

A geometric mechanism for diffusion in a priori unstable Hamiltonian systems: verification in concrete examples

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Outline

- 1 Set up
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- 3 An example
- 4 Proof for the example
 - I: A NHIM with transverse manifolds
 - II: Inner dynamics
 - III: Outer dynamics
 - IV: Construction of a transition chain

Diffusion in a priori unstable systems

- **Diffusion** Changes of order 1 in the actions (instabilities) for arbitrarily small perturbations of integrable systems.
- **A priori unstable Hamiltonian systems:**

$$H_\epsilon(p, q, I, \varphi, t) = H_0(p, q, I) + \epsilon h(p, q, I, \varphi, t; \epsilon)$$

For $\epsilon = 0$, $H_0(p, q, I)$ is autonomous ($H_0 = E = \text{constant}$) **integrable** but with some **saddle** variables p, q . Typical example: one rotor (or more) plus one (or more) pendulum.

- **Main question:** What happens to $E(t)$ for small $\epsilon \neq 0$? Is there **global instability**?: $E(t) - E(0) = \mathcal{O}(1)$ or even $E(t) \rightarrow \infty$?
- **Many contributions:** [Chierchia-Gallavotti94-98], [Berti-Biasco-Bolle03], [Marco-Sauzin03] [Treschev04], [Cheng-Yan04], [Gidea-Llave06], [Delshams-Llave-Seara06], [Delshams-H09].

Instability for a priori unstable Hamiltonian systems

We consider a 2π -periodic in time perturbation of a **pendulum** and a **rotor** described by the non-autonomous Hamiltonian,

$$\begin{aligned} H_\epsilon(p, q, I, \varphi, t) &= H_0(p, q, I) + \epsilon h(p, q, I, \varphi, t; \epsilon) \\ &= P_\pm(p, q) + \frac{1}{2}I^2 + \epsilon h(p, q, I, \varphi, t; \epsilon) \end{aligned} \quad (1)$$

where $(p, q, I, \varphi, t) \in (\mathbb{R} \times \mathbb{T})^2 \times \mathbb{T}$ and

$$P_\pm(p, q) = \pm \left(\frac{1}{2}p^2 + V(q) \right) \quad (2)$$

and $V(q)$ is a 2π -periodic function. We will refer to $P_\pm(p, q)$ as the *pendulum*.

Main result for a priori unstable systems

Theorem

Consider the Hamiltonian (1) where V and h are uniformly C^{r+2} for $r \geq r_0$, sufficiently large. Assume also that

H1 The potential $V : \mathbb{T} \rightarrow \mathbb{R}$ has a unique global maximum at $q = 0$ which is non-degenerate. Denote by $(q_0(t), p_0(t))$ an orbit of the pendulum $P_{\pm}(p, q)$ homoclinic to $(0, 0)$.

H2 The Melnikov potential, associated to h (and to the homoclinic orbit (p_0, q_0)):

$$\mathcal{L}(I, \varphi, s) = - \int_{-\infty}^{+\infty} (h(p_0(\sigma), q_0(\sigma), I, \varphi + I\sigma, s + \sigma; 0) - h(0, 0, I, \varphi + I\sigma, s + \sigma; 0)) d\sigma \quad (3)$$

satisfies concrete non-degeneracy conditions.

H3 The perturbation term h satisfies concrete non-degeneracy conditions.

Then, there is $\epsilon^* > 0$ such that for $0 < |\epsilon| < \epsilon^*$, and for any interval $[I_-^*, I_+^*]$, there exists a trajectory $\tilde{x}(t)$ of the system (1) such that for some $T > 0$,

$$I(\tilde{x}(0)) \leq I_-^*; \quad I(\tilde{x}(T)) \geq I_+^*.$$

- Hypotheses **H1**, **H2** and **H3** are \mathcal{C}^2 **generic**.
- The previous Theorem was proven in [DLS06], assuming that h was a trigonometric polynomial in the variables (φ, s) , which is a non-generic assumption, and without this assumption in [DG09].

[DLS06] A. Delshams, R. de la Llave and T.M. Seara. A geometric mechanism for diffusion in Hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model. *Mem. Amer. Math. Soc.*, 179 (844), 2006.

[DG09] A. Delshams, G. H. Geography of resonances and Arnold diffusion in a priori unstable Hamiltonian systems. *Nonlinearity*, 22 (1997–2077), 2009.

Idea of the proof

- Find a big invariant object: a **NIHM** (normally hyperbolic invariant manifold: a global version of a center manifold) $\tilde{\Lambda}$ with **transverse** associated stable and unstable manifolds along a homoclinic manifold Γ : $\mathcal{W}^u(\tilde{\Lambda}) \pitchfork_{\Gamma} \mathcal{W}^s(\tilde{\Lambda})$.
- Compute the invariant objects (typically tori \mathcal{T}) which may prevent instability for the **inner dynamics** of the NHIM.
- Compute the **scattering map** $S : \tilde{\Lambda} \rightarrow \tilde{\Lambda}$ on the NHIM associated to Γ and consider it as an **outer** dynamics on the NHIM (a second dynamics on Γ).
- Check that $S(\mathcal{T}_{I_i}) \pitchfork \mathcal{T}_{I_{i+1}}$ for a sequence of tori $\{\mathcal{T}_{I_i}\}_{i=1}^N$ with $|I_N - I_1| = \mathcal{O}(1)$, and construct a **transition chain** of whiskered tori, i.e. $\mathcal{W}^u(\mathcal{T}_{I_i}) \pitchfork \mathcal{W}^s(\mathcal{T}_{I_{i+1}})$.
- Standard shadowing methods provide an orbit that follows closely the **transition chain**.

An example

Consider the Hamiltonian

$$H_\epsilon(p, q, I, \varphi, t) = \left(\frac{p^2}{2} + \cos q - 1 \right) + \frac{I^2}{2} + \epsilon f(q)g(\varphi, t). \quad (4)$$

with

$$f(q) = \cos(q), \quad (5)$$

and

$$g(\varphi, t) = \sum_{(k,l) \in \mathbb{N}^2} a_{k,l} \cos(k\varphi - lt - \sigma_{k,l}). \quad (6)$$

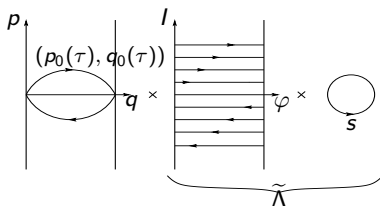
with

$$\hat{\alpha} \rho^{\beta k} r^{\beta l} \leq |a_{k,l}| \leq \alpha \rho^k r^l, \quad (7)$$

where $1 \leq \beta < 2$ and $0 < \hat{\alpha} < \alpha$. Moreover, $0 < \rho, r < 1$ are real numbers that will be chosen small enough.

DH11 A. Delshams and G. Huguet. A geometric mechanism of diffusion: Rigorous verification in a priori unstable Hamiltonian systems. *J. Diff. Eq.* 250(5): 2601–2623, 2011

$$\epsilon = 0$$



- Normally hyperbolic invariant manifold (3D)

$$\tilde{\Lambda} = \{(0, 0, I, \varphi, s) : (I, \varphi, s) \in \mathbb{R} \times \mathbb{T}^2\}$$

- Invariant manifolds (4D):

$$W^s \tilde{\Lambda} = W^u \tilde{\Lambda} = \{(p_0(\tau), q_0(\tau), I, \varphi, s) : \tau \in \mathbb{R}, I \in [I_-, I_+], (\varphi, s) \in \mathbb{T}^2\}$$

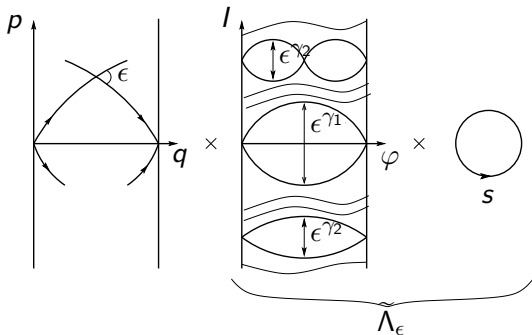
where

$$q_0(t) = 4 \arctan e^{\pm t}, \quad p_0(t) = 2 / \cosh t.$$

is the **separatrix** for positive p of the standard pendulum

$$P(p, q) = p^2/2 + \cos q - 1.$$

$$0 < \epsilon \ll 1$$



- $\tilde{\Lambda}$ persists without any deformation ($\tilde{\Lambda}_\epsilon = \tilde{\Lambda}$).
- $W^s \tilde{\Lambda}_\epsilon \cap W^u \tilde{\Lambda}_\epsilon$ along a homoclinic manifold Γ_ϵ .
- the sequence of invariant tori in the NHIM is destroyed creating gaps of size up to $\mathcal{O}(\epsilon)$.

Main result

Theorem

Assume that $r > 2(m+1)^2$ and $m \geq 10$, then there exists a *discrete sequence* of invariant tori $\{\mathcal{T}_i\}_{i=1}^N$ in $\tilde{\Lambda}_\epsilon$ such that:

- They are distributed along the actions in the interval (I_-, I_+) .
- They are $\mathcal{O}(\epsilon^{1+\eta})$ -closely spaced in terms of the action variables, where $0 < \eta \ll 1$.
- They are given by the *level sets* defined by equation $F(I, \varphi, s; \epsilon) = E$, where F is a \mathcal{C}^2 function F which has different expressions depending on the region of the phase space where invariant tori lie:
 - *Flat tori region*. Primary KAM tori.
 - *Big gaps region*. Primary KAM tori and Secondary KAM tori.

Proof: Averaging procedure + KAM Theorem.

Resonant averaging in our example

- Consider the Hamiltonian restricted to $\tilde{\Lambda}_\epsilon$

$$K(I, \varphi, s, \epsilon) = \frac{I^2}{2} + \epsilon g(\varphi, s)$$

where $(I, \varphi, s) \in \mathbb{R} \times \mathbb{T}^2$ and

$$g(\varphi, s) = \sum_{(k,l) \in \mathbb{N}^2} a_{k,l} \cos(k\varphi + ls).$$

- Look for a **change of coordinates** that eliminates the dependence on the angle variables at order ϵ .
- Does it always exist? No, it fails at resonances.
- Resonances** for this system, values of I such that

$$(I, 1) \cdot (k, l) = 0 \Rightarrow I = -\frac{l}{k}, \quad \text{for some } (k, l) \in \mathbb{N}^2$$

- **Remark:** The perturbation g has **infinite harmonics** in the angles (φ, s) .
- Consider the **truncated Fourier series** of g at order M

$$g^{[\leq M]}(I, \varphi, s; \epsilon) = \sum_{\substack{(k, l) \in \mathbb{N}^2, \\ |k| + |l| \leq M}} a_{k, l} \cos(k\varphi + ls),$$

where $M = M(\epsilon) \sim \epsilon^{-\rho}$, $\rho < 1/n$, (n regularity required for the application of KAM Theorem).

- Apply averaging method to Hamiltonian

$$K_0(I, \varphi, s, \epsilon) = \frac{I^2}{2} + \epsilon g^{[\leq M]}(\varphi, s).$$

- **Set of resonances** up to order 1:

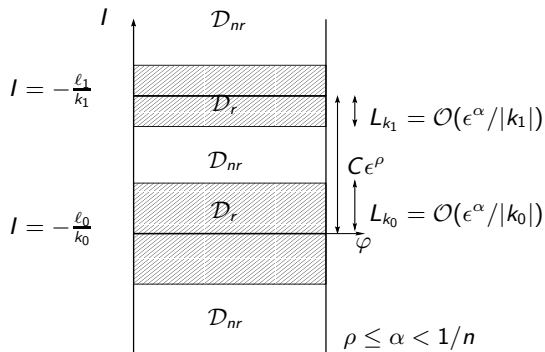
$$\mathcal{R}_1 = \{I = -\frac{l}{k} \in \mathbb{Q} \cap (I_-, I_+) : |k| + |l| \leq M\}$$

It is a finite set but depends on M and therefore on ϵ .

Two types of domains in the NHIM

- Non-resonant region up to order 1
- Resonant region up to order 1: One connected component

$$\mathcal{D}^{k_0, l_0} \equiv \left[-\frac{\ell_0}{k_0} - L_{k_0}, -\frac{\ell_0}{k_0} + L_{k_0} \right] \times \mathbb{T}^2$$



- Non-resonant region up to order 1:

$$K(I, \varphi, s, \epsilon) = \frac{I^2}{2} + \epsilon + \mathcal{O}(\epsilon^2)$$

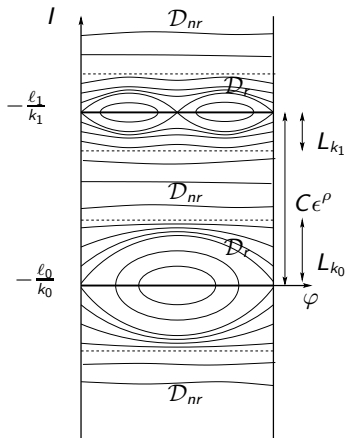
- Resonant region up to order 1:

Keep the harmonics corresponding to the resonance $I = -\ell_0/k_0$

$$K(I, \varphi, s, \epsilon) = \frac{I^2}{2} + \epsilon(1 + U^{k_0, \ell_0}(\theta)) + \mathcal{O}(\epsilon^2)$$

where $\theta = k_0\varphi + \ell_0 s$ and

$$U^{k_0, \ell_0}(\theta) = a_{k_0, \ell_0} \cos(\theta) + \sum_{t=2}^M a_{tk_0, t\ell_0} \cos(t\theta)$$



- H3' $U^{k_0, \ell_0}(\theta, 0)$ has a non-degenerate global **maximum** at $\theta = 0 \Rightarrow K$ is close to a pendulum

- Repeat the averaging procedure up to order m

$$K_m(I, \varphi, s; \epsilon) = Z^m(I, \varphi, s; \epsilon) + \epsilon^{m+1} R^m(I, \varphi, s; \epsilon)$$

- New resonances and resonant regions appear

- Non-resonant region** up to order m , D_{nr}^m :

$$Z^m(I, \varphi, s; \epsilon) = \frac{I^2}{2} + \epsilon \tilde{Z}^m(I; \epsilon).$$

(like a **rotor**)

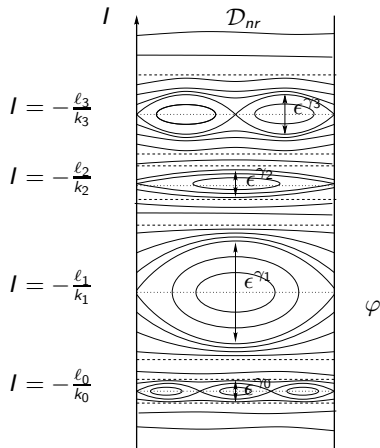
- Connected component \mathcal{D}^{k_0, l_0} of the **resonant region** D_r^m :

$$Z^m(I, \varphi, s; \epsilon) = \frac{I^2}{2} + \epsilon \tilde{Z}^m(I; \epsilon) + \epsilon^j U_m^{(k_0, l_0)}(\theta; \epsilon).$$

where $\theta = k_0 \varphi + l_0 s$ (like an **integrable pendulum**)

- The **size of the gaps** depends on the order of the resonance and the size of the Fourier coefficient associated to it.
- The error coming from the tail $g^{[>M]}$ of the Fourier series can be neglected provided that the perturbation g is regular enough.

Invariant objects in $\tilde{\Lambda}_\epsilon$ given by the integrable Hamiltonian $Z^m(I, \varphi, s; \epsilon)$



We consider two regions in $\tilde{\Lambda}_\epsilon$:

- The **big gaps region**.
Resonant regions with gaps of size bigger than ϵ .
Primary KAM tori and secondary KAM tori.
- **Flat tori region**.
Non resonant regions and resonant regions with gaps smaller than ϵ .
Primary KAM tori.

KAM Theorem

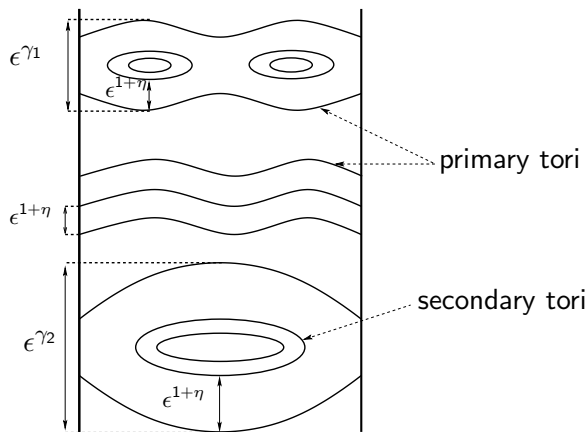
Which of them survive and at what distance when we add the perturbation term?

KAM theorem (Quantitative version of a KAM Theorem by Herman)

- **Flat tori region.** We can apply KAM Theorem straightforwardly. Tori at a distance $\epsilon^{1+\eta}$, with $0 < \eta \ll 1$ (**$m = 2$ is enough**).
- **Big gaps region.** The integrable system is close to a pendulum. We need to write the Hamiltonian in **action-angle variables**. Tori at a distance $\epsilon^{1+\eta}$, with $0 < \eta \ll 1$ (**$m = 10$ is enough**).

For $I \in [I_-, I_+]$, there exists in $\tilde{\Lambda}_\epsilon$ a **discrete sequence** of invariant tori \mathcal{T}_i (some primary and some secondary) which are $\epsilon^{1+\eta}$ -closely spaced, with $0 < \eta \ll 1$. They are given by the leaves L_E^F of a foliation \mathcal{F}_F .

Invariant objects of K in the NHIM $\tilde{\Lambda}_\epsilon$



Consider the **Melnikov potential**

$$\mathcal{L}(I, \varphi, s) = - \int_{-\infty}^{+\infty} (h(p_0(\sigma), q_0(\sigma), I, \varphi + I\sigma, s + \sigma; 0) - h(0, 0, I, \varphi + I\sigma, s + \sigma; 0)) d\sigma \quad (8)$$

H2' Given real numbers $I_- < I_+$, assume that for any value of $I \in (I_-, I_+)$ the map

$$\tau \in \mathbb{R} \mapsto \mathcal{L}(I, \varphi - I\tau, s - \tau)$$

has a **non-degenerate critical point** τ which is locally given by the implicit function theorem in the form

$$\tau = \tau^*(I, \varphi, s),$$

with τ^* a smooth function.

Then, for ϵ small enough, there exists a locally unique point \tilde{z} of the form

$$\tilde{z}(I, \varphi, s; \epsilon) = \tilde{z}(\tau^*(I, \varphi, s), I, \varphi, s; \epsilon) = (p_0(\tau) + \mathcal{O}(\epsilon), q_0(\tau) + \mathcal{O}(\epsilon), I, \varphi, s),$$

such that $W^s(\tilde{\Lambda}_\epsilon) \pitchfork W^u(\tilde{\Lambda}_\epsilon)$ at \tilde{z} .

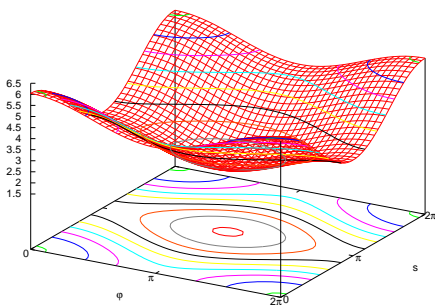
Homoclinic manifold

$$\Gamma_\epsilon = \{z = z(I, \varphi, s; \epsilon) = (p_0(\tau^*), q_0(\tau^*), I, \varphi, s); \tau^* = \tau^*(I, \varphi, s)\}.$$

Graph and level curves of the **first order approximation** in ρ , r of the Melnikov potential

$$\mathcal{L}^{[\leq 1]}(I, \varphi, s) = A_{0,0} + A_{1,0} \cos \varphi + A_{0,1}(I) \cos s,$$

for $0 < A_{1,0} < A_{0,1} < 1$



Four non-degenerate critical points: maximum $(0,0)$, minimum (π,π) and two saddles $(0,\pi)$, $(\pi,0)$.

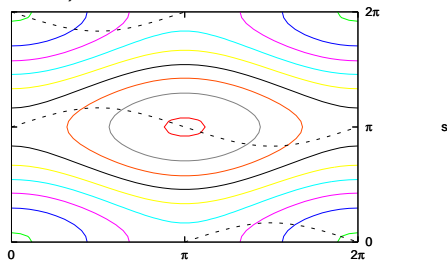
Given a fixed value of I , we first look at the points $(\varphi, s) \in \mathbb{T}^2$ such that $\tau^*(I, \varphi, s) = 0$. They satisfy the implicit equation

$$I \frac{\partial \mathcal{L}}{\partial \varphi}(I, \varphi, s) + \frac{\partial \mathcal{L}}{\partial s}(I, \varphi, s) = 0, \quad (9)$$

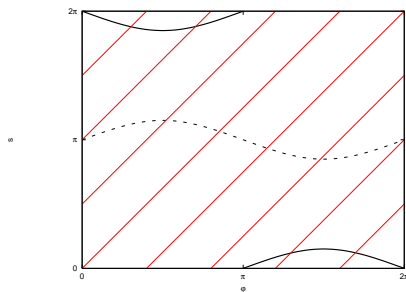
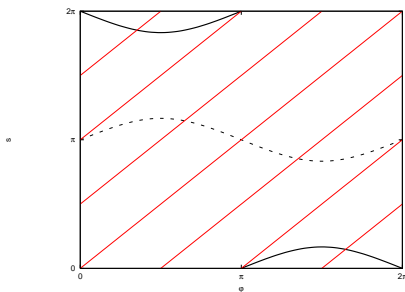
which in our case has the form

$$IA_{1,0}(I) \sin \varphi + A_{0,1} \sin s + \mathcal{O}_2(\rho, r) = 0.$$

Illustration of some level sets of the function \mathcal{L} and the two curves $\mathcal{C}_M, \mathcal{C}_m$ satisfying (9) (dashed black):

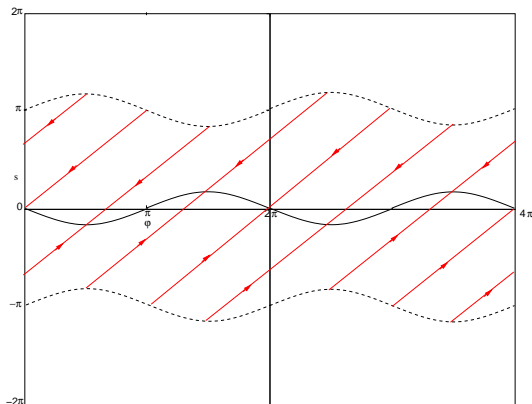


Straight lines $\tau \mapsto (\varphi - I\tau, s - \tau) \in \mathbb{T}^2$ with slope 1 (Left) and 0.8 (Right) and the curves $I \frac{\partial \mathcal{L}}{\partial \varphi}(I, \varphi, s) + \frac{\partial \mathcal{L}}{\partial s}(I, \varphi, s) = 0$: the curve \mathcal{C}_M of the maxima (solid curve) and the curve \mathcal{C}_m of the minima (dashed curve)



If $1.6 |a_{1,0}/a_{0,1}| < 1$, the intersections between the straight lines and the curves \mathcal{C}_M , \mathcal{C}_m are transversal for all I .

We choose $\tau(I, \varphi, s) = \tau_M^*(I, \varphi, s)$ as the real number τ with minimum absolute value $|\tau|$ among all τ satisfying $(\varphi - I\tau, s - \tau) \in \mathcal{C}_M(I)$.



Straight lines $\tau \mapsto (\varphi - I\tau, s - \tau) \in \mathbb{T}^2$ with slope 1 and the curves $I \frac{\partial \mathcal{L}}{\partial \varphi}(I, \varphi, s) + \frac{\partial \mathcal{L}}{\partial s}(I, \varphi, s) = 0$: the curve \mathcal{C}_M of the maxima (solid curve) and the curve \mathcal{C}_m of the minima (dashed curve).

Under condition 1.6 $|a_{1,0}/a_{0,1}| < 1$, one possible domain of definition where $\tau^*(I, \varphi, s)$ is smooth consists of excluding, for any $I \in [I_-, I_+]$, the curve \mathcal{C}_m from the domain of (φ, s) , that is

$$H = \{(I, \varphi, s) : (\varphi, s) \notin \mathcal{C}_m\}. \quad (10)$$

With this particular choice of τ^* , we have chosen a homoclinic manifold Γ to which we can associate an outer dynamics to the NHIM, that we will describe using the scattering map.

$$\Gamma_\epsilon = \{z = z(I, \varphi, s; \epsilon) = (p_0(\tau^*), q_0(\tau^*), I, \varphi, s) + \mathcal{O}(\epsilon); \tau^* = \tau^*(I, \varphi, s)\}.$$

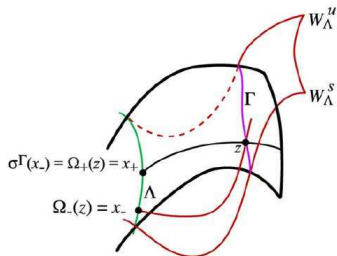
Scattering map (outer map)

$$S_\epsilon : \begin{array}{ccc} H_-^{\Gamma_\epsilon} \subset \tilde{\Lambda}_\epsilon & \rightarrow & H_+^{\Gamma_\epsilon} \subset \tilde{\Lambda}_\epsilon \\ x_- & \mapsto & x_+ \end{array}$$

$x_+ = S_\epsilon(x_-) \Leftrightarrow \exists z \in \Gamma_\epsilon$, such that

$\text{dist}(\Phi_t(z), \Phi_t(x_\pm)) \rightarrow 0 \quad \text{for } t \rightarrow \pm\infty$

- S_ϵ is exact symplectic.



[Delshams, de la Llave, Seara, 2008], [Canalias, Delshams, Masdemont, Roldan, 2006], [Delshams, Masdemont, Roldan, 2008], [Delshams, Gidea, Roldan, 2010]

- The **reduced Poincaré function**

$$\mathcal{L}(I, \varphi - I\tau^*(I, \varphi, s), s - \tau^*(I, \varphi, s)) := \mathcal{L}^*(I, \tilde{\theta}),$$

where $\tilde{\theta} = \varphi - Is$.

- The scattering map is given by the time- ϵ map of a Hamiltonian which is autonomous in first order

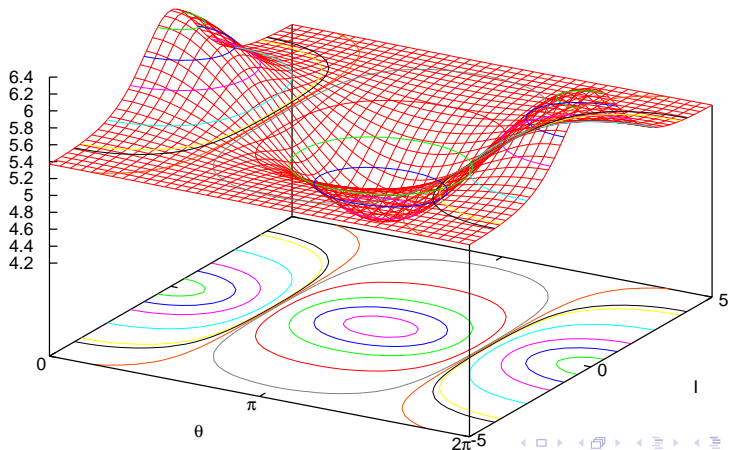
$$\mathcal{S}_\epsilon(I, \varphi, s) = -\mathcal{L}^*(I, \varphi - Is) + \mathcal{O}(\epsilon).$$

- The scattering map is given by:

$$\mathcal{S}_\epsilon(I, \varphi, s) = \left(I - \epsilon \frac{\partial \mathcal{S}_0}{\partial \varphi}(I, \varphi, s) + \mathcal{O}(\epsilon^2), \varphi + \frac{\partial \mathcal{S}_0}{\partial I}(I, \varphi, s) + \mathcal{O}(\epsilon^2), s \right).$$

- The scattering map can jump distances of $\mathcal{O}(\epsilon)$ in terms of the variable I .

Graph and level curves of the reduced Poincaré function $\mathcal{L}^*(I, \tilde{\theta})$, where $\tilde{\theta} = \varphi - Is$, for $a_{1,0} = 1/4$ and $a_{0,1} = 1/2$:



- We combine now the inner and the outer dynamics to construct a **transition chain** along $\tilde{\Lambda}_\epsilon$:

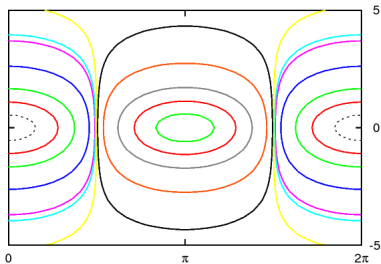
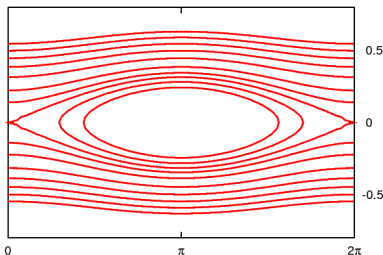
A sequence of whiskered tori $\{\mathcal{T}_i\}_{i=1}^N$ such that

$$W_{\tau_i}^u \pitchfork W_{\tau_{i+1}}^s$$

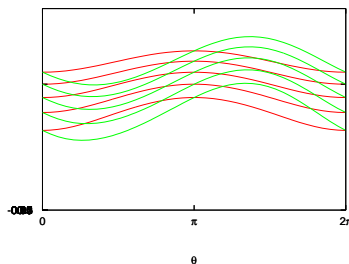
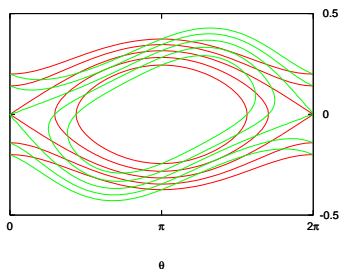
Standard shadowing methods [Fontich-Martin00] provide orbits connecting arbitrary small neighborhoods of τ_1 and τ_N .

- We will use that

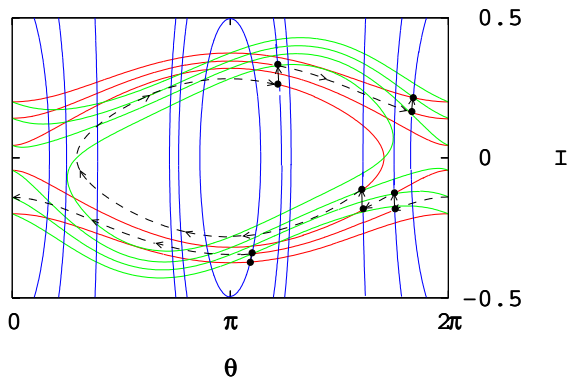
$$S_\epsilon(\tau_i) \pitchfork_{\tilde{\Lambda}_\epsilon} \tau_{i+1} \Rightarrow W_{\tau_i}^u \pitchfork W_{\tau_{i+1}}^s$$



Invariant tori (primary and secondary) in the resonant region around $I = 0$ (red curves) given implicitly by the level sets of the function $F^*(I, \tilde{\theta})$ with $k_0 = 1$, $l_0 = 0$ and $a_{1,0} = 1/2$. Images of these invariant tori (green curves) under the scattering map generated by the reduced Poincaré function $\mathcal{L}^*(I, \tilde{\theta})$:



Combination of two dynamics



- There is an orbit $\tilde{x}(t)$ that **shadows** the transition chain.
Then, there is $\epsilon^* > 0$ such that for $0 < |\epsilon| < \epsilon^*$, and for any interval $[I_-^*, I_+^*]$, $\tilde{x}(t)$ satisfies that, for some $T > 0$,

$$I(\tilde{x}(0)) \leq I_-^*; \quad I(\tilde{x}(T)) \geq I_+^*.$$