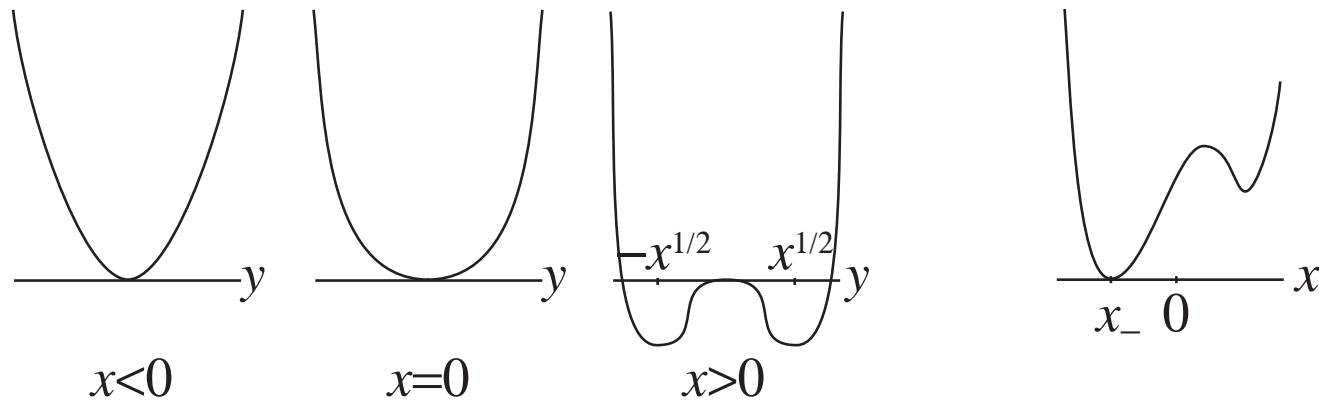
We consider

$$(1) \quad x_{\tau\tau} = g_x(x, y),$$

$$(2) \quad \varepsilon^2 y_{\tau\tau} = g_y(x, y),$$

where

$$(3) \quad g(x, y) = \frac{1}{4}y^4 - \frac{1}{2}xy^2 + h(x).$$



Graph of $(1/4)y^4 - (1/2)xy^2$

Graph of $h(x)$

First-order system

Write (1)–(2) as a first-order system (the slow system) with $u_1 = x$, $u_3 = y$:

$$(4) \quad u_{1\tau} = u_2,$$

$$(5) \quad u_{2\tau} = g_x(u_1, u_3) = -\frac{1}{2}u_3^2 + h'(u_1),$$

$$(6) \quad \varepsilon u_{3\tau} = u_4,$$

$$(7) \quad \varepsilon u_{4\tau} = g_y(u_1, u_3) = u_3^3 - u_1 u_3.$$

In (4)–(7) let $\tau = \varepsilon\sigma$. We obtain the fast system:

$$(8) \quad u_{1\sigma} = \varepsilon u_2,$$

$$(9) \quad u_{2\sigma} = \varepsilon g_x(u_1, u_3) = \varepsilon \left(-\frac{1}{2}u_3^2 + h'(u_1) \right),$$

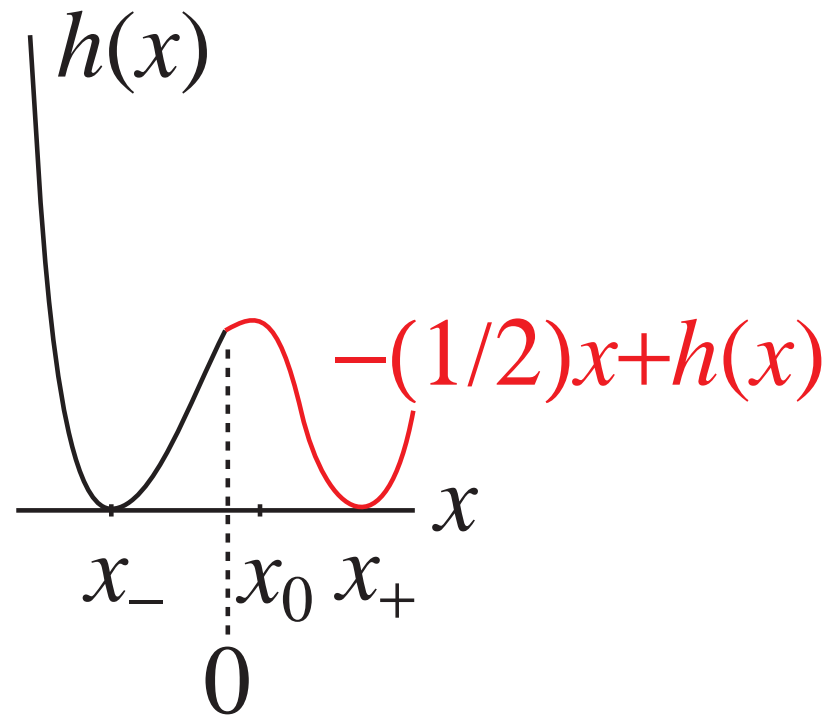
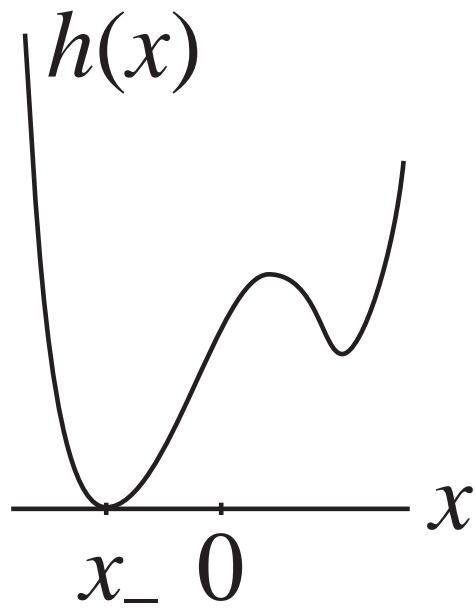
$$(10) \quad u_{3\sigma} = u_4,$$

$$(11) \quad u_{4\sigma} = g_y(u_1, u_3) = u_3^3 - u_1 u_3 = u_3(u_3^2 - u_1).$$

Equilibria of the fast system for $\varepsilon > 0$:

$$(u_1, 0, 0, 0) \text{ with } h'(u_1) = 0, \quad (u_1, 0, \pm u_1^{\frac{1}{2}}, 0) \text{ with } -\frac{1}{2}u_1 + h'(u_1) = 0.$$

Assumptions on h :



Fast system

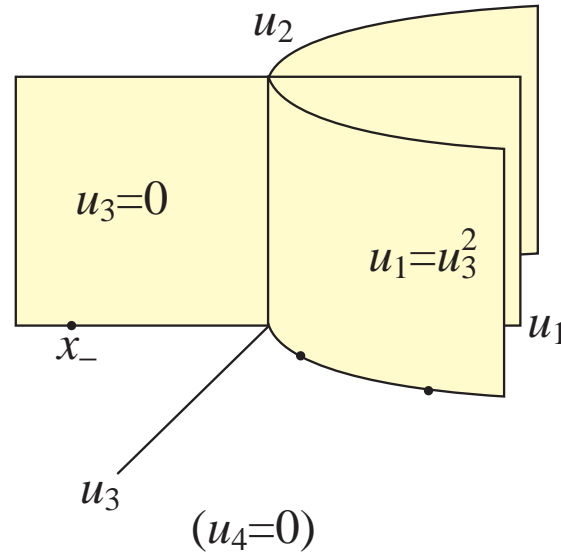
$$u_{1\sigma} = \varepsilon u_2,$$

$$u_{2\sigma} = \varepsilon g_x(u_1, u_3) = \varepsilon \left(-\frac{1}{2}u_3^2 + h'(u_1) \right),$$

$$u_{3\sigma} = u_4,$$

$$u_{4\sigma} = g_y(u_1, u_3) = u_3^3 - u_1 u_3 = u_3(u_3^2 - u_1)$$

Equilibria for $\varepsilon = 0$ (yellow) and for $\varepsilon > 0$ (black dots):



$$(x_-, 0, 0, 0), \quad (x_0, 0, \pm x_0^{\frac{1}{2}}, 0), \quad (x_+, 0, \pm x_+^{\frac{1}{2}}, 0).$$

For each ε , the fast system has the first integral

$$H(u_1, u_2, u_3, u_4) = \frac{1}{2}u_2^2 + \frac{1}{2}u_4^2 - g(u_1, u_3).$$

Note:

$$H(x_-, 0, 0, 0) = H(x_+, 0, x_+^{\frac{1}{2}}, 0) = 0.$$

Goal: show that for small $\varepsilon > 0$, there is a heteroclinic solution of the fast system from $(x_-, 0, 0, 0)$ to $(x_+, 0, x_+^{\frac{1}{2}}, 0)$.

For $\varepsilon > 0$, $(x_-, 0, 0, 0)$ and $(x_+, 0, x_+^{\frac{1}{2}}, 0)$ are hyperbolic equilibria of the fast system with two negative eigenvalues and two positive eigenvalues.

Manifolds of possible outer solutions: The heteroclinic solution should correspond to an intersection of the 2-dimensional manifolds $W_\varepsilon^u(x_-, 0, 0, 0)$ and $W_\varepsilon^s(x_+, 0, x_+^{\frac{1}{2}}, 0)$ that is transverse within the 3-dimensional manifold $H^{-1}(0)$ (which is indeed a manifold away from equilibria).

Fast limit and slow systems

Set $\varepsilon = 0$ in the fast system to obtain the fast limit system:

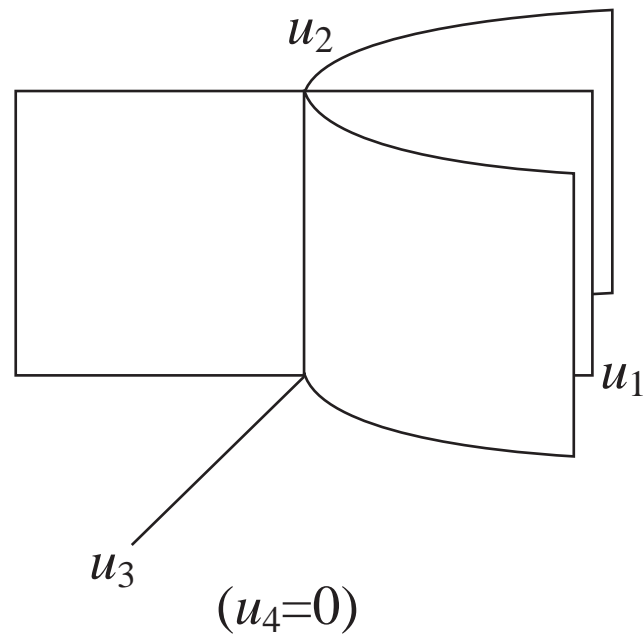
$$(12) \quad u_{1\sigma} = 0,$$

$$(13) \quad u_{2\sigma} = 0,$$

$$(14) \quad u_{3\sigma} = u_4,$$

$$(15) \quad u_{4\sigma} = g_y(u_1, u_3) = u_3(u_3^2 - u_1).$$

Equilibria:

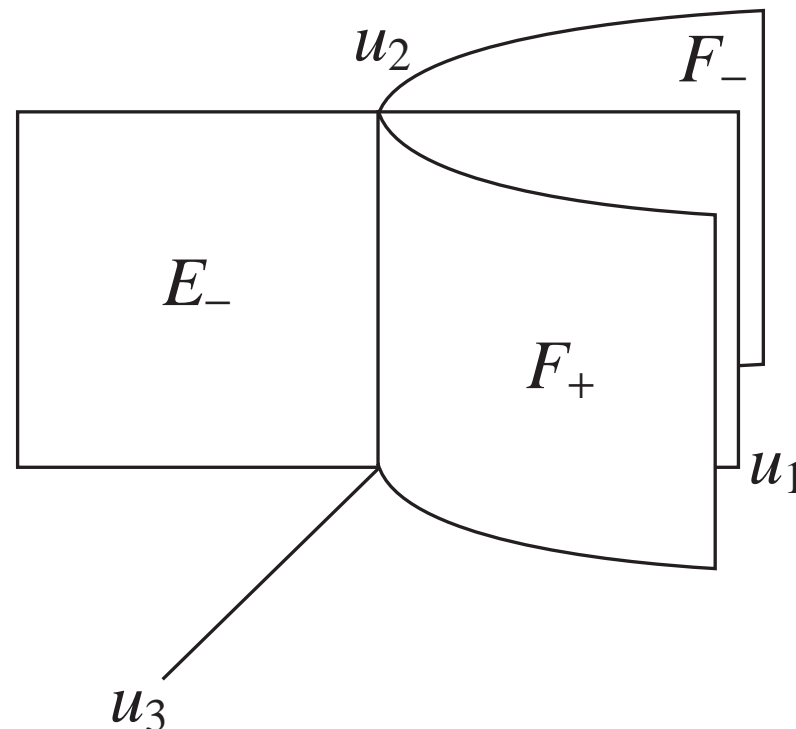


Three manifolds of normally hyperbolic equilibria:

$$E_- = \{(u_1, u_2, 0, 0) : u_1 < 0 \text{ and } u_2 \text{ arbitrary}\},$$

$$F_- = \{(u_1, u_2, -u_1^{\frac{1}{2}}, 0) : u_1 > 0 \text{ and } u_2 \text{ arbitrary}\},$$

$$F_+ = \{(u_1, u_2, u_1^{\frac{1}{2}}, 0) : u_1 > 0 \text{ and } u_2 \text{ arbitrary}\}.$$



Each has one positive eigenvalue and one negative eigenvalue. (On E_+ there are two pure imaginary eigenvalues. On the u_2 -axis all eigenvalues are 0.)

Set $\varepsilon = 0$ in the slow system to obtain the slow limit system:

$$(16) \quad u_{1\tau} = u_2,$$

$$(17) \quad u_{2\tau} = g_x(u_1, u_3) = -\frac{1}{2}u_3^2 + h'(u_1),$$

$$(18) \quad 0 = u_4,$$

$$(19) \quad 0 = g_y(u_1, u_3) = u_3(u_3^2 - u_1).$$

E_{\pm} , F_{\pm} are manifolds of solutions of (18)–(19). Equations (16)–(17) give the slow system on these manifolds.

Slow system on E_- ($u_1 < 0$, u_2 arbitrary):

$$(20) \quad u_{1\tau} = u_2,$$

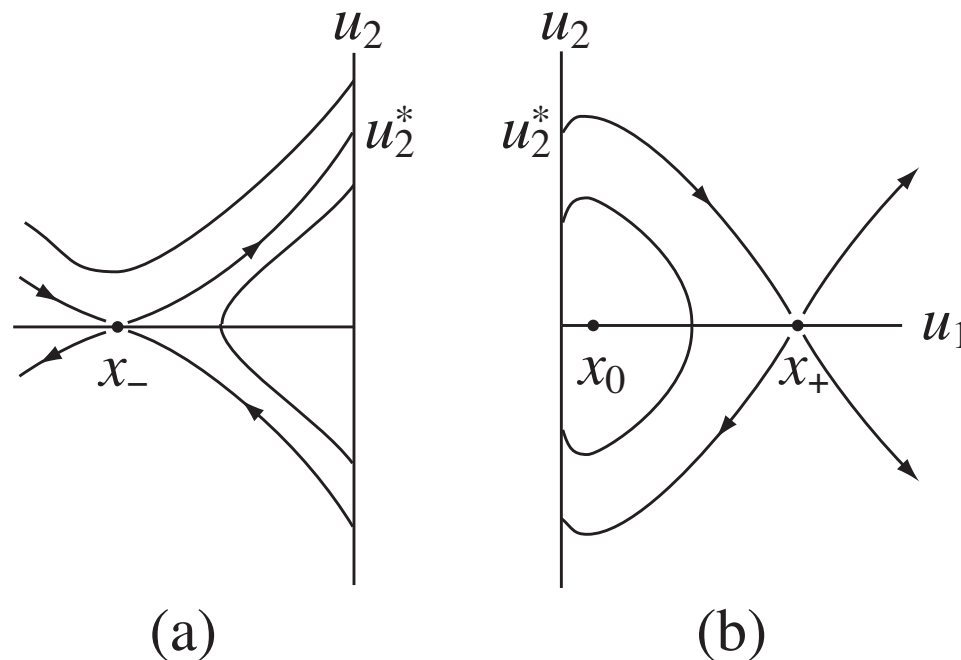
$$(21) \quad u_{2\tau} = g_x(u_1, 0) = h'(u_1).$$

Slow system on F_+ ($u_1 > 0$, u_2 arbitrary):

$$(22) \quad u_{1\tau} = u_2,$$

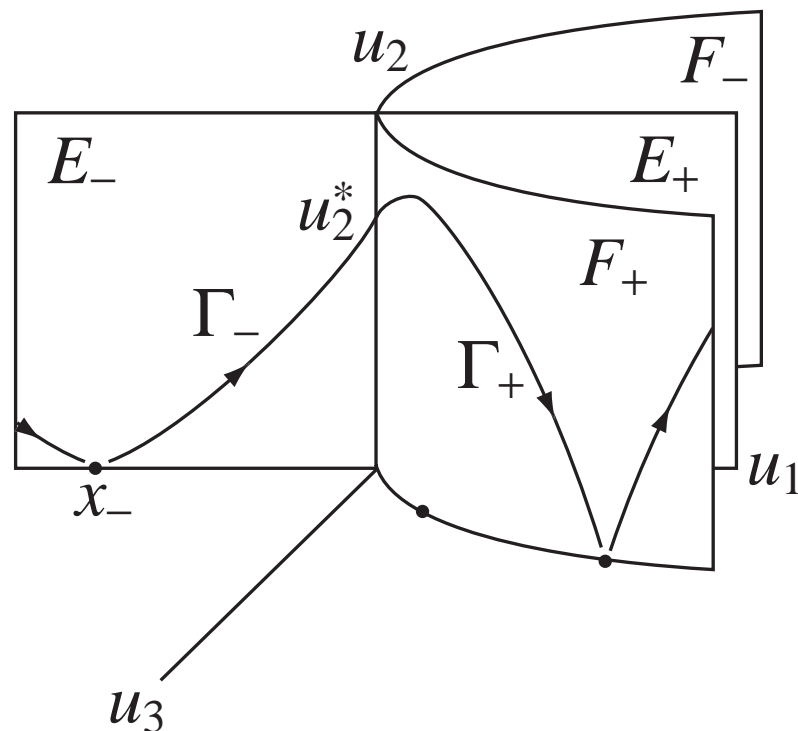
$$(23) \quad u_{2\tau} = g_x(u_1, u_1^{\frac{1}{2}}) = -\frac{1}{2}u_1 + h'(u_1).$$

Phase portraits of slow system on E_- and F_+ in u_1u_2 -coordinates, both extended to $u_1 = 0$:



- In (a), $(x_-, 0)$ is a hyperbolic saddle, and a branch of its unstable manifold meets the u_2 axis at a point $(0, u_2^*)$.
- In (b), $(x_+, 0)$ is a hyperbolic saddle, and a branch of its stable manifold meets the u_2 axis at the same point $(0, u_2^*)$.

Slow limit system on E_- and F_+ :



We want to show that for small $\varepsilon > 0$, there is a heteroclinic solution of the fast system from $(x_-, 0, 0, 0)$ to $(x_+, 0, x_+^{\frac{1}{2}}, 0)$ that is close to $\Gamma_- \cup \Gamma_+$.

Blow-up

To the fast system append the equation $\varepsilon_\sigma = 0$:

$$(24) \quad u_{1\sigma} = \varepsilon u_2,$$

$$(25) \quad u_{2\sigma} = \varepsilon g_x(u_1, u_3) = \varepsilon \left(-\frac{1}{2}u_3^2 + h'(u_1) \right),$$

$$(26) \quad u_{3\sigma} = u_4,$$

$$(27) \quad u_{4\sigma} = g_y(u_1, u_3) = u_3^3 - u_1 u_3,$$

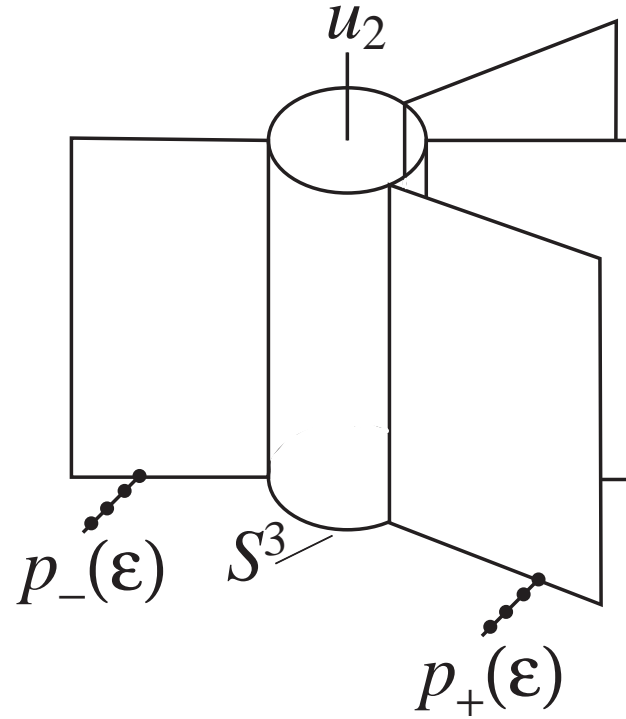
$$(28) \quad \varepsilon_\sigma = 0.$$

For $\varepsilon = 0$, the u_2 -axis consists of equilibria of (24)–(27) that are not normally hyperbolic within $u_1 u_2 u_3 u_4$ -space

In $u_1 u_2 u_3 u_4 \varepsilon$ -space, we blow up the u_2 -axis to the product of the u_2 -axis with a 3-sphere. The 3-sphere is a blow-up of the origin in $u_1 u_3 u_4 \varepsilon$ -space.

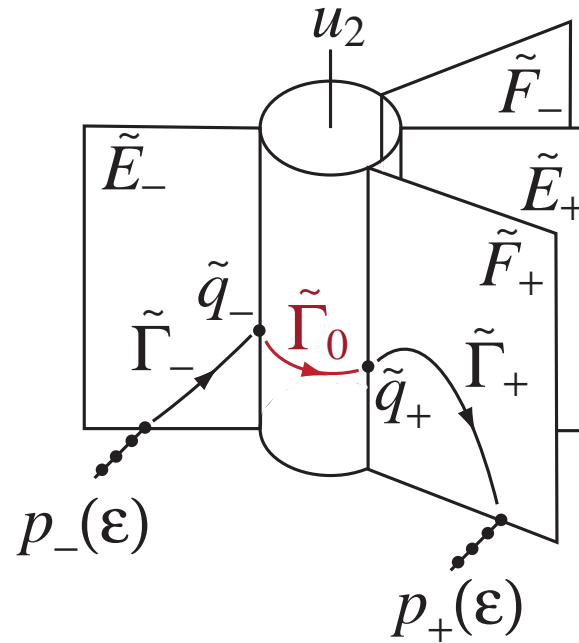
The blowup transformation is a map from $\mathbb{R} \times S^3 \times [0, \infty)$ to $u_1 u_2 u_3 u_4 \varepsilon$ -space. Let $(u_2, (\bar{u}_1, \bar{u}_3, \bar{u}_4, \bar{\varepsilon}), \bar{r})$ be a point of $\mathbb{R} \times S^3 \times [0, \infty)$; we have $\bar{u}_1^2 + \bar{u}_3^2 + \bar{u}_4^2 + \bar{\varepsilon}^2 = 1$. Then

$$(29) \quad u_1 = \bar{r}^2 \bar{u}_1, \quad u_2 = u_2, \quad u_3 = \bar{r} \bar{u}_3, \quad u_4 = \bar{r}^2 \bar{u}_4, \quad \varepsilon = \bar{r}^3 \bar{\varepsilon}.$$



Under this transformation (24)–(28) pulls back to a vector field X on $\mathbb{R} \times S^3 \times [0, \infty)$ for which the cylinder $\bar{r} = 0$ consists entirely of equilibria. The vector field we shall study is $\tilde{X} = \bar{r}^{-1}X$. Division by \bar{r} desingularizes the vector field on the cylinder $\bar{r} = 0$ but leaves it invariant.

Let $p_-(\epsilon)$ (respectively $p_+(\epsilon)$) be the unique point in $\mathbb{R} \times S^3 \times [0, \infty)$ that corresponds to $(x_-, 0, 0, 0, \epsilon)$ (respectively $(x_+, 0, x_+^{\frac{1}{2}}, 0, \epsilon)$). We wish to show that for small $\epsilon > 0$ there is an integral curve of \tilde{X} from $p_-(\epsilon)$ to $p_+(\epsilon)$.



In blow-up space:

- $\tilde{\Gamma}_-$ corresponds to Γ_- and approaches a point $\tilde{q}_- = (u_2^*, \hat{q}_-, 0)$ on the blow-up cylinder.
- $\tilde{\Gamma}_+$ corresponds to Γ_+ and approaches a point $\tilde{q}_+ = (u_2^*, \hat{q}_+, 0)$ on the blow-up cylinder.
- On the blow-up cylinder, each 3-sphere $u_2 = \text{constant}$ is invariant.

Proposition (inner solution). There is an integral curve $\tilde{\Gamma}_0$ of \tilde{X} from \tilde{q}_- to \tilde{q}_+ that lies in the 3-dimensional hemisphere given by $u_2 = u_2^*$, $\bar{r} = 0$, $\bar{\epsilon} > 0$.

Theorem. For small $\epsilon > 0$ there is an integral curve $\tilde{\Gamma}(\epsilon)$ of \tilde{X} from $p_-(\epsilon)$ to $p_+(\epsilon)$. As $\epsilon \rightarrow 0$, $\tilde{\Gamma}(\epsilon) \rightarrow \tilde{\Gamma}_- \cup \tilde{\Gamma}_0 \cup \tilde{\Gamma}_+$.

We shall need three charts on blow-up space:

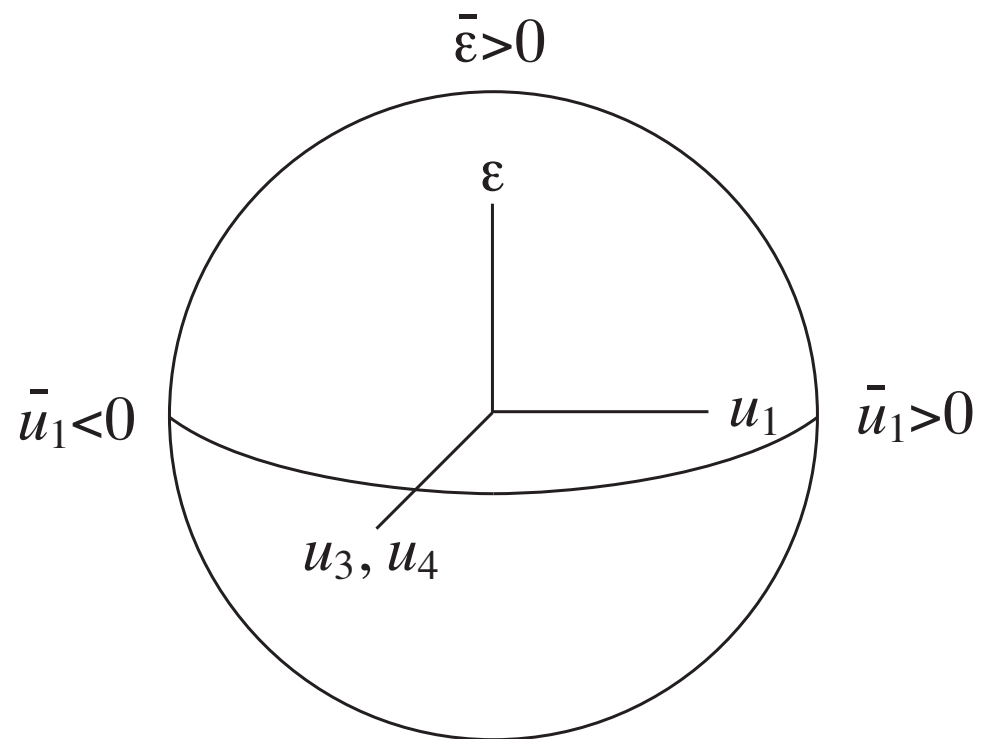


Chart for $\bar{\varepsilon} > 0$

$$(30) \quad u_1 = r^2 b_1, \quad u_2 = u_2, \quad u_3 = r b_3, \quad u_4 = r^2 b_4, \quad \varepsilon = r^3,$$

with $r \geq 0$. After division by r , (24)–(28) becomes

$$(31) \quad b_{1s} = u_2,$$

$$(32) \quad u_{2s} = r^2 \left(-\frac{1}{2} r^2 b_3^2 + h'(r^2 b_1) \right),$$

$$(33) \quad b_{3s} = b_4,$$

$$(34) \quad b_{4s} = b_3^3 - b_1 b_3,$$

$$(35) \quad r_s = 0.$$

Note: $r = 0$ implies $u_{2s} = 0$.

Chart for $\bar{u}_1 < 0$

$$(36) \quad u_1 = -v^2, \quad u_2 = u_2, \quad u_3 = va_3, \quad u_4 = v^2a_4, \quad \varepsilon = v^3\delta,$$

with $v \geq 0$. After division by v , (24)–(28) becomes

$$(37) \quad v_t = -\frac{1}{2}v\delta u_2,$$

$$(38) \quad u_{2t} = v^2\delta\left(-\frac{1}{2}v^2a_3^2 + h'(-v^2)\right),$$

$$(39) \quad a_{3t} = a_4 + \frac{1}{2}\delta u_2 a_3,$$

$$(40) \quad a_{4t} = a_3^3 + a_3 + \delta u_2 a_4,$$

$$(41) \quad \delta_t = \frac{3}{2}\delta^2 u_2.$$

Note: $v = 0$ implies $u_{2t} = 0$.

Chart for $\bar{u}_1 > 0$

$$(42) \quad u_1 = w^2, \quad u_2 = u_2, \quad u_3 = wc_3, \quad u_4 = w^2c_4, \quad \varepsilon = w^3\gamma.$$

with $w \geq 0$. After division by w , (24)–(28) becomes

$$(43) \quad w_t = \frac{1}{2}w\gamma u_2,$$

$$(44) \quad u_{2t} = w^2\gamma\left(-\frac{1}{2}w^2c_3^2 + h'(w^2)\right),$$

$$(45) \quad c_{3t} = c_4 - \frac{1}{2}\gamma u_2 c_3,$$

$$(46) \quad c_{4t} = c_3^3 - c_3 - \gamma u_2 c_4,$$

$$(47) \quad \gamma_t = -\frac{3}{2}\gamma^2 u_2.$$

Note: $w = 0$ implies $u_{2t} = 0$.

Construction of the inner solution $\tilde{\Gamma}_0$

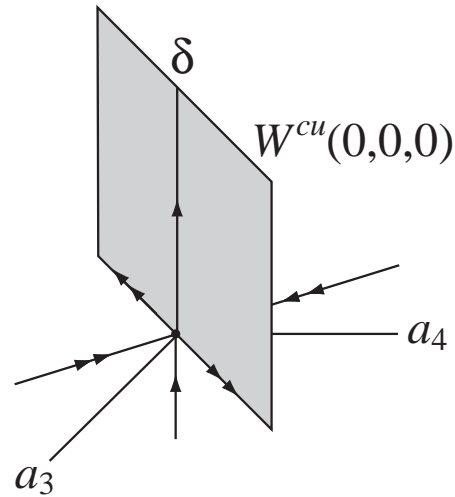
Consider the restriction of the vector field \tilde{X} to the invariant 3-sphere $S = \{u_2^*\} \times S^3 \times \{0\}$, $S^3 = \{(\bar{u}_1, \bar{u}_3, \bar{u}_4, \bar{\epsilon}) : \bar{u}_1^2 + \bar{u}_3^2 + \bar{u}_4^2 + \bar{\epsilon}^2 = 1\}$.

Chart on the open subset of S with $\bar{u}_1 < 0$: use (a_3, a_4, δ) . In this chart:

$$(48) \quad a_{3t} = a_4 + \frac{1}{2}\delta u_2^* a_3,$$

$$(49) \quad a_{4t} = a_3^3 + a_3 + \delta u_2^* a_4,$$

$$(50) \quad \delta_t = \frac{3}{2}\delta^2 u_2^*.$$



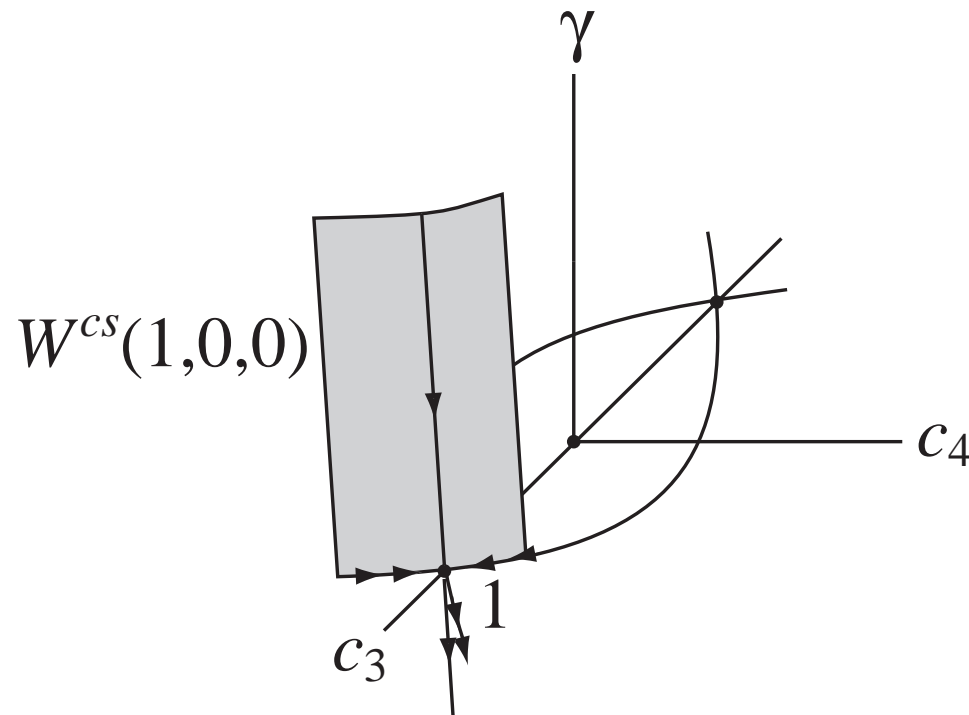
Only equilibrium is $\hat{q}^- = (0,0,0)$. Hyperbolicity is recovered in the a_3 - and a_4 -directions.

Chart on the open subset of S with $\bar{u}_1 > 0$: Use (c_3, c_4, γ) . In this chart:

$$(51) \quad c_{3t} = c_4 - \frac{1}{2}\gamma u_2^* c_3,$$

$$(52) \quad c_{4t} = c_3^3 - c_3 - \gamma u_2^* c_4,$$

$$(53) \quad \gamma_t = -\frac{3}{2}\gamma^2 u_2^*.$$



Three equilibria, $\hat{q}^+ = (1, 0, 0)$. Hyperbolicity is recovered in the c_3 - and c_4 -directions.

Chart on the open subset of S with $\bar{\varepsilon} > 0$: use (b_1, b_3, b_4) . In this chart, we have:

$$(54) \quad b_{1s} = u_2^*,$$

$$(55) \quad b_{3s} = b_4,$$

$$(56) \quad b_{4s} = b_3^3 - b_1 b_3 = b_3(b_3^2 - b_1).$$

The solution of (54) with $b_1(0) = 0$ is $b_1 = u_2^* s$. Substitute into (56) and combining (55) and (56) into a second-order equation:

$$(57) \quad b_{3ss} = b_3(b_3^2 - u_2^* s)$$

By Sourdís and Fife, (57) has a solution $b_3(s)$ with $b_{3s} > 0$ such that

$$(S1) \quad b_3(s) = o\left(|s|^{-\frac{1}{4}} e^{-\frac{2}{3}(u_2^*)^{\frac{1}{2}}|s|^{\frac{3}{2}}}\right) \text{ as } s \rightarrow -\infty,$$

$$(S2) \quad b_3(s) = (u_2^* s)^{\frac{1}{2}} + o(s^{-\frac{5}{2}}) \text{ as } s \rightarrow \infty,$$

$$(S3) \quad b_{3s}(s) \leq C|s|^{-\frac{1}{2}}, s \neq 0.$$

$(u_2^* s, b_3(s), b_{3s}(s))$ is a solution of (54)–(56). It represents an intersection of $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ in the 3-sphere S .

Transversality

$W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ are 2-dimensional submanifolds of the 3-sphere S .

Let $\tilde{\Gamma}_0 = (u_2^*, \hat{\Gamma}_0, 0)$. They intersect along $\hat{\Gamma}_0$.

Proposition. $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ intersect transversally within S along $\hat{\Gamma}_0$.

Proof. The linearization of

$$\begin{aligned} b_{1s} &= u_2^*, \\ b_{3s} &= b_4, \\ b_{4s} &= b_3^3 - b_1 b_3 \end{aligned}$$

along $(u_2^* s, b_3(s), b_{3s}(s))$ is

$$(58) \quad \begin{pmatrix} B_{1s} \\ B_{3s} \\ B_{4s} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -b_3(s) & 3b_3(s)^2 - u_2^* s & 0 \end{pmatrix} \begin{pmatrix} B_1 \\ B_3 \\ B_4 \end{pmatrix}.$$

We must show there are no solutions with appropriate behavior at $s = \pm\infty$ other than multiples of (u_2^*, b_{3s}, b_{3ss}) .

There is a complementary 2-dimensional space of solutions of (58) with $B_1(s) = 0$ and $(B_3(s), B_4(s))$ a solution of

$$(59) \quad \begin{pmatrix} B_{3s} \\ B_{4s} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3b_3(s)^2 - u_2^*s & 0 \end{pmatrix} \begin{pmatrix} B_3 \\ B_4 \end{pmatrix}$$

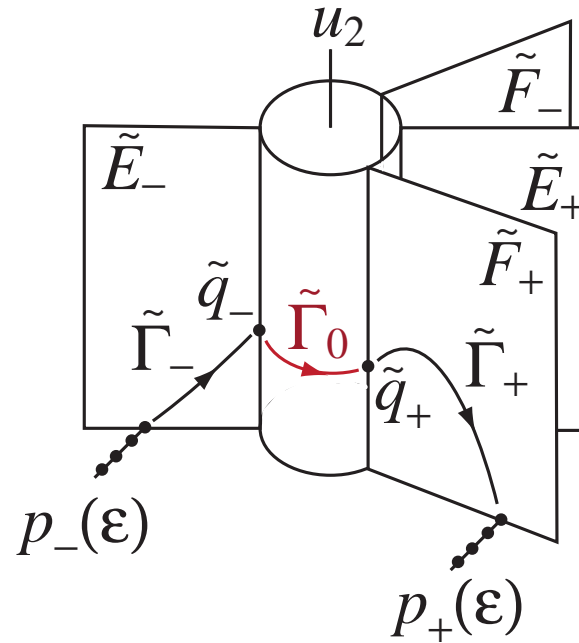
We must show that no nontrivial solution has appropriate behavior at $s = \pm\infty$.

(59) is equivalent to the second order linear system

$$(60) \quad B_{3ss} = (3b_3(s)^2 - u_2^*s)B_3.$$

By Alikakos, Bates, Cahn, Fife, Fusco, and Tanoglu, *Analysis of the corner layer problem in anisotropy*, Discrete Contin. Dyn. Syst. **6** (2006), 237–255, (60) has no nontrivial solutions in L^2 , hence no solution with the correct asymptotic behavior.

Matching

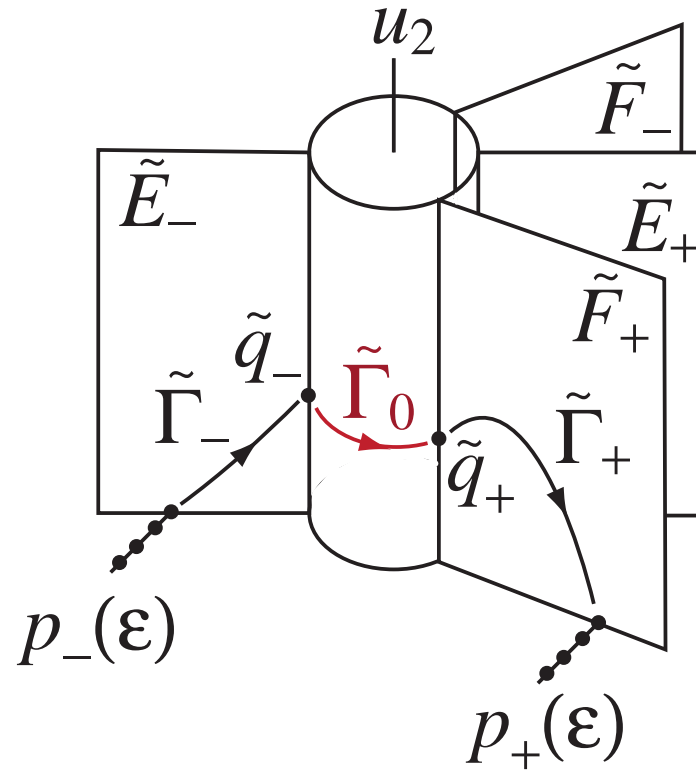


Recall: for each ε , the fast system has the first integral

$$H(u_1, u_2, u_3, u_4) = \frac{1}{2}u_2^2 + \frac{1}{2}u_4^2 - \left(\frac{1}{4}u_3^4 - \frac{1}{2}u_1u_3^2 + h(u_1) \right).$$

H gives rise to a first integral \tilde{H} on blow-up space:

$$\tilde{H}(u_2, (\bar{u}_1, \bar{u}_3, \bar{u}_4, \bar{\varepsilon}), \bar{r}) = \frac{1}{2}u_2^2 + \bar{r}^4 \left(\frac{1}{2}\bar{u}_4^2 - \frac{1}{4}\bar{u}_3^4 + \frac{1}{2}\bar{u}_1\bar{u}_3^2 \right) - h(\bar{r}^2\bar{u}_1).$$



Let N_ε denote the set of points in blow-up space at which $\tilde{H} = 0$ and $\bar{r}^3 \bar{\varepsilon} = \varepsilon$.

Away from equilibria of \tilde{X} , each N_ε is a manifold of dimension 3.

For the vector field \tilde{X} and $\varepsilon > 0$, the equilibria $p_-(\varepsilon)$ and $p_+(\varepsilon)$ have 2-dimensional unstable and stable manifolds.

We will prove the theorem by showing that for small $\varepsilon > 0$, $W^u(p_-(\varepsilon))$ and $W^s(p_+(\varepsilon))$ have a nonempty intersection that is transverse within N_ε .

Chart for $\bar{u}_1 < 0$:

$$\begin{aligned} v_t &= -\frac{1}{2}v\delta u_2, \\ u_{2t} &= v^2\delta\left(-\frac{1}{2}v^2a_3^2 + h'(-v^2)\right), \\ a_{3t} &= a_4 + \frac{1}{2}\delta u_2 a_3, \\ a_{4t} &= a_3^3 + a_3 + \delta u_2 a_4, \\ \delta_t &= \frac{3}{2}\delta^2 u_2. \end{aligned}$$

The 3-dimensional space $a_3 = a_4 = 0$ is invariant, and is normally hyperbolic near the plane of equilibria $a_3 = a_4 = \delta = 0$. One eigenvalue is positive, one is negative.

The plane of equilibria corresponds to E_- . Normal hyperbolicity within $\delta = 0$ is *not* lost at $v = 0$, which corresponds to $u_1 = 0$.

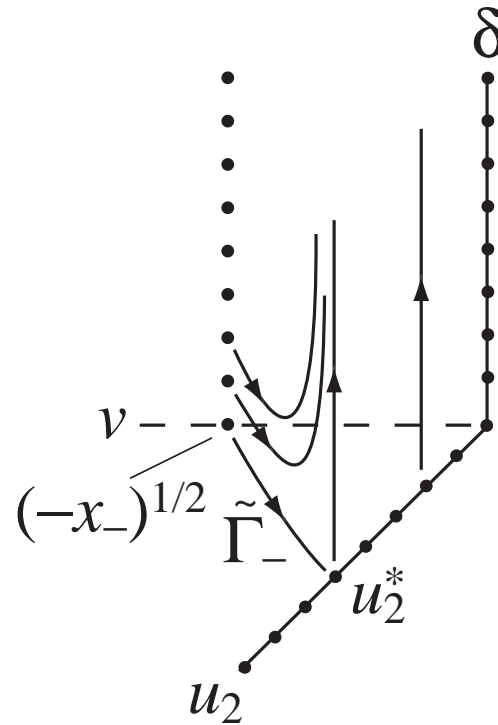
Restrict to $a_3 = a_4 = 0$ and divide by δ :

$$(61) \quad \dot{v} = -\frac{1}{2}vu_2,$$

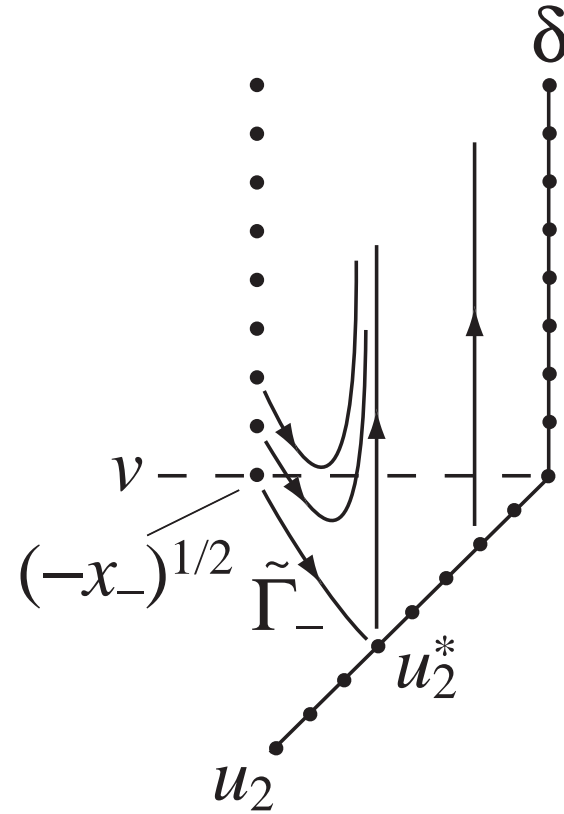
$$(62) \quad \dot{u}_2 = v^2 h'(-v^2),$$

$$(63) \quad \dot{\delta} = \frac{3}{2}\delta u_2.$$

$$\begin{aligned}\dot{v} &= -\frac{1}{2}vu_2, \\ \dot{u}_2 &= v^2 h'(-v^2), \\ \dot{\delta} &= \frac{3}{2}\delta u_2.\end{aligned}$$



Equilibria on the lines $\{(v, u_2, \delta) : v = (-x_-)^{1/2}, u_2 = 0\}$ and $\{(v, u_2, \delta) : v = \delta = 0, u_2 \neq 0\}$ are normally hyperbolic within $vu_2\delta$ -space, with one positive eigenvalue and one negative eigenvalue.



Lemma. As $\delta_0 \rightarrow 0+$, $W^u((-x_-)^{\frac{1}{2}}, 0, \delta_0)$ approaches $W^u(0, u_2^*, 0)$ in the C^1 topology. (Both have dimension 1.) (*Corner Lemma.*)

Lemma. In the chart for $\bar{u}_1 < 0$, as $\delta_0 \rightarrow 0+$, $W^u((-x_-)^{\frac{1}{2}}, 0, 0, 0, \delta_0)$ approaches the manifold of unstable fibers over $W^u(0, u_2^*, 0)$ in the C^1 topology. (Both have dimension 2.)

The latter corresponds to $W^{cu}(\hat{q}_1)$ in $S = \{u_2^*\} \times S^3 \times \{0\}$.

Chart for $\bar{u}_1 > 0$:

$$\begin{aligned} w_t &= \frac{1}{2}w\gamma u_2, \\ u_{2t} &= w^2\gamma\left(-\frac{1}{2}w^2c_3^2 + h'(w^2)\right), \\ c_{3t} &= c_4 - \frac{1}{2}\gamma u_2 c_3, \\ c_{4t} &= c_3^3 - c_3 - \gamma u_2 c_4, \\ \gamma_t &= -\frac{3}{2}\gamma^2 u_2. \end{aligned}$$

The equilibria of the plane $c_3 = 1$, $c_4 = \gamma = 0$ have, transverse to the plane, one positive eigenvalue, one negative eigenvalue, one zero eigenvalue.

Therefore this plane is part of a 3-dimensional normally hyperbolic invariant manifold S_2 , with equations

$$c_3 = 1 + \gamma^2 \tilde{c}_3(w, u_2, \gamma), \quad c_4 = \gamma \tilde{c}_4(w, u_2, \gamma).$$

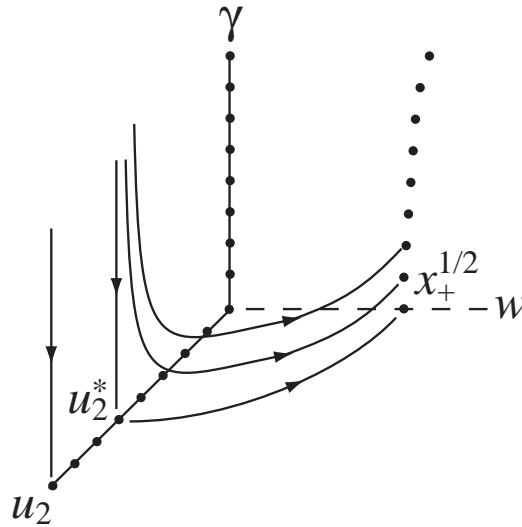
The plane of equilibria corresponds to F_+ . Normal hyperbolicity within $\gamma = 0$ is *not* lost at $w = 0$, which corresponds to $u_1 = 0$.

Restrict to S_2 and divide by γ :

$$(64) \quad w_t = \frac{1}{2} w u_2,$$

$$(65) \quad u_{2t} = w^2 \left(-\frac{1}{2} w^2 (1 + \gamma^2 \tilde{c}_3)^2 + h'(w^2) \right),$$

$$(66) \quad \gamma_t = -\frac{3}{2} \gamma u_2.$$



Lemma. As $\gamma_0 \rightarrow 0+$, $W^s(x_+^{\frac{1}{2}}, 0, \gamma_0)$ approaches $W^s(0, u_2^*, 0)$ in the C^1 topology. (Both have dimension 1.)

Lemma. In the chart for $\bar{u}_1 > 0$, as $\gamma_0 \rightarrow 0+$, $W^s(x_+^{\frac{1}{2}}, 0, 1, 0, \gamma_0)$ approaches the manifold of stable fibers over $W^s(0, u_2^*, 0)$ in the C^1 topology. (Both have dim 2.)

The latter corresponds to $W^{cs}(\hat{q}_+)$ in $S = \{u_2^*\} \times S^3 \times \{0\}$.

Conclusion: in blow-up space,

As $\varepsilon \rightarrow 0+$, $W^u(p_-(\varepsilon))$ approaches $W^{cu}(\hat{q}_-)$ in the C^1 topology.

As $\varepsilon \rightarrow 0+$, $W^s(p_+(\varepsilon))$ approaches $W^{cs}(\hat{q}_+)$ in the C^1 topology.

We showed $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ meet transversally within the 3-sphere $\bar{r} = 0$, $u_2 = u_2^*$, **which is** N_0 .

In the chart for $\bar{\varepsilon} > 0$, H corresponds to

$$H_b(b_1, u_2, b_3, b_4, r) = \frac{1}{2}u_2^2 + r^4\left(\frac{1}{2}b_4^2 - \frac{1}{4}b_3^4 + \frac{1}{2}b_1b_3^2\right) + h(r^2b_1).$$

N_0 corresponds to the set of (b_1, u_2, b_3, b_4, r) such that $H_b = 0$ and $r = 0$. The functions H_b and r have linearly independent gradients provided $u_2 \neq 0$. Therefore, where $u_2 \neq 0$, the sets $N_{\frac{1}{\varepsilon^3}} = N_r$ depend smoothly on r . Since $W^{cu}(\hat{q}_-)$ and $W^{cs}(\hat{q}_+)$ meet transversally within N_0 , it follows that $W^u(p_-(\varepsilon))$ and $W^s(p_+(\varepsilon))$ meet transversally within N_ε for ε small.

