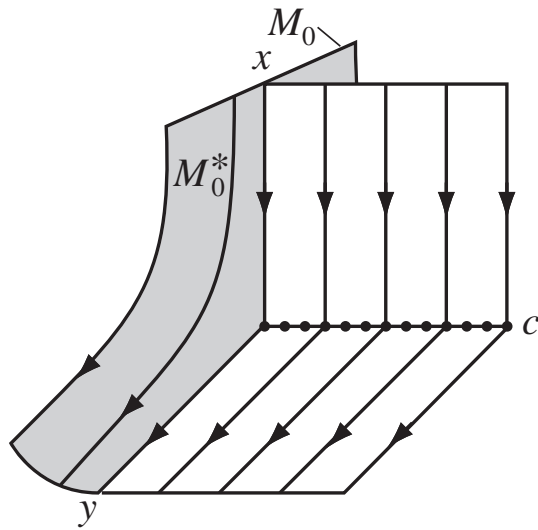
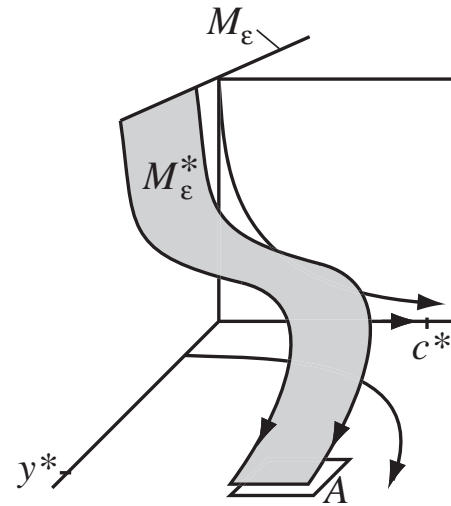


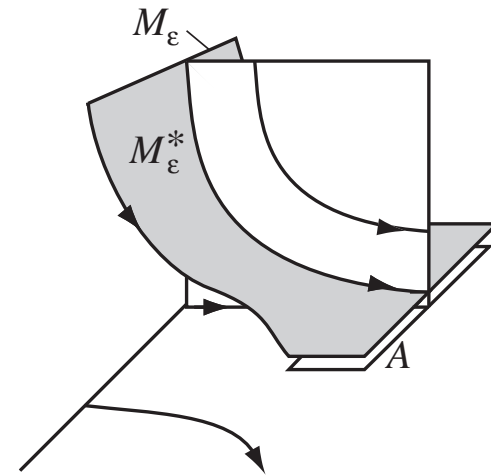
Exchange Lemmas



(a)



(b)



(c)

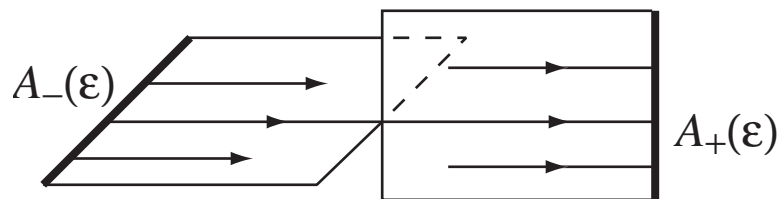
Steve Schechter
North Carolina State University

Plan

- (1) Boundary value problems
- (2) Fitzhugh-Nagumo Equation
- (3) Exchange Lemma of Jones and Kopell
- (4) Exchange Lemma of Jones and Tin
- (5) General Exchange Lemma
- (6) Loss-of-stability turning points: Liu's Exchange Lemma
- (7) Gain-of-stability turning points
- (8) Basis of proof: Generalized Deng's Lemma

Boundary Value Problems

$$\dot{\xi} = F(\xi, \varepsilon), \quad \xi(t_-) \in A_-(\varepsilon), \quad \xi(t_+) \in A_+(\varepsilon),$$



To show existence of a solution: show that the manifold of solutions that start on $A_-(\varepsilon)$ and the manifold of solutions that end on $A_+(\varepsilon)$ meet transversally.

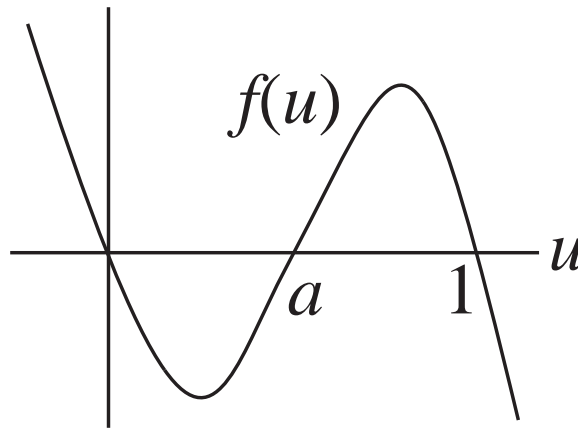
Remarks

- The problem with $\varepsilon = 0$ may be degenerate in some major way .
- Such problems are called *singularly perturbed*.
- The geometric approach to these problems, which focuses on *tracking manifolds of potential solutions* rather than on asymptotic expansions of solutions, is called *geometric singular perturbation theory* (Fenichel, Kopell, Jones, ...).

Fitzhugh-Nagumo Equation

$$\begin{aligned}u_t &= u_{xx} + f(u) - w \\w_t &= \varepsilon(u - \gamma w)\end{aligned}$$

$$f(u) = u(u - a)(1 - u), \quad 0 < a < \frac{1}{2}, \quad 0 < \gamma, \quad 0 < \varepsilon \ll 1.$$



u = voltage potential across nerve axon membrane.

w = negative feedback effects.

Is there a traveling wave $(u, w)(\xi)$, $\xi = x - ct$ for some velocity c , such that $(u, w)(\xi) \rightarrow (0, 0)$ as $\xi \rightarrow \pm\infty$?

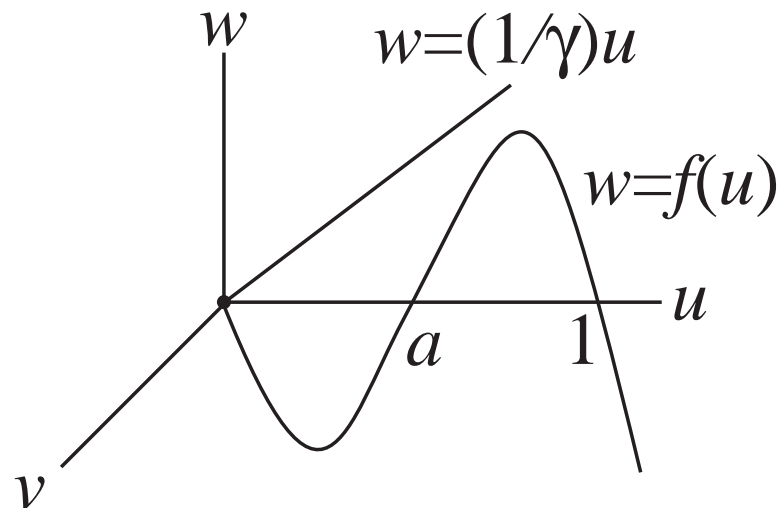
$$\begin{aligned}-cu_\xi &= u_{\xi\xi} + f(u) - w \\-cw_\xi &= \varepsilon(u - \gamma w)\end{aligned}$$

$$\begin{aligned} -cu_\xi &= u_{\xi\xi} + f(u) - w \\ -cw_\xi &= \varepsilon(u - \gamma w) \end{aligned}$$

Write as a first-order system, make c a variable:

$$\begin{aligned} u_\xi &= v \\ v_\xi &= -cv - f(u) + w \\ w_\xi &= \frac{\varepsilon}{c}(\gamma w - u) \\ c_\xi &= 0 \end{aligned}$$

Slow-fast system, 2 slow variables, 2 fast variables. Equilibria for $\varepsilon > 0$: $v = 0$, $w = f(u)$, $w = \frac{1}{\gamma}u$. For γ small there is just the origin for each c :



Origin has 1 negative eigenvalue and 2 eigenvalues with positive real part.

Set $\varepsilon = 0$:

$$u_\xi = v$$

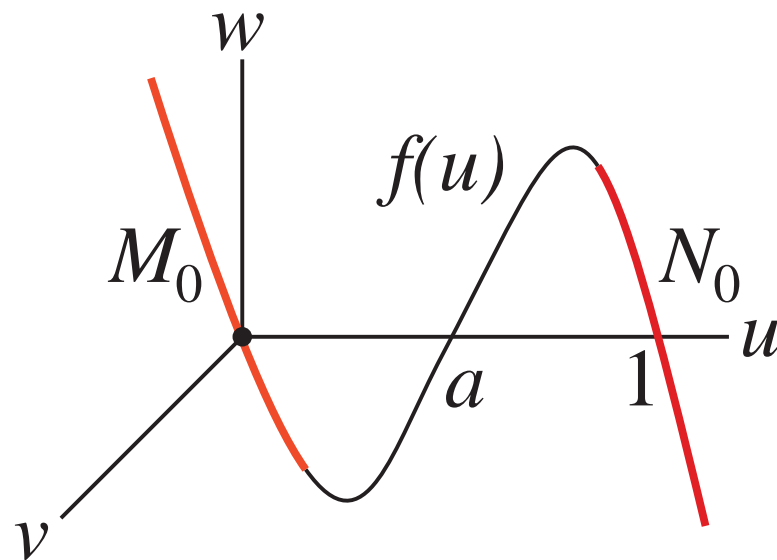
$$v_\xi = -cv - f(u) + w$$

$$w_\xi = 0$$

$$c_\xi = 0$$

2-dimensional manifold of equilibria $v = 0$, $w = f(u)$, c arbitrary.

2-dimensional normally hyperbolic manifolds of equilibria with 1 positive eigenvalue, 1 negative eigenvalue:



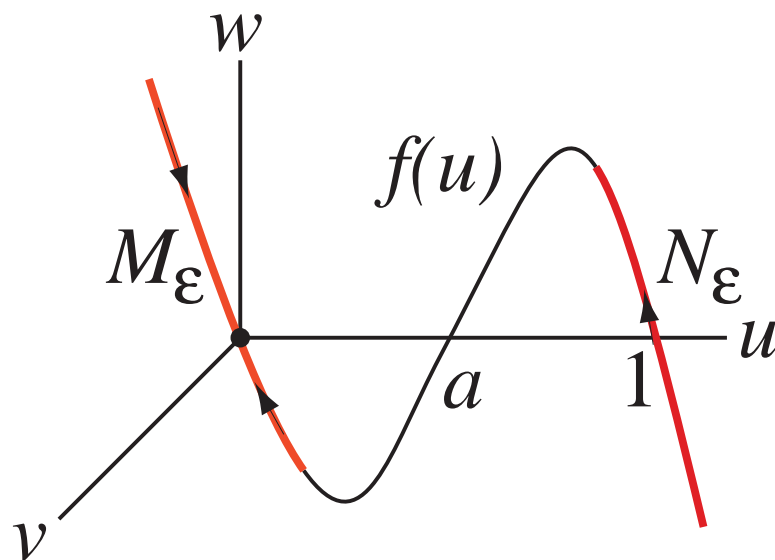
$$u = "f^{-1}(w)", \quad v = 0, \quad c \text{ arbitrary.}$$

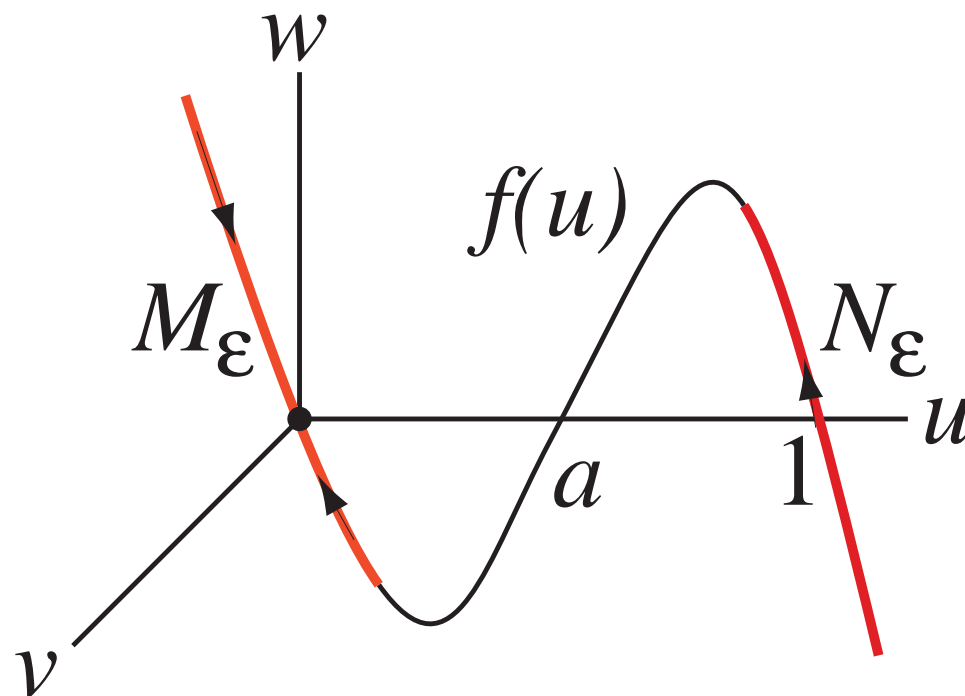
$$\begin{aligned}
 u_\xi &= v \\
 v_\xi &= -cv - f(u) + w \\
 w_\xi &= \frac{\varepsilon}{c}(\gamma w - u) \\
 c_\xi &= 0
 \end{aligned}$$

For $\varepsilon > 0$, M_0 and N_0 persist as normally hyperbolic invariant manifolds. Differential equation on them:

$$\begin{aligned}
 w_\xi &= \frac{\varepsilon}{c}(\gamma w - f^{-1}(w)) + O(\varepsilon^2) \\
 c_\xi &= 0
 \end{aligned}$$

For $c < 0$ and γ small:





Let $J = \{(0, 0, 0, c) : c \text{ arbitrary}\}$, a line of equilibria.

$J \subset M_\varepsilon$ for all ε .

$W_\varepsilon^u(J)$ and $W_\varepsilon^{cs}(J) = W^s(M_\varepsilon)$ exist and depends smoothly on ε from the theory of normally hyperbolic invariant manifolds.

Look for a solution that at time t_- is in $W_\varepsilon^u(J)$ (dimension = 2) and at time t_+ is in $W^{cs}(M_\varepsilon)$ (dimension=3).

Back to $\varepsilon = 0$:

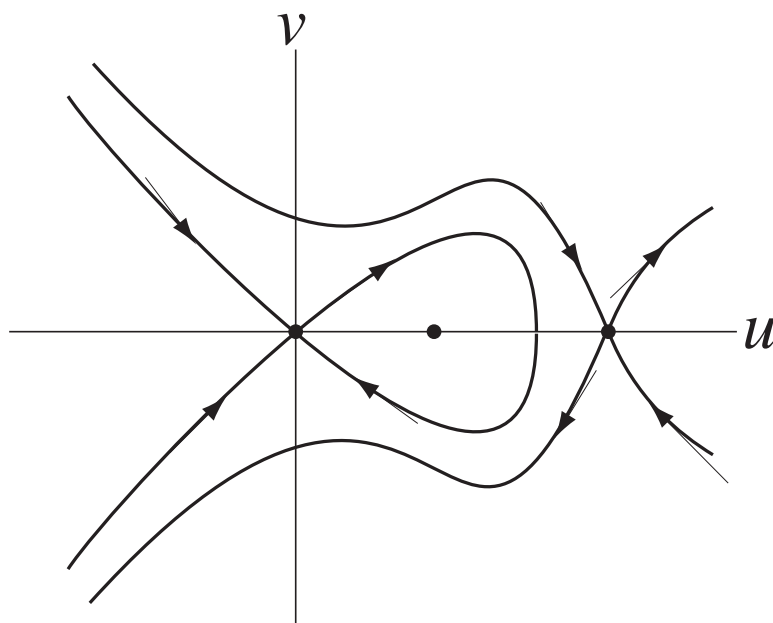
$$u_\xi = v$$

$$v_\xi = -cv - f(u) + w$$

$$w_\xi = 0$$

$$c_\xi = 0$$

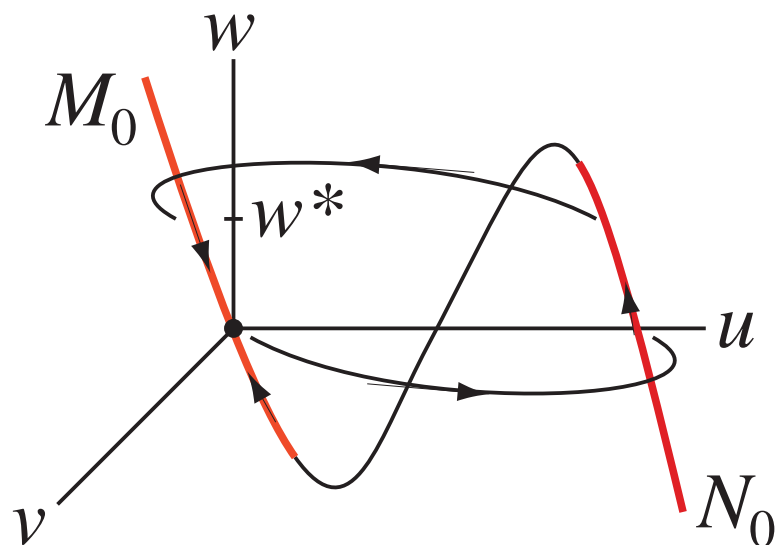
Phase portrait for $(w, c) = (0, 0)$



For $(w, c) = (0, c^*)$, $c^* < 0$, $(0,0)$ connects to $(1,0)$

For $(w, c) = (w^*, c^*)$, $w^* > 0$, right equilibrium connects to left equilibrium.

Slow and fast orbits, $c = c^*$:



Is there a true homoclinic orbit nearby for $\varepsilon > 0$?

1. For $\varepsilon = 0$: $W^u_\varepsilon(J)$ (dimension = 2) intersects $W^s(N_0)$ (dimension = 3) transversally in an orbit with $c = c^*$.

2. Therefore for ε small, $W^u_\varepsilon(J)$ intersects $W^s(N_\varepsilon)$ transversally in an orbit with $c = c(\varepsilon)$, $c(0) = c^*$.

3. For $\varepsilon = 0$: $W^u(N_0) \cap \{c = c^*\}$ (dimension = 2) intersects $W^s(M_0)$ (dimension = 3) transversally in an orbit with $w = w^*$.

4. Therefore for ε small, $W^u(N_\varepsilon) \cap \{c = c(\varepsilon)\}$ intersects $W^s(M_\varepsilon)$ transversally in an orbit with “ w near w^* .”

5. For ε small, does $W^u_\varepsilon(J)$ become close to $W^u(N_\varepsilon) \cap \{c = c(\varepsilon)\}$? If so we are done.

Exchange Lemma of Jones and Kopell

Slow-Fast Systems

$$\dot{a} = f(a, b, \varepsilon), \quad \dot{b} = \varepsilon g(a, b, \varepsilon), \quad (a, b) \in \mathbb{R}^n \times \mathbb{R}^m.$$

Set $\varepsilon = 0$:

$$\dot{a} = f(a, b, 0), \quad \dot{b} = 0.$$

Assume:

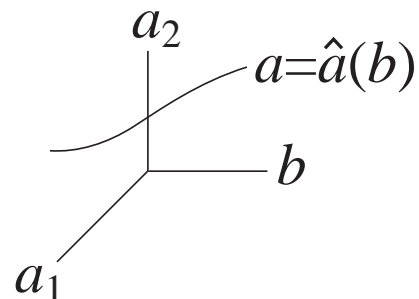
(1) $f(\hat{a}(b), b, 0) = 0$.

(2) $D_a f(\hat{a}(b), b, 0)$ has

- k eigenvalues with negative real part.
- l eigenvalues with positive real part.
- $k + l = n$.

(3) $g(\hat{a}(b), b, 0) \neq 0$.

(1) and (2) say the m -dimensional manifold $a = \hat{a}(b)$ is a normally hyperbolic manifold of equilibria for $\varepsilon = 0$.



After a change of coordinates:

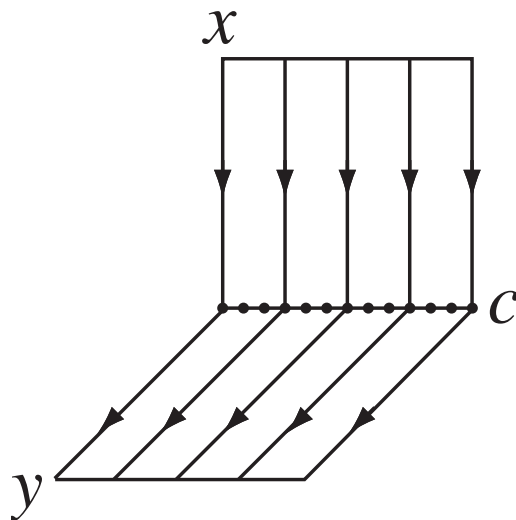
$$\dot{x} = A(x, y, c, \varepsilon)x,$$

$$\dot{y} = B(x, y, c, \varepsilon)y,$$

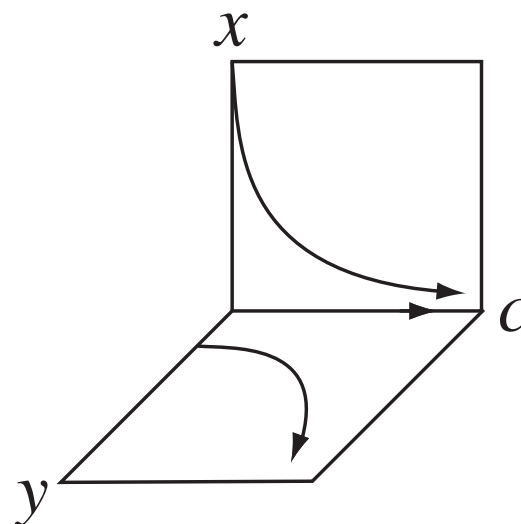
$$\dot{c} = \varepsilon((1, 0, \dots, 0) + L(x, y, c, \varepsilon)xy),$$

$(x, y, c) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^m$ (contracting \times expanding \times center),
 $A(0, 0, c, 0)$ has eigenvalues with negative real part,
 $B(0, 0, c, 0)$ has eigenvalues with positive real part.

Note that on xc -space and yc -space, \dot{c} depends on only on c and ε . Gives “fast foliation.”



Flow with $\varepsilon = 0$.

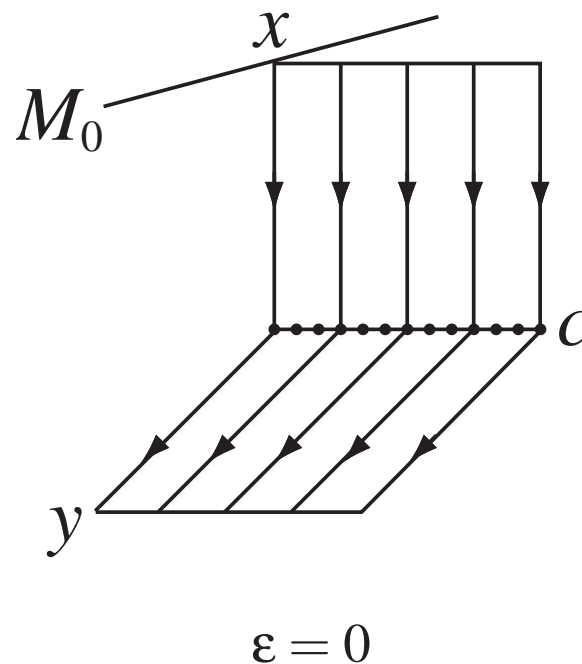


Flow with $\varepsilon > 0$.

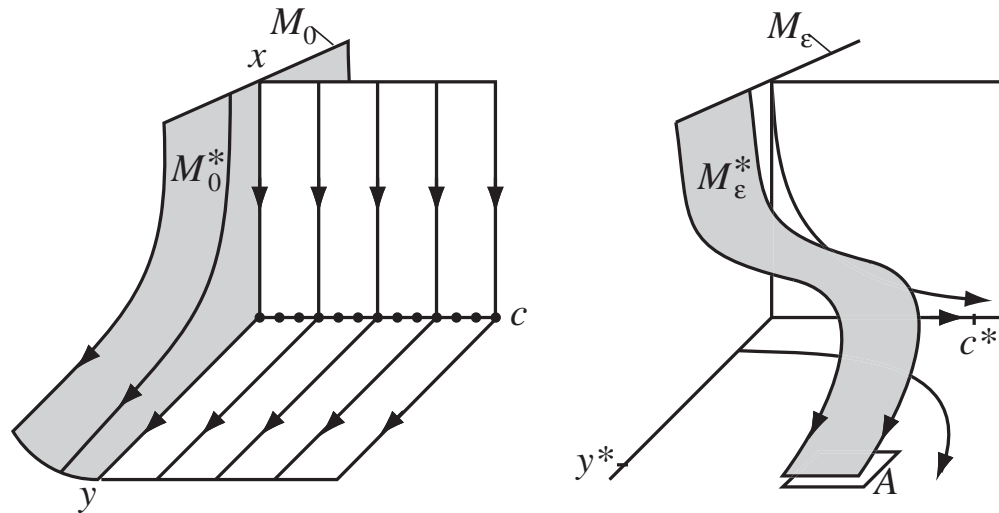
Exchange Lemma of Jones and Kopell

Assume:

- (1) For each ε , M_ε is a submanifold of xyz -space of dimension l .
- (2) $M = \{(x, y, c, \varepsilon) : (x, y, c) \in M_\varepsilon\}$ is itself a manifold.
- (3) For each ε , M_ε meets xc -space transversally in a point $(x(\varepsilon), 0, 0)$.



Under the forward flow, each M_ε becomes a manifold M_ε^* of dimension $l + 1$.



Theorem 1 (Exchange Lemma of Jones and Kopell, 1994). Let

$$c^* = (c_1^*, 0, \dots, 0) \in \mathbb{R}^m$$

with $0 < c_1^*$. Let $y^* \neq 0$. Let A be a small neighborhood of (y^*, c_1^*) in yc_1 -space. Then for small $\varepsilon_0 > 0$ there are smooth functions $\tilde{x} : A \times [0, \varepsilon_0) \rightarrow \mathbb{R}^k$ and $\tilde{c} : A \times [0, \varepsilon_0) \rightarrow \mathbb{R}^{m-1}$ such that:

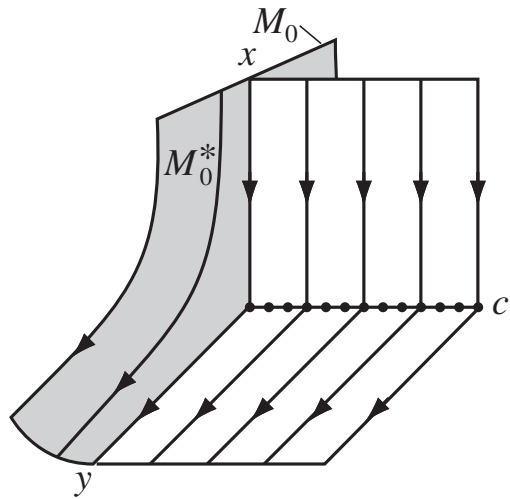
- (1) For $0 < \varepsilon < \varepsilon_0$, $\{(x, y, c_1, \dots, c_m) : (y, c_1) \in A, x = \tilde{x}(y, c_1, \varepsilon), \text{ and } (c_2, \dots, c_m) = \tilde{c}(y, c_1, \varepsilon)\}$ is contained in M_ε^* .
- (2) $\tilde{x}(y, c_1, 0) = 0$.
- (3) $\tilde{c}(y, c_1, 0) = \tilde{c}(0, c_1, \varepsilon) = 0$.
- (4) As $\varepsilon \rightarrow 0$, $(\tilde{x}, \tilde{c}) \rightarrow 0$ exponentially, along with its derivatives with respect to all variables.

Transversality to xc -space is “exchanged” for closeness to yc -space.

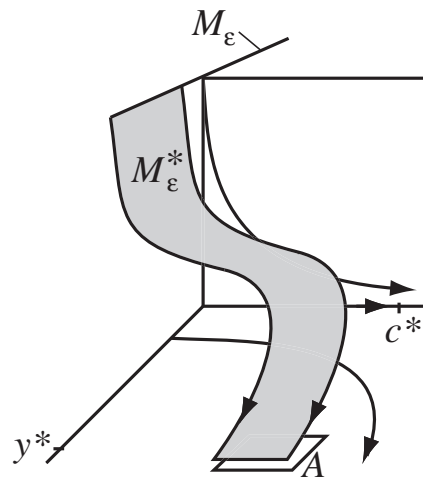
Brunovsky's Reformulation of Jones and Kopell's Exchange Lemma as an Inclination Lemma

Theorem 2 (1999). Let $c^* = (c_1^*, 0, \dots, 0) \in \mathbb{R}^m$ with $0 < c_1^*$. Let A be a small neighborhood of $(0, c_1^*)$ in yc_1 -space. Then for small $\varepsilon_0 > 0$ there are smooth functions $\tilde{x} : A \times [0, \varepsilon_0) \rightarrow \mathbb{R}^k$ and $\tilde{c} : A \times [0, \varepsilon_0) \rightarrow \mathbb{R}^{m-1}$ such that:

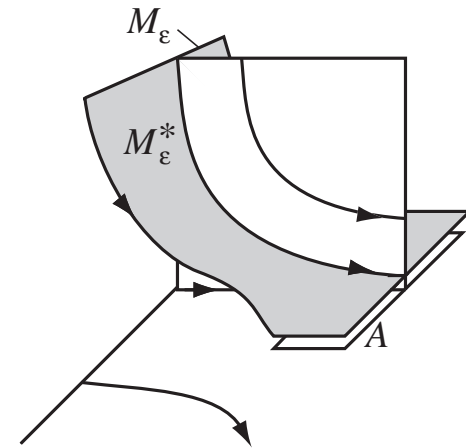
- (1) For $0 < \varepsilon < \varepsilon_0$, $\{(x, y, c_1, \dots, c_m) : (y, c_1) \in A, x = \tilde{x}(y, c_1, \varepsilon), \text{ and } (c_2, \dots, c_m) = \tilde{c}(y, c_1, \varepsilon)\}$ is contained in M_ε^* .
- (2) $\tilde{x}(y, c_1, 0) = 0$.
- (3) $\tilde{c}(y, c_1, 0) = \tilde{c}(0, c_1, \varepsilon) = 0$.
- (4) As $\varepsilon \rightarrow 0$, $(\tilde{x}, \tilde{c}) \rightarrow 0$ exponentially, along with its derivatives with respect to all variables.



(a)



(b)



(c)

Exchange Lemma of Jones and Tin

Consider again:

$$\begin{aligned}\dot{x} &= A(x, y, c, \varepsilon)x, \\ \dot{y} &= B(x, y, c, \varepsilon)y, \\ \dot{c} &= \varepsilon((1, 0, \dots, 0) + L(x, y, c, \varepsilon)xy),\end{aligned}$$

$(x, y, c) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^m$ (contracting \times expanding \times center),
 $A(0, 0, c, 0)$ has eigenvalues with negative real part,
 $B(0, 0, c, 0)$ has eigenvalues with positive real part.

Assume:

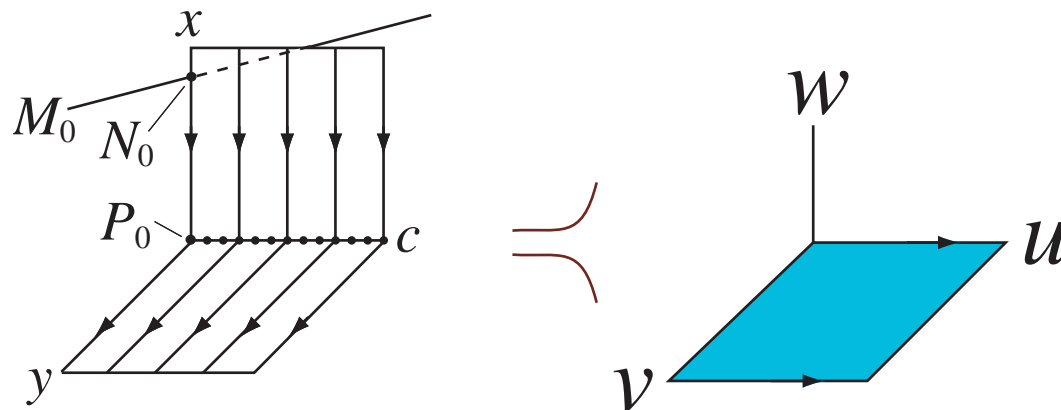
- (1) For each ε , M_ε is a submanifold of xyz -space of dimension $l + p$, $0 \leq p \leq m - 1$.
- (2) $M = \{(x, y, c, \varepsilon) : (x, y, c) \in M_\varepsilon\}$ is itself a manifold.
- (3) M_0 meets xc -space transversally in a manifold N_0 of dimension p .
- (4) N_0 projects smoothly to a submanifold P_0 of c -space of dimension p .
- (5) The vector $(1, 0, \dots, 0)$ is not tangent to P_0 .

Then:

- (1) Each M_ε meets xc -space transversally in a manifold N_ε of dimension p .
- (2) N_ε projects smoothly to a submanifold P_ε of c -space of dimension p .
- (3) The vector $(1, 0, \dots, 0)$ is not tangent to P_ε .

After a change of coordinates $c = (u, v, w) \in \mathbb{R} \times \mathbb{R}^p \times \mathbb{R}^{m-1-p}$ that takes each P_ε to v -space, the system can be put in the form

$$\begin{aligned}\dot{x} &= A(x, y, u, v, w, \varepsilon)x, \\ \dot{y} &= B(x, y, u, v, w, \varepsilon)y, \\ \dot{u} &= \varepsilon(1 + e(x, y, u, v, w, \varepsilon)xy), \\ \dot{v} &= \varepsilon F(x, y, u, v, w, \varepsilon)xy, \\ \dot{w} &= \varepsilon G(x, y, u, v, w, \varepsilon)xy.\end{aligned}$$



Under the forward flow, each M_ε becomes a manifold M_ε^* of dimension $l + p + 1$. Each P_ε becomes a manifold P_ε^* of dimension $p + 1$, which in our coordinates is just uv -space.

Theorem 3 (Exchange Lemma of Jones and Tin, 2009). Let $0 < u^*$. Let A be a small neighborhood of $(0, u^*, 0)$ in yuv -space. Then for small $\varepsilon_0 > 0$ there are smooth function $\tilde{x} : A \times [0, \varepsilon_0) \rightarrow \mathbb{R}^k$ and $\tilde{w} : A \times [0, \varepsilon_0) \rightarrow \mathbb{R}^{m-p-1}$ such that:

- (1) For $0 < \varepsilon < \varepsilon_0$, $\{(x, y, u, v, w) : (y, u, v) \in A \text{ and } (x, w) = (\tilde{x}, \tilde{w})(y, u, v, \varepsilon)\}$ is contained in M_ε^* .
- (2) $\tilde{x}(y, u, v, 0) = 0$.
- (3) $\tilde{w}(y, u, v, 0) = \tilde{w}(0, u, v, \varepsilon) = 0$.
- (4) As $\varepsilon \rightarrow 0$, $(\tilde{x}, \tilde{w}) \rightarrow 0$ exponentially, along with its derivatives with respect to all variables.

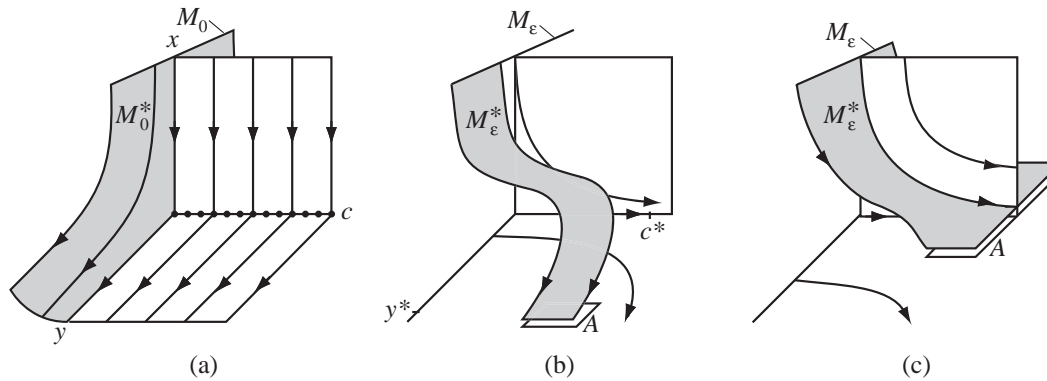
Remark

The theorem also applies to

$$\begin{aligned}\dot{x} &= A(x, y, c, \varepsilon)x, \\ \dot{y} &= B(x, y, c, \varepsilon)y, \\ \dot{c} &= \varepsilon(1, 0, \dots, 0) + L(x, y, c, \varepsilon)xy,\end{aligned}$$

It is really about perturbations of systems with a family of normally hyperbolic equilibria, not about slow-fast systems.

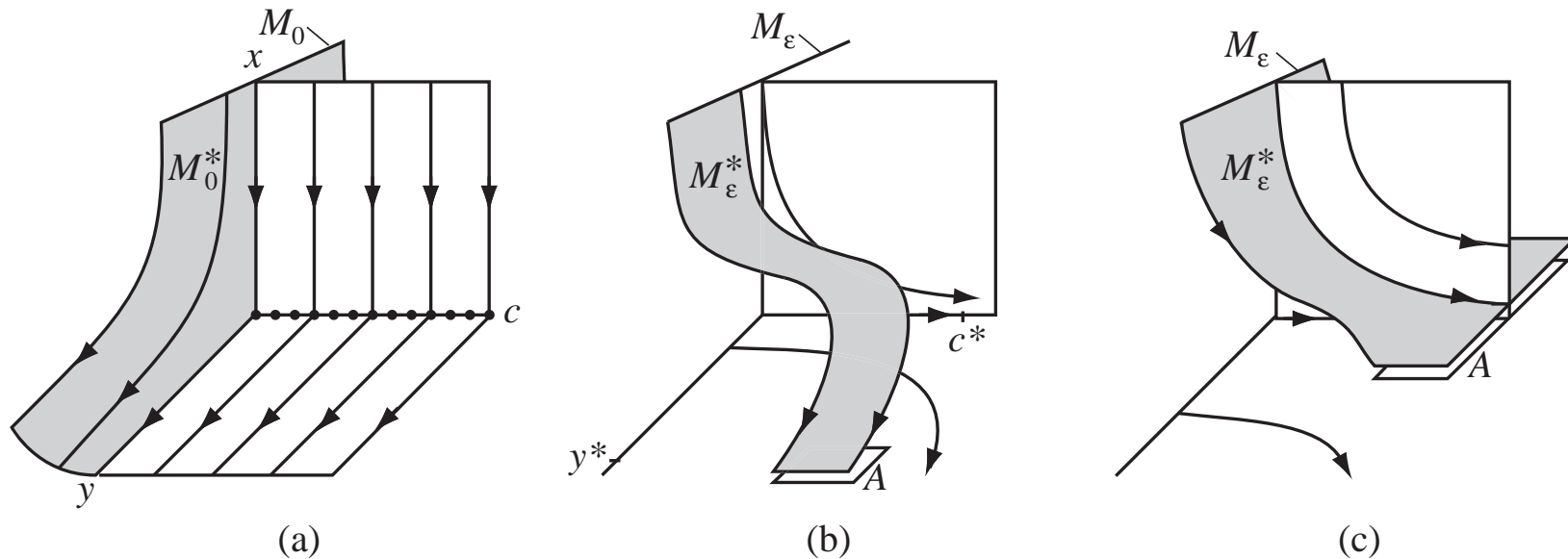
General Exchange Lemma



Important Features of the Exchange Lemma

- (1) There is a normally hyperbolic invariant manifold (c -space) and a small parameter ϵ .
- (2) There is a collection of submanifolds M_ϵ of xyz -space such that $M = \{(x, y, c, \epsilon) : (x, y, c) \in M_\epsilon\}$ is itself a manifold. M_ϵ meets xc -space transversally in a manifold N_ϵ (in picture, a point).
- (3) N_ϵ projects along the stable fibration of xc space to a submanifold P_ϵ of c -space of the same dimension (in picture, a point).
- (4) For $\epsilon > 0$, the vector field is not tangent to P_ϵ .
- (5) For small $\epsilon > 0$, the flow on c -space is followed for a long time.
- (6) It takes P_ϵ to a set P_ϵ^* of dimension one greater. As $\epsilon \rightarrow 0$, the limit of $P_\epsilon^* \neq$ where the limiting DE takes P_0 . Nevertheless, the limit of P_ϵ^* exists and has the same dimension. Call it P_0^* .
- (7) As $\epsilon \rightarrow 0$, $M_\epsilon^* \rightarrow W^u(P_0^*)$.

General Exchange Lemma (S., 2007). (1)–(6) plus technical assumptions imply (7).



What's the point?

- To understand the flow on the normally hyperbolic invariant manifold for $\epsilon > 0$ may require rectification, blowing-up, etc.
- Once you've done this work, the General Exchange Lemma deals with the remaining dimensions.

Loss-of-Stability Turning Points: Liu's Exchange Lemma

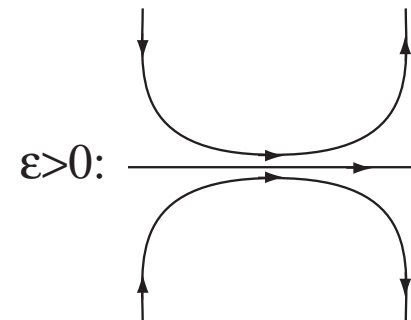
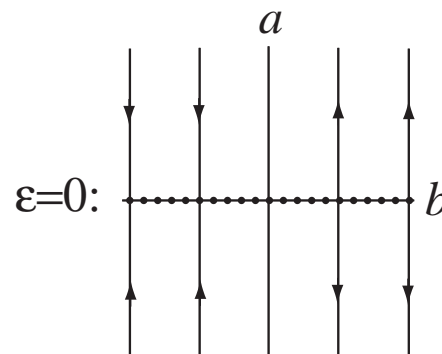
Liu considers a slow-fast system

$$\begin{aligned}\dot{a} &= f(a, b, \varepsilon), \\ \dot{b} &= \varepsilon g(a, b, \varepsilon),\end{aligned}$$

with $a \in \mathbb{R}^{k+l+1}$ and $b \in \mathbb{R}^{m-1}$, $m \geq 2$. Assume:

- (1) $f(0, b, \varepsilon) = 0$. (Hence for each ε , b -space is invariant, and for $\varepsilon = 0$ it consists of equilibria.)
- (2) $D_a f(0, b, 0)$ has
 - k eigenvalues with negative real part;
 - l eigenvalues with positive real part;
 - a last eigenvalue $\nu(b)$ such that $\nu(0) = 0$.
- (3) $D\nu(0)g(0, 0, 0) > 0$.

$k+l=0, m=2$



After a change of coordinates:

$$\dot{x} = A(x, y, z, c, \varepsilon)x,$$

$$\dot{y} = B(x, y, z, c, \varepsilon)y,$$

$$\dot{z} = h(z, c, \varepsilon)z + k(x, y, z, c, \varepsilon)xy,$$

$$\dot{c} = \varepsilon((1, 0, \dots, 0) + l(z, c, \varepsilon)z + L(x, y, z, c, \varepsilon)xy),$$

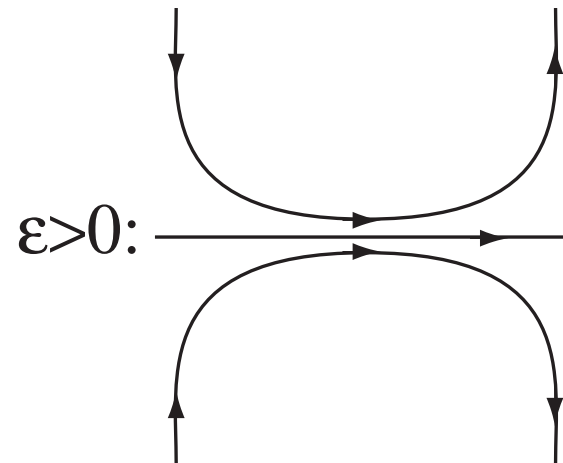
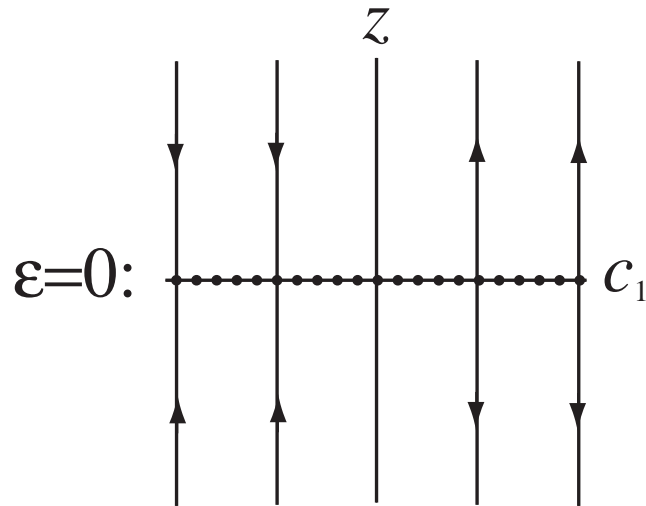
$$(x, y, z, c) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}^{m-1},$$

$A(0, 0, 0, c, 0)$ has eigenvalues with negative real part,

$B(0, 0, 0, c, 0)$ has eigenvalues with positive real part,

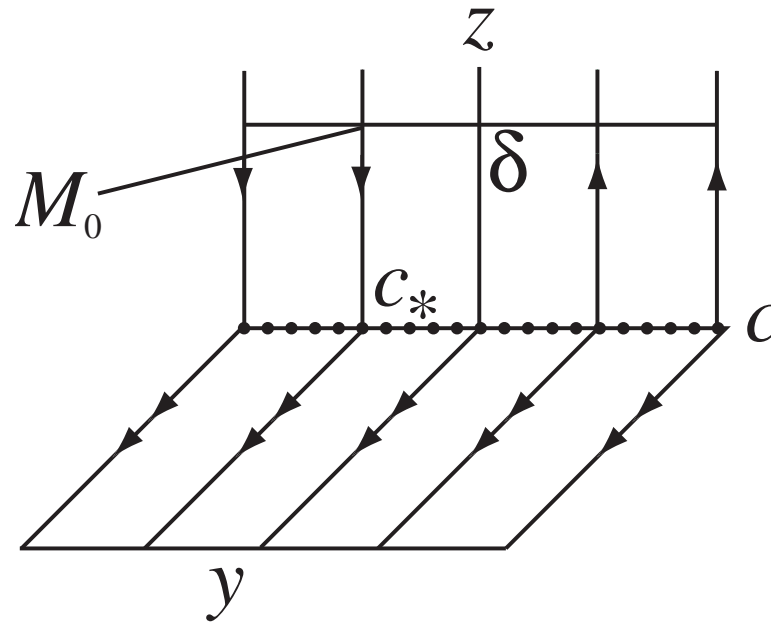
$$h(0, (0, c_2, \dots, c_{m-1}), 0) = 0,$$

$$\frac{\partial h}{\partial c_1} > 0.$$



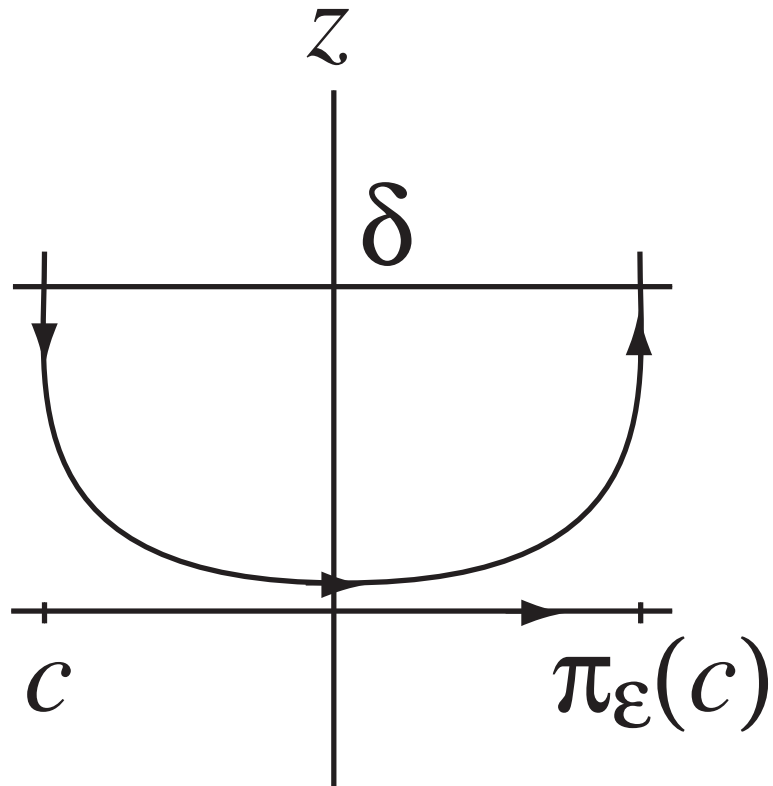
Assume:

- (1) $m = 2$ (for simplicity, so c -space is one-dimensional).
- (2) For each ε , M_ε is a submanifold of $xyzc$ -space of dimension l .
- (3) $M = \{(x, y, z, c, \varepsilon) : (x, y, z, c) \in M_\varepsilon\}$ is itself a manifold.
- (4) M_0 meets xzc -space transversally at a point $(x_*, 0, \delta, c_*)$ with $\delta \neq 0$ and $c_* < 0$.
We may assume that $M \subset \{(x, y, z, c, \varepsilon) : z = \delta\}$.



Each M_ε meets xzc -space transversally at $(x, y, z, c) = (x(\varepsilon), 0, c(\varepsilon), \delta)$ with $(x(0), c(0)) = (x_*, c_*)$.

For $\varepsilon > 0$ define Poincaré maps on $z = \delta$ by $c \rightarrow \pi_\varepsilon(c)$.



Define π_0 implicitly by

$$\int_c^{\pi_0(c)} h(0, u, 0) du = 0.$$

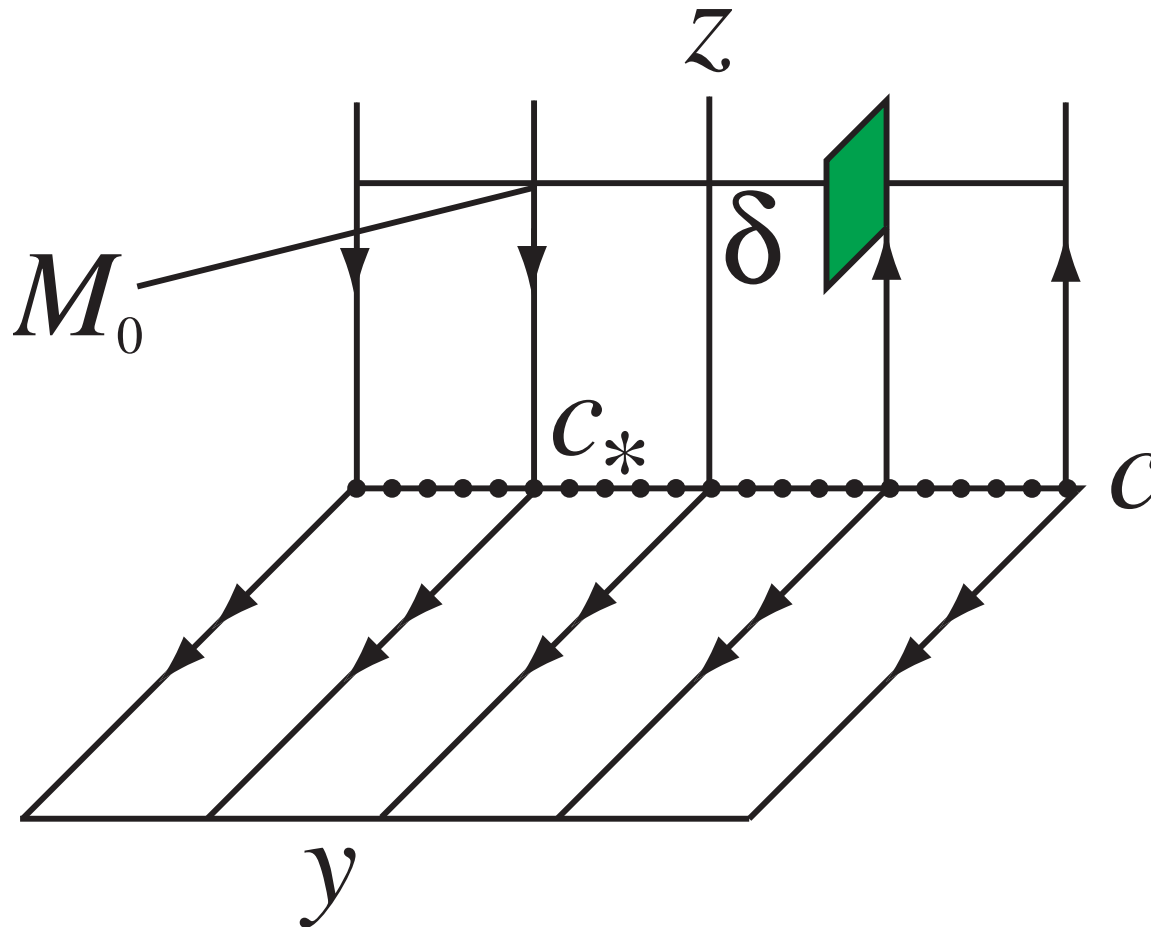
$\pi_\varepsilon \rightarrow \pi_0$, along with its derivatives, as $\varepsilon \rightarrow 0$ (De Maesschalck, 2008).

Under the forward flow, each M_ε becomes a manifold M_ε^* of dimension $l + 1$.

Theorem 4 (Liu's Exchange Lemma, 2000). In zc -space, consider a short integral curve C_ε through $(z, c) = (\delta, \pi_\varepsilon(c(\varepsilon)))$. Let

$$A_\varepsilon = \{(x, y, z, c) : x = 0, \|y\| \text{ is small}, (z, c) \in C_\varepsilon\}.$$

Then M_ε^* is close to A_ε . As $\varepsilon \rightarrow 0$ the distance goes to 0 exponentially.



Gain-of-Stability Turning Points (Rarefactions in the Dafermos Regularization)

Consider the system

$$\begin{aligned}\dot{u} &= v, \\ \dot{v} &= (A(u) - xI)v, \\ \dot{x} &= \varepsilon,\end{aligned}$$

with $(u, v, x) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ and $A(u)$ an $n \times n$ matrix.

Let $n = k + l + 1$. Assume that on an open set U in \mathbb{R}^n :

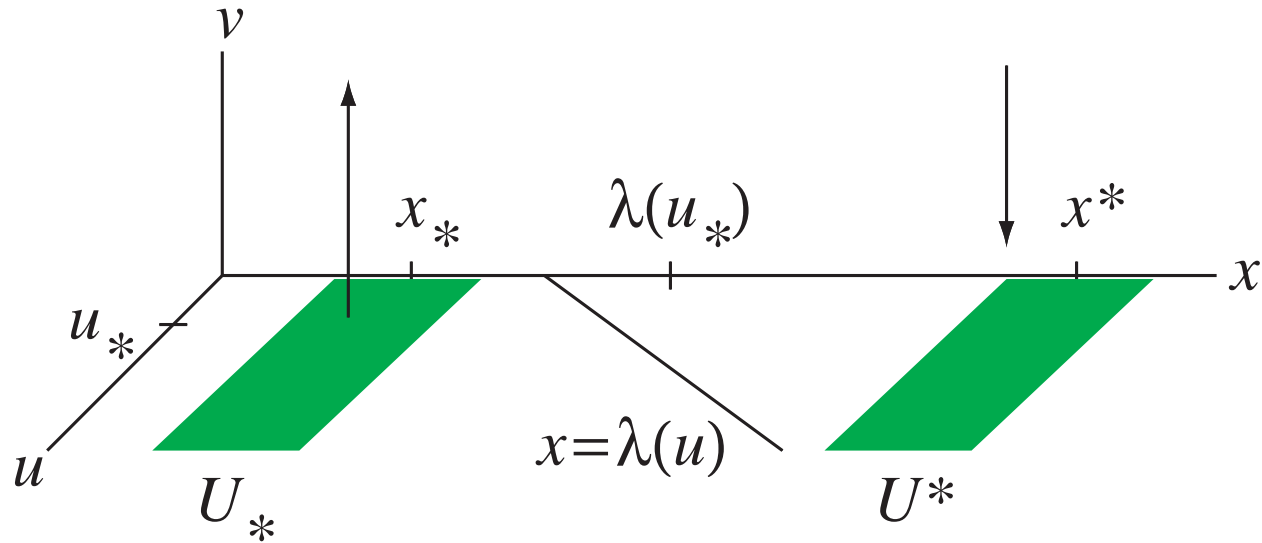
- There are numbers $\lambda_1 < \lambda_2$ such that $A(u)$ has
 - k eigenvalues with real part less than λ_1 ,
 - l eigenvalues with real part greater than λ_2 ,
 - a simple real eigenvalue $\lambda(u)$ with $\lambda_1 < \lambda(u) < \lambda_2$.
- $A(u)$ has an eigenvector $r(u)$ for the eigenvalue $\lambda(u)$ such that $D\lambda(u)r(u) = 1$.

Notice ux -space is invariant for every ε . For $\varepsilon = 0$ it consists of equilibria, but loses normal hyperbolicity along the surface $x = \lambda(u)$.

Choose $u_* \in U$, x_* , x^* such that $\lambda_1 < x_* < \lambda(u_*) < x^* < \lambda_2$. Let

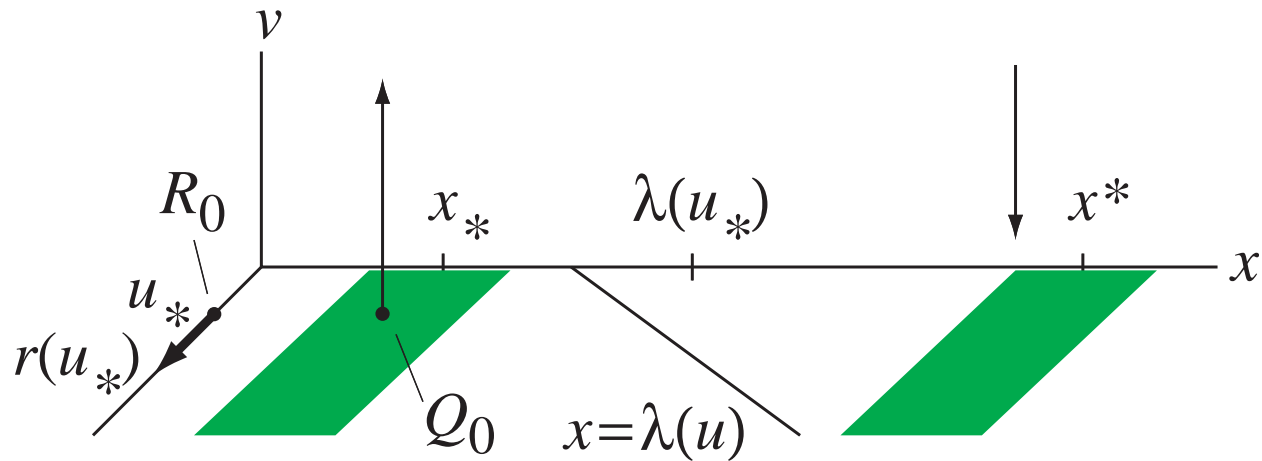
$$U_* = \{(u, v, x) : u \in U, v = 0, |x - x_*| < \delta\},$$

$$U^* = \{(u, v, x) : u \in U, v = 0, |x - x^*| < \delta\}.$$



For $\varepsilon = 0$, U_* and U^* are normally hyperbolic manifolds of equilibria of dimension $n + 1$. For U_* , the stable and unstable manifolds of each point have dimensions k and $l + 1$ respectively; for U^* , the stable and unstable manifolds of each point have dimensions $k + 1$ and l respectively.

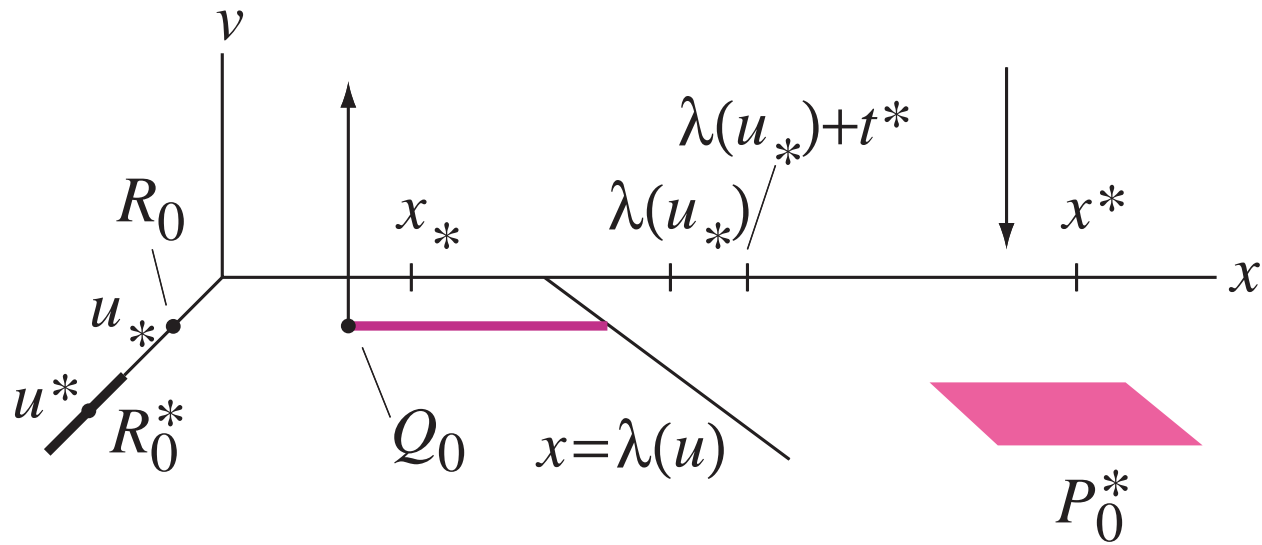
For $\varepsilon > 0$, U_* and U^* are normally hyperbolic invariant manifolds on which the system reduces to $\dot{u} = 0$, $\dot{x} = \varepsilon$.



For each $\varepsilon \geq 0$, let M_ε be a submanifold of uvx -space of dimension $l + 1 + p$, $0 \leq p \leq n - 1$. Assume:

- $M = \{(u, v, x, \varepsilon) : (u, v, x) \in M_\varepsilon\}$ is itself a manifold.
- M_0 is transverse to $W_0^s(U_*)$ at a point in the stable fiber of $(u_*, 0, x_*)$. The intersection of M_0 and $W_0^s(U_*)$ is a smooth manifold S_0 of dimension p .
- S_0 projects smoothly to a submanifold Q_0 of ux -space of dimension p .
- The vector $(\dot{u}, \dot{x}) = (0, 1)$ is not tangent to Q_0 . Therefore Q_0 projects smoothly to a submanifold R_0 of u -space of dimension p .
- $r(u_*)$ is not tangent to R_0 .

Under the flow, each M_ε becomes a manifold M_ε^* of dimension $l + 2 + p$.



Let $\phi(t, u)$ be the flow of $\dot{u} = r(u)$. Choose $t^* > 0$ such that $\lambda(u_*) + t^* < x^*$. Let

$$R_0^* = \cup_{|t-t^*| < \delta} \phi(t, R_0), \quad P_0^* = \{(u, v, x) : u \in R_0^*, v = 0, |x - x^*| < \delta\}.$$

R_0^* and P_0^* have dimensions $p + 1$ and $p + 2$ respectively.

Let $u^* = \phi(t^*, u_*)$.

Theorem 7. Near $(u^*, 0, x^*)$, M_ε^* is close to $W_0^u(P_0^*)$.

Generalized Deng's Lemma

In the literature, there are three ways to prove exchange lemmas:

- Jones and Kopell's approach, which is to follow the tangent space to M_ε forward using the extension of the linearized differential equation to differential forms.
- Brunovsky's approach, which is to locate M_ε^* by solving a boundary value problem in Silnikov variables.
- Krupa–Sandstede–Szmolyan approach (1997), using Lin's method.

We follow Brunovsky's approach, which is based on work of Bo Deng (1990). Brunovsky generalized a lemma of Deng that gives estimates on solutions of boundary value problems in Silnikov variables. Our proof of the Generalized Exchange Lemma is based on a further generalization of Deng's Lemma.

Let $(x, y, c) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R}^m$. Let V be an open subset of \mathbb{R}^m . On a neighborhood of $\{0\} \times \{0\} \times V$, consider the C^{r+1} differential equation

$$\begin{aligned}\dot{x} &= A(x, y, c)x, \\ \dot{y} &= B(x, y, c)y, \\ \dot{c} &= C(c) + E(x, y, c)xy.\end{aligned}$$

Let $\phi(t, c)$ be the flow of $\dot{c} = C(c)$. For each $c \in V$ there is a maximal interval I_c containing 0 such that $\phi(t, c) \in V$ for all $t \in I_c$. Let the linearized solution operator of the system, with $\varepsilon = 0$, along the solution $(0, 0, \phi(t, c^0))$ be

$$\begin{pmatrix} \bar{x}(t) \\ \bar{y}(t) \\ \bar{c}(t) \end{pmatrix} = \begin{pmatrix} \Phi^s(t, s, c^0) & 0 & 0 \\ 0 & \Phi^u(t, s, c^0) & 0 \\ 0 & 0 & \Phi^c(t, s, c^0) \end{pmatrix} \begin{pmatrix} \bar{x}(s) \\ \bar{y}(s) \\ \bar{c}(s) \end{pmatrix}$$

Assume:

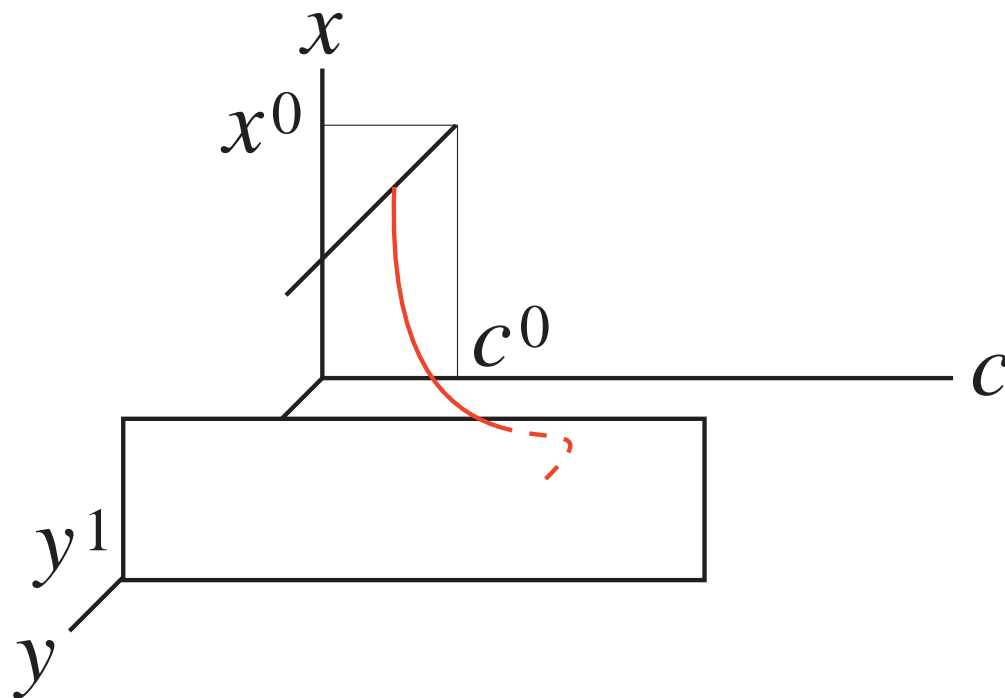
(E1) There are numbers $\lambda_0 < 0 < \mu_0$, $\beta > 0$, and $M > 0$ such that for all $c^0 \in N$ and $s, t \in I_{c^0}$,

$$\begin{aligned}\|\Phi^s(t, s, c^0)\| &\leq Me^{\lambda_0(t-s)} && \text{if } t \geq s, \\ \|\Phi^u(t, s, c^0)\| &\leq Me^{\mu_0(t-s)} && \text{if } t \leq s, \\ \|\Phi^c(t, s, c^0)\| &\leq Me^{\beta|t-s|} && \text{for all } t, s.\end{aligned}$$

(E2) $\lambda_0 + r\beta < 0 < \lambda_0 + \mu_0 - r\beta$.

We wish to study solutions of Silnikov's boundary value problem on an interval $0 \leq t \leq \tau$:

$$x(0) = x^0, \quad y(\tau) = y^1, \quad c(0) = c^0.$$



We denote the solution of Silnikov's boundary value problem

$$x(0) = x^0, \quad y(\tau) = y^1, \quad c(0) = c^0.$$

by $(x, y, c)(t, \tau, x^0, y^1, c^0)$.

Theorem 9 (Generalized Deng's Lemma, S. 2008). Let V_0 and V_1 be compact subsets of V such that $V_0 \subset \text{Int}(V_1)$. For each $c^0 \in V_0$ let J_{c^0} be the maximal interval such that $\phi(t, c^0) \in \text{Int}(V_1)$ for all $t \in J_{c^0}$. Then for λ and μ a little closer to 0 than λ_0 and μ_0 , there is a number $\delta_0 > 0$ such that if $\|x^0\| \leq \delta_0$, $\|y^1\| \leq \delta_0$, $c^0 \in V_0$, and $\tau > 0$ is in J_{c^0} , then Silnikov's boundary value problem has a solution $(x, y, c)(t, \tau, x^0, y^1, c^0)$ on the interval $0 \leq t \leq \tau$. Moreover, there is a number $K > 0$ such that for all (t, τ, x^0, y^1, c^0) as above,

$$\begin{aligned} \|x(t, \tau, x^0, y^1, c^0)\| &\leq Ke^{\lambda t}, \\ \|y(t, \tau, x^0, y^1, c^0)\| &\leq Ke^{\mu(t-\tau)}, \\ \|c(t, \tau, x^0, y^1, c^0) - \phi(t, c^0)\| &\leq Ke^{\lambda t + \mu(t-\tau)}. \end{aligned}$$

In addition, if \mathbf{i} is any $|\mathbf{i}|$ -tuple of integers between 1 and $2 + k + l + m$, with $1 \leq |\mathbf{i}| \leq r$, then

$$\begin{aligned} \|D_{\mathbf{i}}x(t, \tau, x^0, y^1, c^0)\| &\leq Ke^{(\lambda + |\mathbf{i}|\beta)t}, \\ \|D_{\mathbf{i}}y(t, \tau, x^0, y^1, c^0)\| &\leq Ke^{(\mu - |\mathbf{i}|\beta)(t-\tau)}, \\ \|D_{\mathbf{i}}c(t, \tau, x^0, y^1, c^0) - D_{\mathbf{i}}\phi(t, c^0)\| &\leq Ke^{(\lambda + |\mathbf{i}|\beta)t + (\mu - |\mathbf{i}|\beta)(t-\tau)}. \end{aligned}$$

Proof of Deng's Lemma

$$\begin{aligned}\dot{x} &= A(x, y, c)x, \\ \dot{y} &= B(x, y, c)y, \\ \dot{c} &= C(c) + E(x, y, c)xy.\end{aligned}$$

Let $c = \phi(t, c^0) + z$. Rewrite as

$$\begin{aligned}\dot{x} &= A(t, c^0)x + f(t, c^0, x, y, z), \\ \dot{y} &= B(t, c^0)y + g(t, c^0, x, y, z), \\ \dot{z} &= C(t, c^0)z + \theta(t, c^0, z) + h(t, c^0, x, y, z),\end{aligned}$$

with $A(t, c^0), B(t, c^0), C(t, c^0)$ linear. Silnikov's problem:

$$x(0) = x^0, \quad y(\tau) = y^1, \quad c(0) = c^0.$$

$(x(t), y(t), c(t))$ is a solution of Silnikov's problem if and only if $c(t) = \phi(t, c^0) + z(t)$ and $\eta(t) = (x(t), y(t), z(t))$ satisfies

$$\begin{aligned}x(t) &= \Phi^s(t, 0, c^0)x^0 + \int_0^t \Phi^s(t, s, c^0)f(s, c^0, \eta(s)) ds, \\ y(t) &= \Phi^u(t, \tau, c^0)y^1 + \int_\tau^t \Phi^u(t, s, c^0)g(s, c^0, \eta(s)) ds, \\ z(t) &= \int_0^t \Phi^c(t, s, c^0)(\theta(s, c^0, z(s)) + h(s, c^0, \eta(s))) ds,\end{aligned}$$

$$\begin{aligned}
x(t) &= \Phi^s(t, 0, c^0)x^0 + \int_0^t \Phi^s(t, s, c^0)f(s, c^0, \eta(s)) ds, \\
y(t) &= \Phi^u(t, \tau, c^0)y^1 + \int_\tau^t \Phi^u(t, s, c^0)g(s, c^0, \eta(s)) ds, \\
z(t) &= \int_0^t \Phi^c(t, s, c^0)(\theta(s, c^0, z(s)) + h(s, c^0, \eta(s))) ds,
\end{aligned}$$

Regard the right-hand side as a map from a weighted space of functions on $[0, \tau]$ into itself. Show there is a fixed point and estimate derivatives (which are fixed points of inhomogeneous linear maps).

How Exchange Lemmas are proved

Consider the Jones-Kopell Exchange Lemma. Situation: $(x, y, u, v) \in \mathbb{R}^k \times \mathbb{R}^l \times \mathbb{R} \times \mathbb{R}^{m-1}$,

$$\begin{aligned}\dot{x} &= Ax, \\ \dot{y} &= By, \\ \dot{u} &= \varepsilon + Cxy, \\ \dot{v} &= Exy,\end{aligned}$$

with A, B, C, E functions of $(x, y, u, v, \varepsilon)$. Eigenvalues of A have negative real part, eigenvalues of B have positive real part.

M is parameterized by y and ε :

$$\begin{aligned}x &= x(\varepsilon) + L(y, \varepsilon)y, \\ u &= M(y, \varepsilon)y, \\ v &= N(y, \varepsilon)y.\end{aligned}$$

Given $\varepsilon > 0$ and (y^1, u^1) near (y, u^*) . Find y^0 such that if we set

$$(1) \quad x^0 = x(\varepsilon) + L(y^0, \varepsilon)y^0,$$

$$(2) \quad u^0 = M(y^0, \varepsilon)y^0,$$

$$(3) \quad v^0 = N(y^0, \varepsilon)y^0,$$

$$(4) \quad \tau = \frac{u^1 - u^0}{\varepsilon}$$

then the solution of Silnikov's boundary value problem with

$$x(0) = x^0, \quad y(\tau) = y^1, \quad u(0) = u^0, \quad v(0) = v^0$$

has $y(0) = y_0$. Then M^* includes the graph of

$$(x(\tau, \tau, x^0, y^1, u^0, v^0), v(\tau, \tau, x^0, y^1, u^0, v^0)).$$

Note that the arguments depend on (y^1, u^1, ε) . Now estimate (x, v) and their derivatives using Deng's Lemma.

To find y^0 as a function of (y^1, u^1, ε) , consider the mapping $(y^0, y^1, u^1, \varepsilon) \rightarrow y(0)$ given by the above construction.

Show that for fixed (y^1, u^1, ε) , this mapping is a contraction of a closed ball in y^0 -space. Find the fixed point.