A geometric mechanism for diffusion in a priori unstable Hamiltonian systems: verification in concrete examples

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Outline

1. Set up
2. A priori unstable systems
3. An example
4. Proof for the example
   - I: A NHIM with transverse manifolds
   - II: Inner dynamics
   - III: Outer dynamics
   - IV: Construction of a transition chain

Diffusion in a priori unstable systems

- Diffusion Changes of order 1 in the actions (instabilities) for arbitrarily small perturbations of integrable systems.
- A priori unstable Hamiltonian systems:

$$H_\epsilon(p,q,I,\varphi,t) = H_0(p,q,I) + \epsilon h(p,q,I,\varphi,t;\epsilon)$$

  For $\epsilon = 0$, $H_0(p,q,I)$ is autonomous ($H_0 = E =$constant) integrable but with some saddle variables p, q. Typical example: one rotor (or more) plus one (or more) pendulum.
- Main question: What happens to $E(t)$ for small $\epsilon \neq 0$? Is there global instability?: $E(t) - E(0) = \mathcal{O}(1)$ or even $E(t) \longrightarrow \infty$?
- Many contributions: [Chierchia-Gallavotti94-98], [Berti-Biasco-Bolle03], [Marco-Sauzin03] [Treschev04], [Cheng-Yan04], [Gidea-Llave06], [Delshams-Llave-Seara06], [Delshams-H09].

Instability for a priori unstable Hamiltonian systems

We consider a 2π-periodic in time perturbation of a pendulum and a rotor described by the non-autonomous Hamiltonian,

$$\begin{aligned} H_\epsilon(p,q,I,\varphi,t) &= H_0(p,q,I) + \epsilon h(p,q,I,\varphi,t;\epsilon) \\ &= P_\pm(p,q) + \tfrac{1}{2}I^2 + \epsilon h(p,q,I,\varphi,t;\epsilon) \end{aligned} \tag{1}$$

where $(p,q,I,\varphi,t) \in (\mathbb{R}\times\mathbb{T})^2 \times \mathbb{T}$ and

$$P_\pm(p,q) = \pm\left(\frac{1}{2}p^2 + V(q)\right) \tag{2}$$

and $V(q)$ is a 2π-periodic function. We will refer to $P_\pm(p,q)$ as the *pendulum*.

Main result for a priori unstable systems

Theorem

Consider the Hamiltonian (1) where V and h are uniformly $\mathcal{C}^{r+2}$ for $r \geq r_0$, sufficiently large. Assume also that

H1 *The potential $V : \mathbb{T} \to \mathbb{R}$ has a unique global maximum at $q = 0$ which is non-degenerate. Denote by $(q_0(t), p_0(t))$ an orbit of the pendulum $P_\pm(p, q)$ homoclinic to $(0, 0)$.*

H2 *The Melnikov potential, associated to h (and to the homoclinic orbit (p_0, q_0)):*

$$\begin{aligned} \mathcal{L}(I, \varphi, s) \quad = \quad & -\int_{-\infty}^{+\infty} (h(p_0(\sigma), q_0(\sigma), I, \varphi + I\sigma, s + \sigma; 0) \\ & -h(0, 0, I, \varphi + I\sigma, s + \sigma; 0))d\sigma \end{aligned} \tag{3}$$

satisfies concrete non-degeneracy conditions.

H3 *The perturbation term h satisfies concrete non-degeneracy conditions.*

Then, there is $\epsilon^* > 0$ such that for $0 < |\epsilon| < \epsilon^*$, and for any interval $[I_-^*, I_+^*]$, there exists a trajectory $\widetilde{x}(t)$ of the system (1) such that for some $T > 0$,

$$I(\widetilde{x}(0)) \leq I_-^*; \qquad I(\widetilde{x}(T)) \geq I_+^*.$$

- Hypotheses **H1**, **H2** and **H3** are $\mathcal{C}^2$ generic.
- The previous Theorem was proven in [DLS06], assuming that h was a trigonometric polynomial in the variables (φ, s), which is a non-generic assumption, and without this assumption in [DG09].

[DLS06] A. Delshams, R. de la Llave and T.M. Seara. A geometric mechanism for diffusion in Hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model. *Mem. Amer. Math. Soc.*, 179 (844), 2006.
[DG09] A. Delshams, G. H. Geography of resonances and Arnold diffusion in *a priori* unstable Hamiltonian systems. *Nonlinearity*, 22 (1997–2077), 2009.

Idea of the proof

- Find a big invariant object: a NIHM (normally hyperbolic invariant manifold: a global version of a center manifold) $\widetilde{\Lambda}$ with transverse associated stable and unstable manifolds along a homoclinic manifold Γ: $\mathcal{W}^u(\widetilde{\Lambda}) \pitchfork_\Gamma \mathcal{W}^s(\widetilde{\Lambda})$.
- Compute the invariant objects (typically tori $\mathcal{T}$) which may prevent instability for the inner dynamics of the NHIM.
- Compute the scattering map $S : \widetilde{\Lambda} \to \widetilde{\Lambda}$ on the NHIM associated to Γ and consider it as an outer dynamics on the NHIM (a second dynamics on Γ).
- Check that $S(\mathcal{T}_{I_i}) \pitchfork \mathcal{T}_{I_{i+1}}$ for a sequence of tori $\{\mathcal{T}_{I_i}\}_{i=1}^N$ with $|I_N - I_1| = \mathcal{O}(1)$, and construct a transition chain of whiskered tori, i.e. $\mathcal{W}^u(\mathcal{T}_{I_i}) \pitchfork \mathcal{W}^s(\mathcal{T}_{I_{i+1}})$.
- Standard shadowing methods provide an orbit that follows closely the transition chain.

An example

Consider the Hamiltonian

$$H_\epsilon(p,q,I,\varphi,t) = \left(\frac{p^2}{2} + \cos q - 1\right) + \frac{I^2}{2} + \epsilon f(q) g(\varphi,t). \tag{4}$$

with

$$f(q) = \cos(q), \tag{5}$$

and

$$g(\varphi,t) = \sum_{(k,l)\in\mathbb{N}^2} a_{k,l} \cos(k\varphi - lt - \sigma_{k,l}). \tag{6}$$

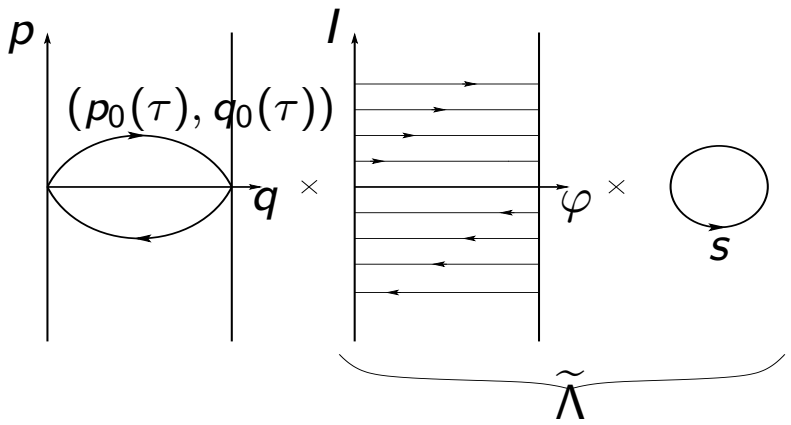
with

$$\hat{\alpha}\rho^{\beta k} r^{\beta l} \leq |a_{k,l}| \leq \alpha \rho^k r^l, \tag{7}$$

where $1 \leq \beta < 2$ and $0 < \hat{\alpha} < \alpha$. Moreover, $0 < \rho, r < 1$ are real numbers that will be chosen small enough.

DH11 A. Delshams and G. Huguet. A geometric mechanism of diffusion: Rigorous verification in a priori unstable Hamiltonian systems. *J. Diff. Eq.* 250(5): 2601–2623, 2011

$\epsilon = 0$



- Normally hyperbolic invariant manifold (3D)

$$\tilde{\Lambda} = \{(0, 0, I, \varphi, s) : (I, \varphi, s) \in \mathbb{R} \times \mathbb{T}^2\}$$

- Invariant manifolds (4D):

$$W^s\widetilde{\Lambda} = W^u\widetilde{\Lambda} = \{(p_0(\tau), q_0(\tau), I, \varphi, s) : \tau \in \mathbb{R}, I \in [I_-, I_+], (\varphi, s) \in \mathbb{T}^2\}$$

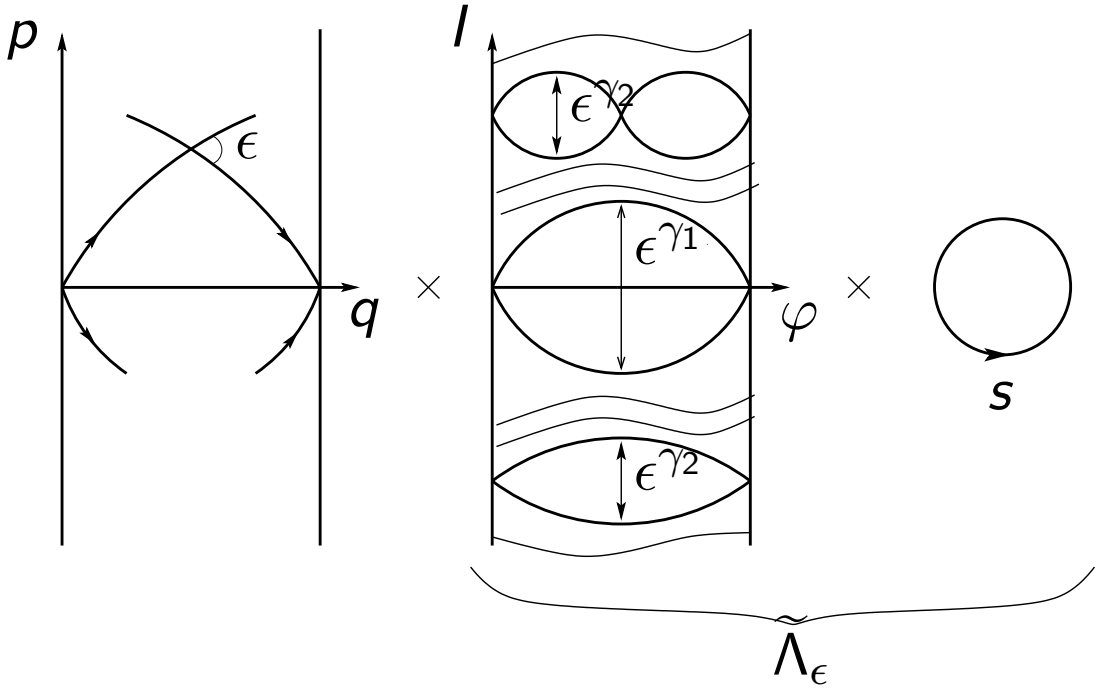
where

$$q_0(t) = 4 \arctan e^{\pm t}, \quad p_0(t) = 2/\cosh t.$$

is the separatrix for positive p of the standard pendulum

$$P(p, q) = p^2/2 + \cos q - 1.$$

$0 < \epsilon \ll 1$



- $\widetilde{\Lambda}$ persists without any deformation ($\widetilde{\Lambda}_\epsilon = \widetilde{\Lambda}$).
- $W^s\widetilde{\Lambda}_\epsilon \pitchfork W^u\widetilde{\Lambda}_\epsilon$ along a homoclinic manifold Γ_ϵ.
- the sequence of invariant tori in the NHIM is destroyed creating gaps of size up to $\mathcal{O}(\epsilon)$.

Main result

Theorem

Assume that $r > 2(m+1)^2$ and $m \geq 10$, then there exists a discrete sequence of invariant tori $\{\mathcal{T}_i\}_{i=1}^N$ in $\widetilde{\Lambda}_\epsilon$ such that:

- *They are distributed along the actions in the interval (I_-, I_+).*
- *They are $\mathcal{O}(\epsilon^{1+\eta})$-closely spaced in terms of the action variables, where $0 < \eta \ll 1$.*
- *They are given by the level sets defined by equation $F(I, \varphi, s; \epsilon) = E$, where F is a $\mathcal{C}^2$ function F which has different expressions depending on the region of the phase space where invariant tori lie:*
  - *Flat tori region. Primary KAM tori.*
  - *Big gaps region. Primary KAM tori and Secondary KAM tori.*

Proof: Averaging procedure + KAM Theorem.

Resonant averaging in our example

- Consider the Hamiltonian restricted to $\widetilde{\Lambda}_\epsilon$

$$K(I,\varphi,s,\epsilon) = \frac{I^2}{2} + \epsilon g(\varphi,s)$$

where $(I,\varphi,s) \in \mathbb{R}\times\mathbb{T}^2$ and

$$g(\varphi,s) = \sum_{(k,l)\in\mathbb{N}^2} a_{k,l}\cos(k\varphi + ls).$$

- Look for a change of coordinates that eliminates the dependence on the angle variables at order ϵ.
- Does it always exist? No, it fails at resonances.
- **Resonances** for this system, values of I such that

$$(I,1)\cdot(k,l) = 0 \Rightarrow I = -\frac{l}{k}, \quad \text{for some } (k,l)\in\mathbb{N}^2$$

- **Remark:** The perturbation g has infinite harmonics in the angles (φ, s).
- Consider the truncated Fourier series of g at order M

$$g^{[\leq M]}(I, \varphi, s; \epsilon) = \sum_{\substack{(k,l)\in\mathbb{N}^2, \\ |k|+|l|\leq M}} a_{k,l} \cos(k\varphi + ls),$$

where $M = M(\epsilon) \sim \epsilon^{-\rho}$, $\rho < 1/n$, (n regularity required for the application of KAM Theorem).

- Apply averaging method to Hamiltonian

$$K_0(I, \varphi, s, \epsilon) = \frac{I^2}{2} + \epsilon g^{[\leq M]}(\varphi, s).$$

- **Set of resonances** up to order 1:

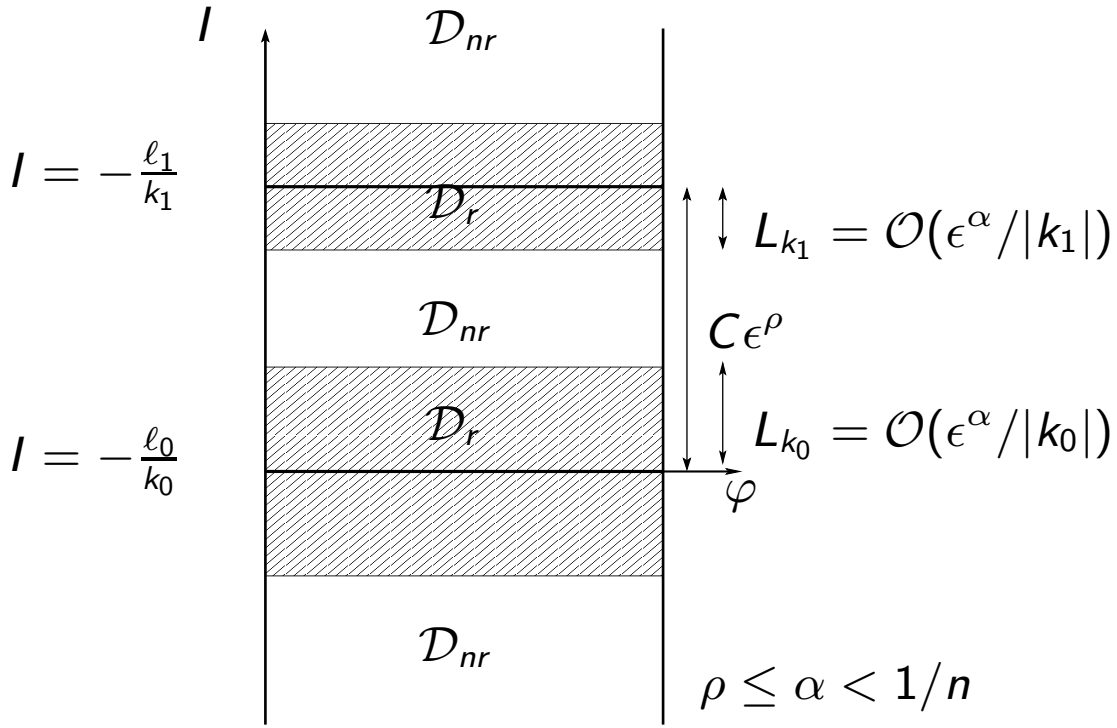
$$\mathcal{R}_1 = \{I = -\frac{l}{k} \in \mathbb{Q} \cap (I_-, I_+) : |k| + |l| \leq M\}$$

It is a finite set but depends on M and therefore on ϵ.

Two types of domains in the NHIM

- Non-resonant region up to order 1
- Resonant region up to order 1: One connected component

$$\mathcal{D}^{k_0,l_0} \equiv \left[-\frac{\ell_0}{k_0} - L_{k_0}, -\frac{\ell_0}{k_0} + L_{k_0}\right] \times \mathbb{T}^2$$



- Non-resonant region up to order 1:

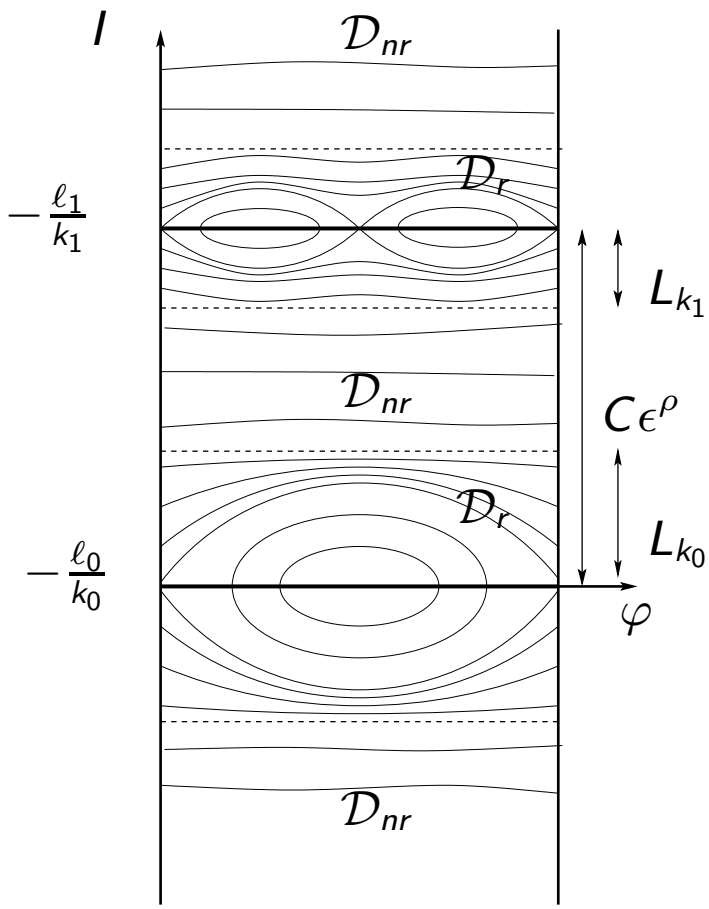
$$K(I,\varphi,s,\epsilon) = \frac{I^2}{2} + \epsilon + \mathcal{O}(\epsilon^2)$$

- Resonant region up to order 1:
  Keep the harmonics corresponding to the resonance $I = -\ell_0/k_0$

$$K(I,\varphi,s,\epsilon) = \frac{I^2}{2} + \epsilon(1 + U^{k_0,l_0}(\theta)) + \mathcal{O}(\epsilon^2)$$

where $\theta = k_0\varphi + \ell_0 s$ and

$$U^{k_0,\ell_0}(\theta) = a_{k_0,l_0}\cos(\theta) + \sum_{t=2}^{M} a_{tk_0,tl_0}\cos(t\theta)$$



H3' $U^{k_0,\ell_0}(\theta,0)$ has a non-degenerate global maximum at $\theta = 0 \Rightarrow K$ is close to a pendulum

- Repeat the averaging procedure up to order m

$$K_m(I,\varphi,s;\epsilon) = Z^m(I,\varphi,s;\epsilon) + \epsilon^{m+1} R^m(I,\varphi,s;\epsilon)$$

- New resonances and resonant regions appear
  - Non-resonant region up to order m, D^m_{nr}:

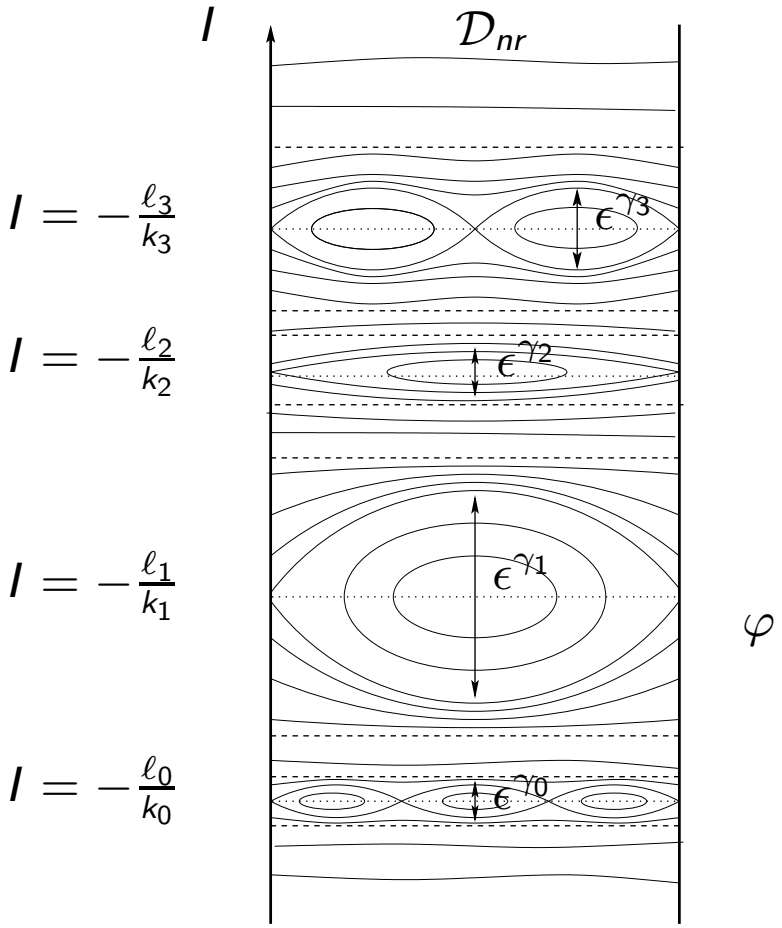
$$Z^m(I,\varphi,s;\epsilon) = \frac{I^2}{2} + \epsilon \widetilde{Z}^m(I;\epsilon).$$

    (like a rotor)
  - Connected component $\mathcal{D}^{k_0,l_0}$ of the resonant region D^m_r:

$$Z^m(I,\varphi,s;\epsilon) = \frac{I^2}{2} + \epsilon \widetilde{Z}^m(I;\epsilon) + \epsilon^j U_m^{(k_0,l_0)}(\theta;\epsilon).$$

    where $\theta = k_0\varphi + l_0 s$ (like an integrable pendulum)
- The size of the gaps depends on the order of the resonance and the size of the Fourier coefficient associated to it.
- The error coming from the tail $g^{[>M]}$ of the Fourier series can be neglected provided that the perturbation g is regular enough.

Invariant objects in $\widetilde{\Lambda}_\epsilon$ given by the integrable Hamiltonian $Z^m(I, \varphi, s; \epsilon)$



We consider two regions in $\widetilde{\Lambda}_\epsilon$:

- The big gaps region.
  Resonant regions with gaps of size bigger than ϵ.
  Primary KAM tori and secondary KAM tori.
- Flat tori region.
  Non resonant regions and resonant regions with gaps smaller than ϵ.
  Primary KAM tori.

KAM Theorem

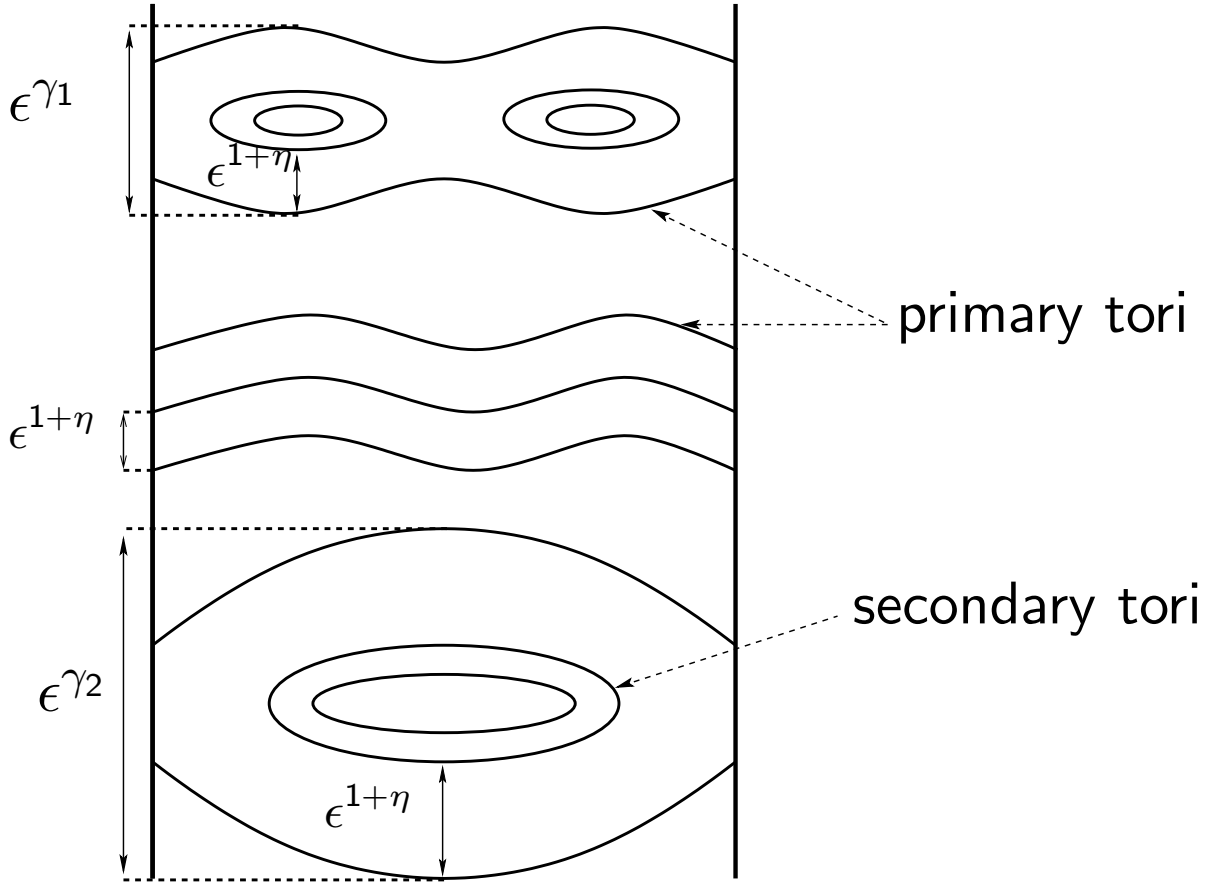
Which of them survive and at what distance when we add the perturbation term?

KAM theorem (Quantitative version of a KAM Theorem by Herman)

- Flat tori region. We can apply KAM Theorem straightforwardly. Tori at a distance $\epsilon^{1+\eta}$, with $0 < \eta \ll 1$ ($m = 2$ is enough).
- Big gaps region. The integrable system is close to a pendulum. We need to write the Hamiltonian in action-angle variables. Tori at a distance $\epsilon^{1+\eta}$, with $0 < \eta \ll 1$ ($m = 10$ is enough).

For $I \in [I_-, I_+]$, there exists in $\widetilde{\Lambda}_\epsilon$ a discrete sequence of invariant tori $\mathcal{T}_i$ (some primary and some secondary) which are $\epsilon^{1+\eta}$-closely spaced, with $0 < \eta \ll 1$. They are given by the leaves L_E^F of a foliation $\mathcal{F}_F$.

Invariant objects of K in the NHIM $\widetilde{\Lambda}_\epsilon$



Consider the Melnikov potential

$$\mathcal{L}(I,\varphi,s) = -\int_{-\infty}^{+\infty} (h(p_0(\sigma),q_0(\sigma),I,\varphi+I\sigma,s+\sigma;0) - h(0,0,I,\varphi+I\sigma,s+\sigma;0))d\sigma \tag{8}$$

H2' Given real numbers $I_- < I_+$, assume that for any value of $I \in (I_-, I_+)$ the map

$$\tau \in \mathbb{R} \mapsto \mathcal{L}(I,\varphi - I\tau, s-\tau)$$

has a non-degenerate critical point τ which is locally given by the implicit function theorem in the form

$$\tau = \tau^*(I,\varphi,s),$$

with τ^* a smooth function.

Then, for ϵ small enough, there exists a locally unique point $\tilde{z}$ of the form

$$\tilde{z}(I,\varphi,s;\epsilon) = \tilde{z}(\tau^*(I,\varphi,s),I,\varphi,s;\epsilon) = (p_0(\tau)+\mathcal{O}(\epsilon), q_0(\tau)+\mathcal{O}(\epsilon), I,\varphi,s),$$

such that $W^s(\tilde{\Lambda}_\epsilon) \pitchfork W^u(\tilde{\Lambda}_\epsilon)$ at $\tilde{z}$.

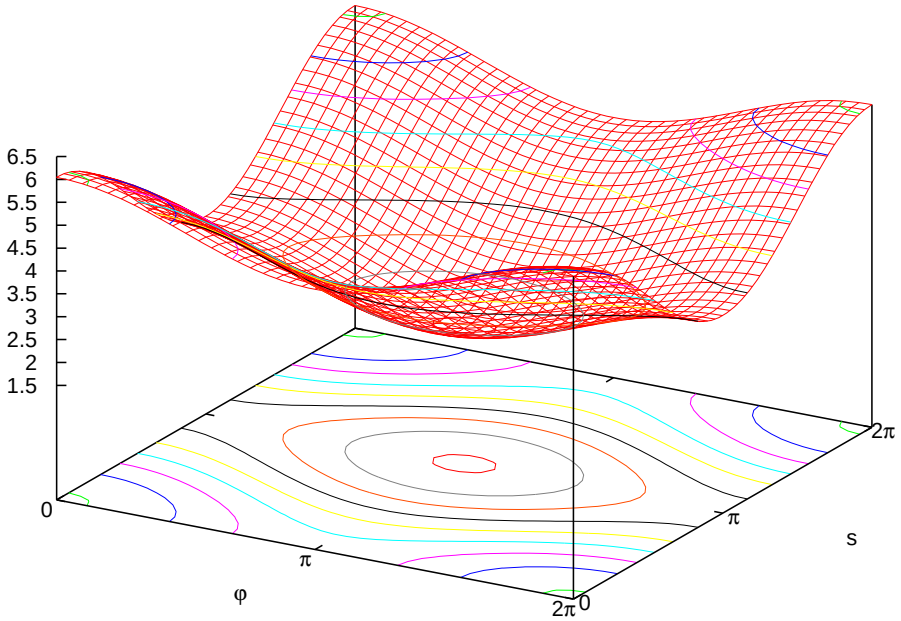
Homoclinic manifold

$$\Gamma_\epsilon = \{z = z(I,\varphi,s;\epsilon) = (p_0(\tau^*), q_0(\tau^*), I,\varphi,s); \tau^* = \tau^*(I,\varphi,s)\}.$$

Graph and level curves of the first order approximation in ρ, r of the Melnikov potential

$$\mathcal{L}^{[\leq 1]}(I, \varphi, s) = A_{0,0} + A_{1,0} \cos \varphi + A_{0,1}(I) \cos s,$$

for $0 < A_{1,0} < A_{0,1} < 1$



Four non-degenerate critical points: maximum $(0, 0)$, minimum (π, π) and two saddles $(0, \pi)$, $(\pi, 0)$.

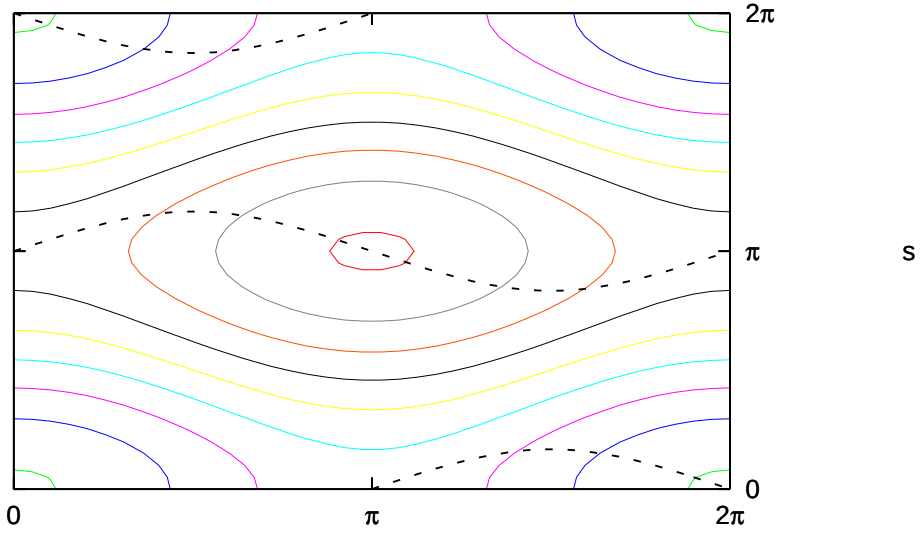
Given a fixed value of I, we first look at the points $(\varphi, s) \in \mathbb{T}^2$ such that $\tau^*(I, \varphi, s) = 0$. They satisfy the implicit equation

$$I \frac{\partial \mathcal{L}}{\partial \varphi}(I, \varphi, s) + \frac{\partial \mathcal{L}}{\partial s}(I, \varphi, s) = 0, \tag{9}$$

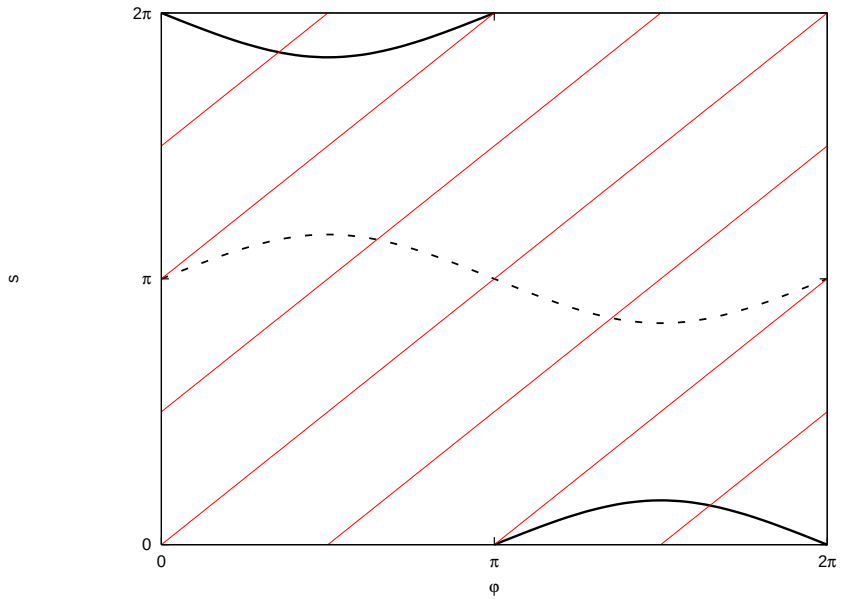
which in our case has the form

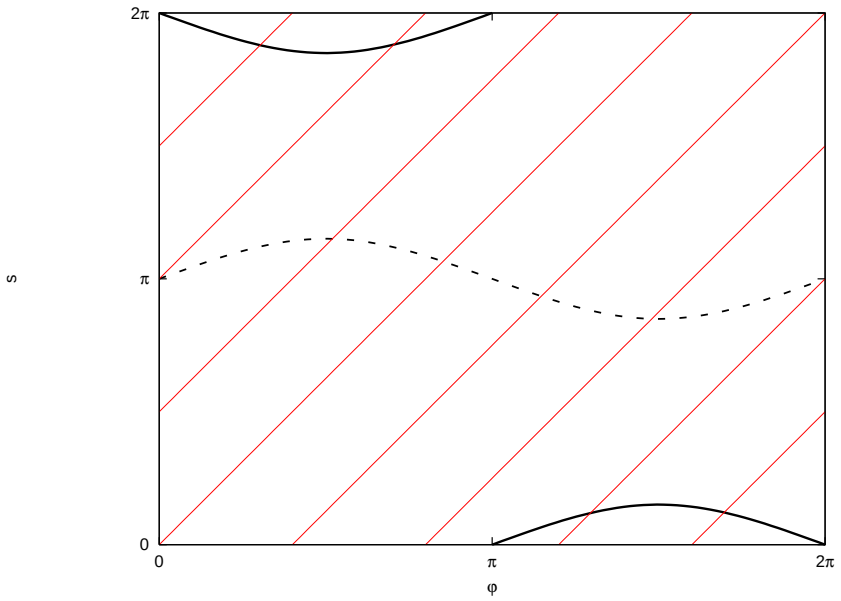
$$I A_{1,0}(I) \sin \varphi + A_{0,1} \sin s + \mathcal{O}_2(\rho, r) = 0.$$

Illustration of some level sets of the function $\mathcal{L}$ and the two curves $\mathcal{C}_M$, $\mathcal{C}_m$ satisfying (9) (dashed black):



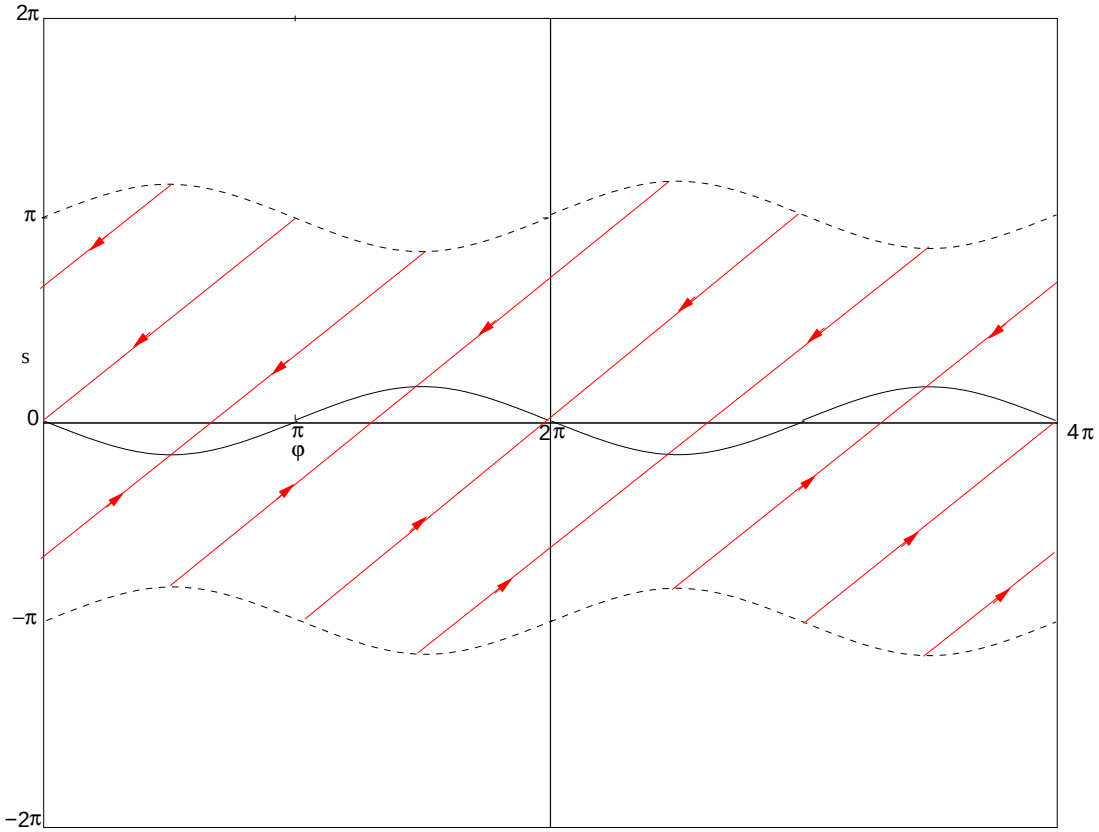
Straight lines $\tau \mapsto (\varphi - I\tau, s - \tau) \in \mathbb{T}^2$ with slope 1 (Left) and 0.8 (Right) and the curves $I\frac{\partial \mathcal{L}}{\partial \varphi}(I, \varphi, s) + \frac{\partial \mathcal{L}}{\partial s}(I, \varphi, s) = 0$: the curve $\mathcal{C}_M$ of the maxima (solid curve) and the curve $\mathcal{C}_m$ of the minima (dashed curve)





If $1.6\,|a_{1,0}/a_{0,1}| < 1$, the intersections between the straight lines and the curves $\mathcal{C}_M$, $\mathcal{C}_m$ are transversal for all I.

We choose $\tau(I,\varphi,s) = \tau^*_M(I,\varphi,s)$ as the real number τ with minimum absolute value $|\tau|$ among all τ satisfying $(\varphi - I\tau, s-\tau) \in \mathcal{C}_M(I)$.



Straight lines $\tau \mapsto (\varphi - I\tau, s-\tau) \in \mathbb{T}^2$ with slope 1 and the curves $I\frac{\partial \mathcal{L}}{\partial \varphi}(I,\varphi,s) + \frac{\partial \mathcal{L}}{\partial s}(I,\varphi,s) = 0$: the curve $\mathcal{C}_M$ of the maxima (solid curve) and the curve $\mathcal{C}_m$ of the minima (dashed curve).

Under condition 1.6 $|a_{1,0}/a_{0,1}| < 1$, one possible domain of definition where $\tau^*(I, \varphi, s)$ is smooth consists of excluding, for any $I \in [I_-, I_+]$, the curve $\mathcal{C}_m$ from the domain of (φ, s), that is

$$H = \{(I, \varphi, s) : (\varphi, s) \notin \mathcal{C}_m\}. \tag{10}$$

With this particular choice of τ^*, we have chosen a homoclinic manifold Γ to which we can associate an outer dynamics to the NHIM, that we will describe using the scattering map.

$$\Gamma_\epsilon = \{z = z(I, \varphi, s; \epsilon) = (p_0(\tau^*), q_0(\tau^*), I, \varphi, s) + \mathcal{O}(\epsilon); \tau^* = \tau^*(I, \varphi, s)\}.$$

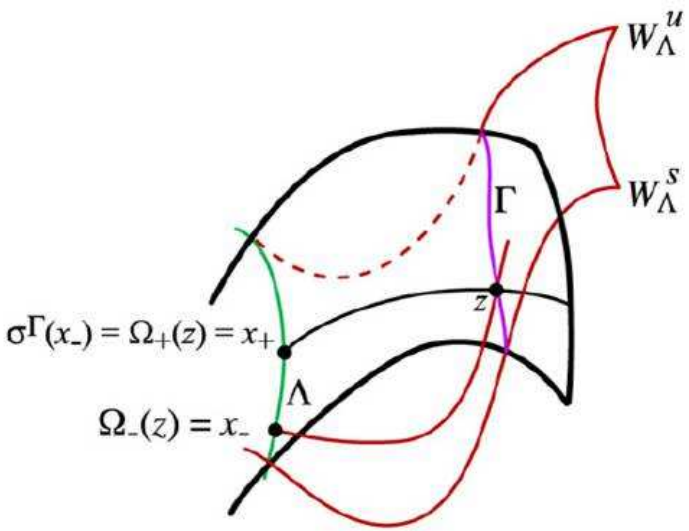
Scattering map (outer map)

$$S_\epsilon : \quad H_-^{\Gamma_\epsilon} \subset \widetilde{\Lambda}_\epsilon \quad \to \quad H_+^{\Gamma_\epsilon} \subset \widetilde{\Lambda}_\epsilon$$
$$x_- \quad \mapsto \quad x_+$$

$x_+ = S_\epsilon(x_-) \Leftrightarrow \exists\, z \in \Gamma_\epsilon$, such that

$$\mathrm{dist}(\Phi_t(z), \Phi_t(x_\pm)) \to 0 \qquad \text{for} \quad t \to \pm\infty$$

- S_ϵ is exact symplectic.



[Delshams, de la Llave, Seara, 2008], [Canalias, Delshams, Masdemont, Roldan, 2006], [Delshams, Masdemont, Roldan, 2008], [Delshams, Gidea, Roldan, 2010]

- The reduced Poincaré function

$$\mathcal{L}(I, \varphi - I\tau^*(I, \varphi, s), s - \tau^*(I, \varphi, s)) := \mathcal{L}^*(I, \widetilde{\theta}),$$

where $\widetilde{\theta} = \varphi - Is$.

- The scattering map is given by the time-ϵ map of a Hamiltonian which is autonomous in first order

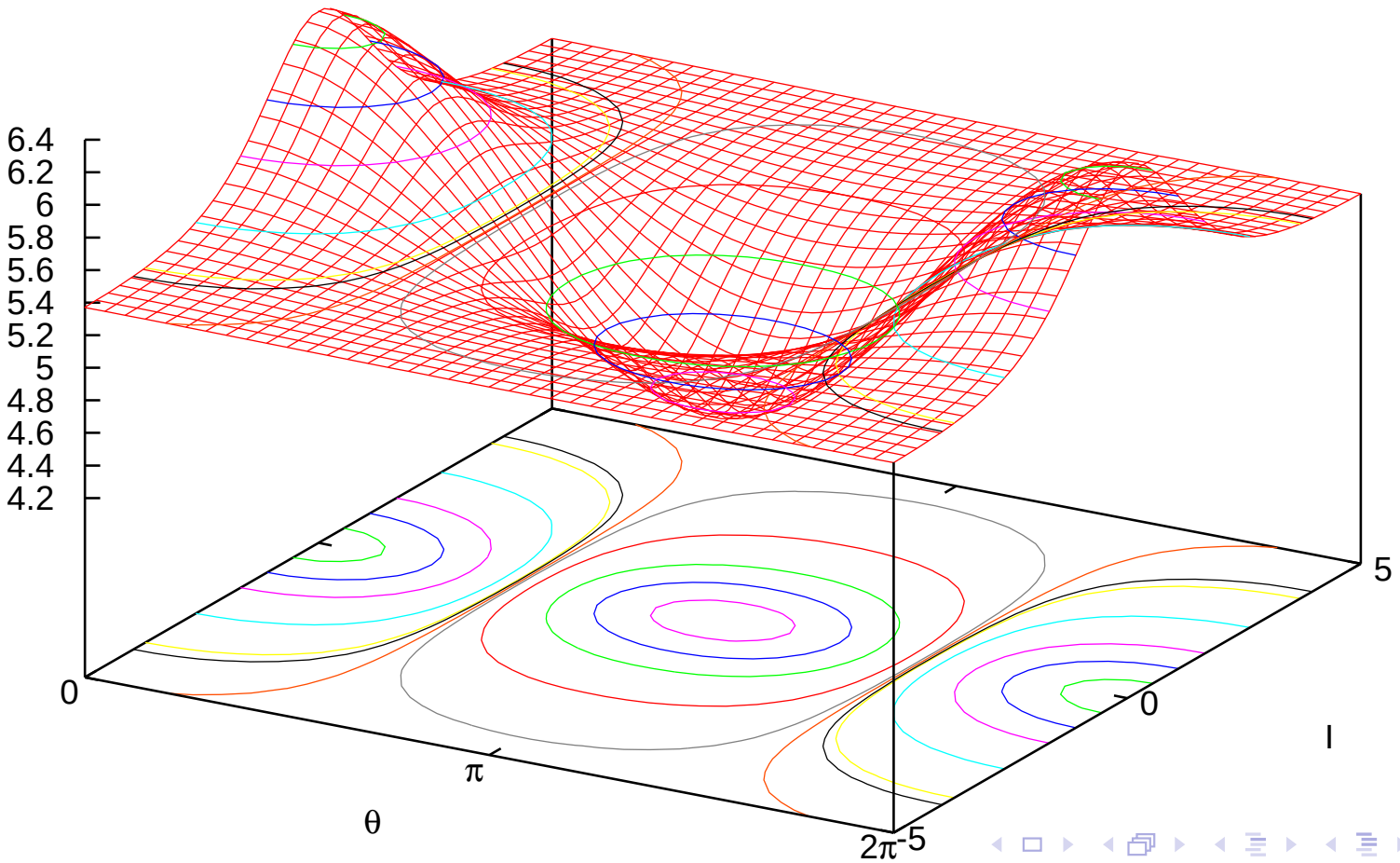
$$\mathcal{S}_\epsilon(I, \varphi, s) = -\mathcal{L}^*(I, \varphi - Is) + \mathcal{O}(\epsilon).$$

- The scattering map is given by:

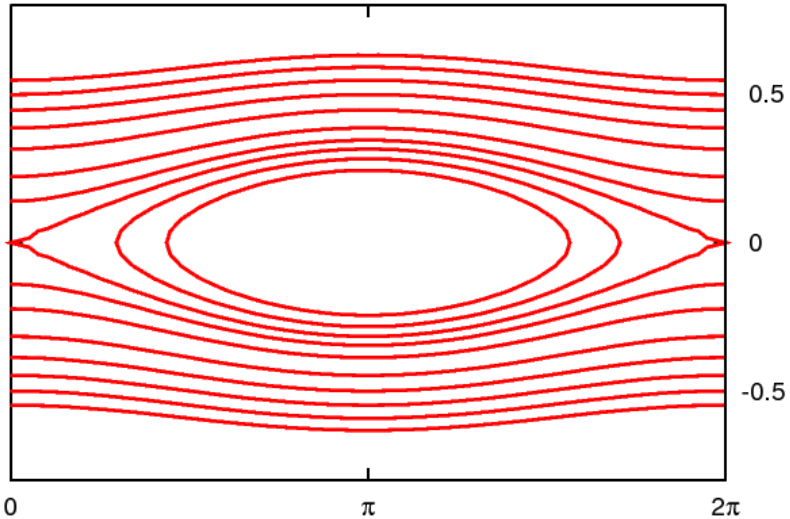
$$S_\epsilon(I, \varphi, s) = \left(I - \epsilon \frac{\partial \mathcal{S}_0}{\partial \varphi}(I, \varphi, s) + \mathcal{O}(\epsilon^2), \varphi + \frac{\partial \mathcal{S}_0}{\partial I}(I, \varphi, s) + \mathcal{O}(\epsilon^2), s \right).$$

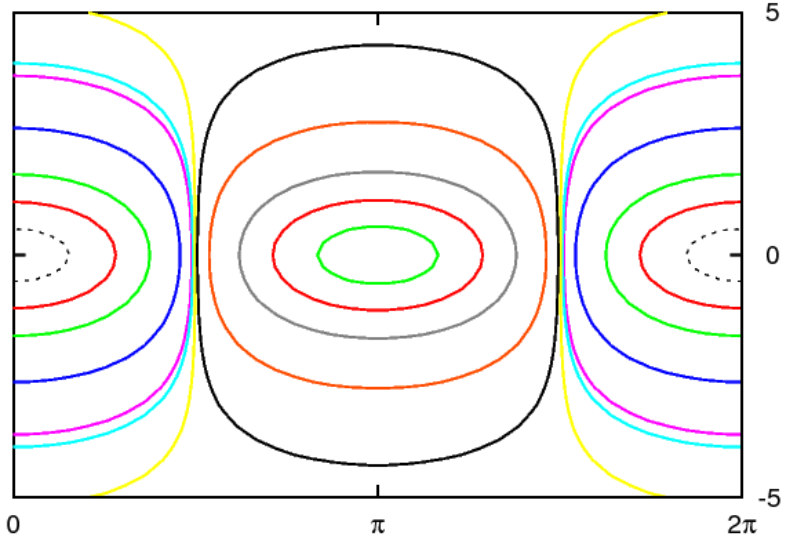
- The scattering map can jump distances of $\mathcal{O}(\epsilon)$ in terms of the variable I.

Graph and level curves of the reduced Poincaré function $\mathcal{L}^*(I, \widetilde{\theta})$, where $\widetilde{\theta} = \varphi - Is$, for $a_{1,0} = 1/4$ and $a_{0,1} = 1/2$:

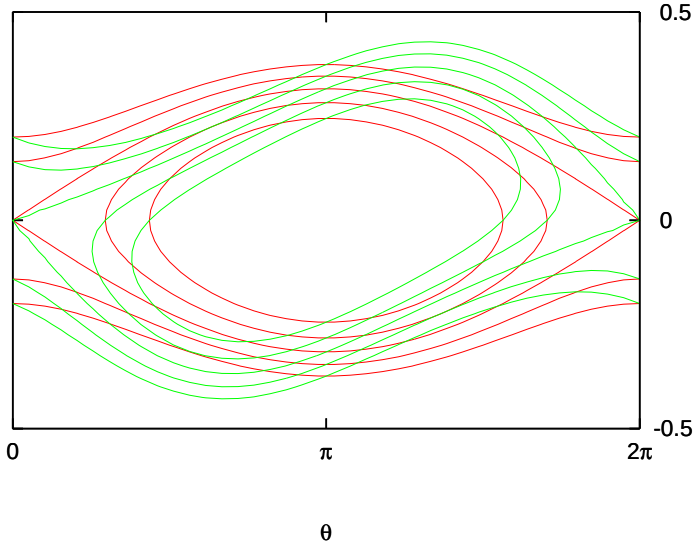


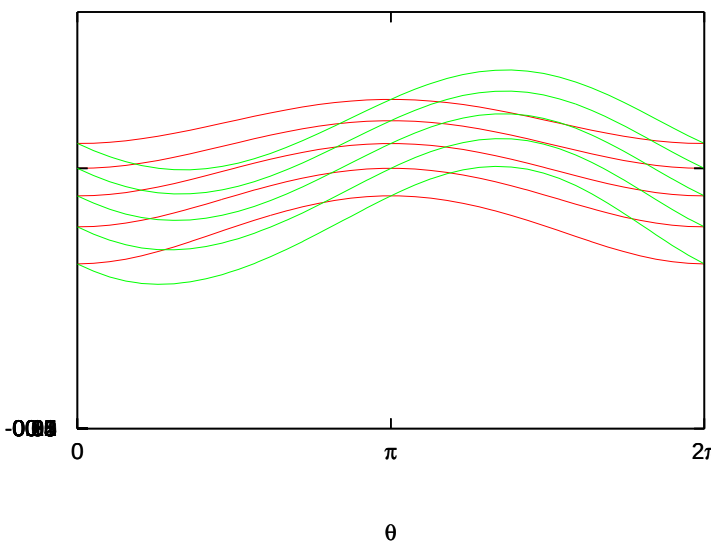
- We combine now the inner and the outer dynamics to construct a transition chain along $\widetilde{\Lambda}_\epsilon$:
  A sequence of whiskered tori $\{\mathcal{T}_i\}_{i=1}^N$ such that
  $$W^u_{\tau_i} \pitchfork W^s_{\tau_{i+1}}$$
  Standard shadowing methods [Fontich-Martin00] provide orbits connecting arbitrary small neighborhoods of τ_1 and τ_N.
- We will use that
  $$S_\epsilon(\tau_i) \pitchfork_{\widetilde{\Lambda}_\epsilon} \tau_{i+1} \Rightarrow W^u_{\tau_i} \pitchfork W^s_{\tau_{i+1}}$$



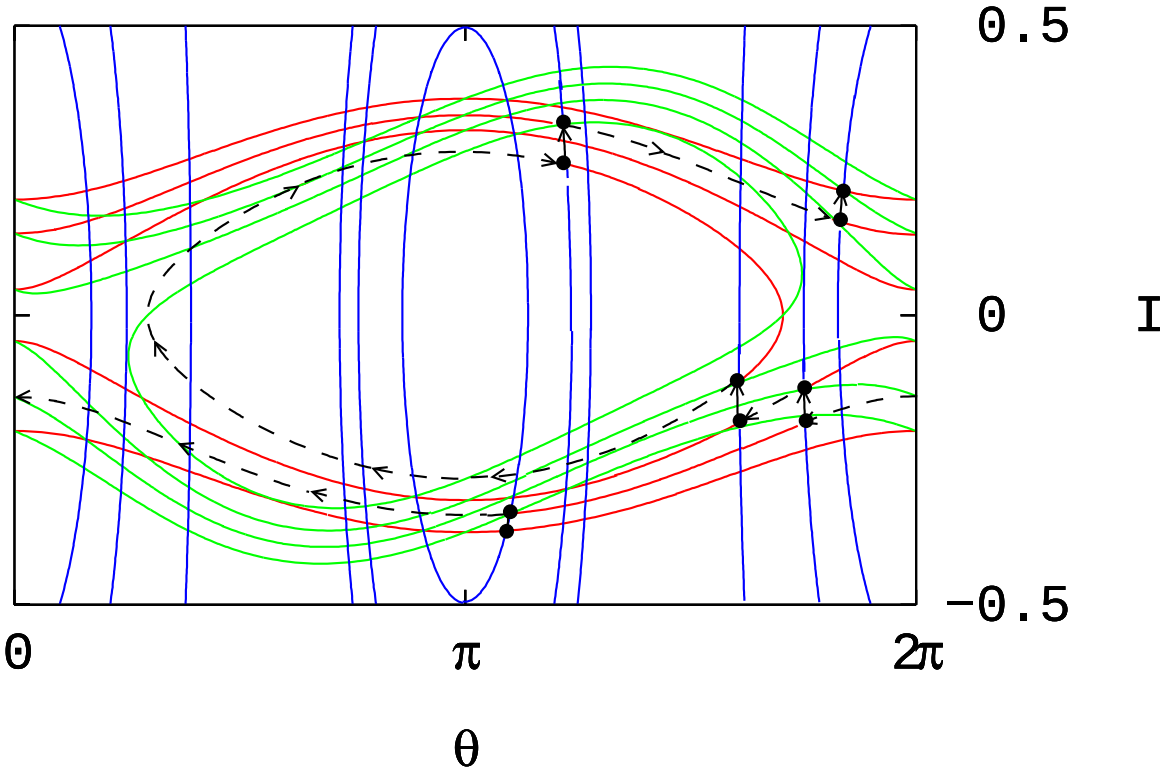


Invariant tori (primary and secondary) in the resonant region around $I = 0$ (red curves) given implicitly by the level sets of the function $F^*(I, \widetilde{\theta})$ with $k_0 = 1$, $l_0 = 0$ and $a_{1,0} = 1/2$. Images of these invariant tori (green curves) under the scattering map generated by the reduced Poincaré function $\mathcal{L}^*(I, \widetilde{\theta})$:





Combination of two dynamics



- There is an orbit $\widetilde{x}(t)$ that shadows the transition chain.
  Then, there is $\epsilon^* > 0$ such that for $0 < |\epsilon| < \epsilon^*$, and for any interval $[I_-^*, I_+^*]$, $\widetilde{x}(t)$ satisfies that, for some $T > 0$,

$$I(\widetilde{x}(0)) \leq I_-^*; \qquad I(\widetilde{x}(T)) \geq I_+^*.$$