

# Fast numerical algorithms to compute invariant tori in Hamiltonian Systems

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# Outline

- 1 Introduction: importance of invariant tori
- 2 Set up and invariance equations
- 3 Whiskered tori

# Invariant torus

- Invariant torus of dimension  $\ell$  = quasi-periodic solution with  $\ell$  independent frequencies (primary and secondary, maximal and whiskered).
- **Importance.** Together with their connections organize the long term behavior of the system (celestial mechanics, chemistry, ...)
- **Goal** Develop numerical algorithms following the theoretical results of KAM Theorem without Action-Angle variables ([LGJV05],[FLS08]) and implement them numerically.

[LGJV05] R. de la Llave, A. González, A. Jorba and J. Villanueva. KAM theory without action-angle variables, *Nonlinearity*,18(2):855–895,2005.

[FLS09] E. Fontich, R. de la Llave and Y. Sire. Construction of invariant whiskered tori by a parameterization method. I. Maps and flows in finite dimensions. *J. Differential Equations*,246(8):3136–3213, 2009.

[HLS11] G. Huguet, R. de la Llave and Y. Sire. Computation of whiskered invariant tori and their associated manifolds: New fast algorithms. Accepted to *Discrete Contin. Dyn. Syst. – Ser. A*, 2011. Preprint at [arXiv:1004.5231](https://arxiv.org/abs/1004.5231)

# The invariance equation

- Consider a map  $F$  **exact symplectic** defined on  $(U \subset \mathbb{R}^d) \times \mathbb{T}^d$ .
- Assume that  $\omega \in \mathbb{R}^\ell$  is fixed and **Diophantine**, i.e. for some  $\nu, \tau > 0$ ,

$$|\omega \cdot k - n|^{-1} \leq \nu |k|^\tau \quad \forall k \in \mathbb{Z}^\ell - \{0\}, n \in \mathbb{Z}$$

- We seek for an embedding  $K : \mathbb{T}^\ell \rightarrow \mathbb{R}^d \times \mathbb{T}^d$  that satisfies the **invariance equation**

$$F \circ K - K \circ T_\omega = 0$$

where  $T_\omega(\theta) = \theta + \omega$ .

- The dynamics of  $F$  restricted on the invariant torus (range of  $K$ ) is conjugated to a rigid rotation of frequency  $\omega$ .

# Whiskered tori

- The **tangent space**  $T_{K(\theta)}\mathcal{M}$ , where  $\mathcal{M} = \mathbb{R}^{2d-\ell} \times \mathbb{T}^\ell$ , has an **invariant analytic splitting**:

$$T_{K(\theta)}\mathcal{M} = \mathcal{E}_{K(\theta)}^c \oplus \mathcal{E}_{K(\theta)}^s \oplus \mathcal{E}_{K(\theta)}^u \quad (1)$$

such that, there exist  $0 < \mu_1, \mu_2 < 1$ ,  $\mu_3 > 1$  satisfying  $\mu_1\mu_3 < 1$ ,  $\mu_2\mu_3 < 1$  and  $C > 0$  such that for all  $n \geq 1$  and  $\theta \in \mathbb{T}^\ell$

$$v \in \mathcal{E}_{K(\theta)}^s \iff |(DF \circ K \circ T_\omega^{n-1}(\theta) \times \dots \times DF \circ K(\theta))v| \leq C\mu_1^n |v|$$

$$v \in \mathcal{E}_{K(\theta)}^u \iff |(DF^{-1} \circ K \circ T_\omega^{-(n-1)}(\theta) \times \dots \times DF^{-1} \circ K(\theta))v| \leq C\mu_2^n |v|$$

$$v \in \mathcal{E}_{K(\theta)}^c \iff |(DF \circ K \circ T_\omega^{n-1}(\theta) \times \dots \times DF \circ K(\theta))v| \leq C\mu_3^n |v|$$

$$|(DF^{-1} \circ K \circ T_\omega^{-(n-1)}(\theta) \times \dots \times DF^{-1} \circ K(\theta))v| \leq C\mu_3^n |v|$$

- Associate to this splitting the **projections**  $\Pi_{K(\theta)}^c$ ,  $\Pi_{K(\theta)}^s$  and  $\Pi_{K(\theta)}^u$  over the invariant spaces  $\mathcal{E}_{K(\theta)}^c$ ,  $\mathcal{E}_{K(\theta)}^s$  and  $\mathcal{E}_{K(\theta)}^u$ , respectively, which are analytic with respect to  $\theta$ .

- Consider the linearized equation for the Newton method

$$(DF \circ K)\Delta - \Delta \circ T_\omega = -E$$

- The **linearized equation** for the Newton method splits in three:

$$DF(K(\theta))\Delta^{c,s,u}(\theta) - \Delta^{c,s,u} \circ T_\omega(\theta) = -E^{c,s,u}(\theta) \quad (2)$$

where  $\Delta^{s,c,u}(\theta) = \Pi_{K(\theta)}^{s,c,u}\Delta(\theta)$  and  $E^{s,c,u}(\theta) = \Pi_{K(\theta)}^{s,c,u}E(\theta)$

- We solve  $\Delta^c$  using the **automatic reducibility**.
- For the **hyperbolic spaces**, we compute  $\Delta^{s,u}$  **iteratively**:

$$\Delta^s(\theta) = \tilde{E}^s \circ T_{-\omega}(\theta) + \sum_{k=1}^{\infty} (DF \circ K \circ T_{-\omega}(\theta) \cdots DF \circ K \circ T_{-k\omega}(\theta)) (\tilde{E}^s \circ T_{-(k+1)\omega}(\theta))$$

$$\Delta^u(\theta) = - \sum_{k=0}^{\infty} (DF^{-1} \circ K(\theta) \cdots DF^{-1} \circ K \circ T_{k\omega}(\theta)) (\tilde{E}^u \circ T_{k\omega}(\theta))$$

- The **contraction** of the cocycles guarantees the uniform convergence of these series.

# Algorithm for the whiskered invariant tori

## Algorithm

*Consider given  $F$ ,  $\omega$ ,  $K_0$  and an approximate solution  $K$  (resp.  $K, \lambda$ ), perform the following operations:*

- A) Compute the invariant splittings and the projections associated to the cocycle  $Z(\theta) = DF \circ K(\theta)$  and  $\omega$ .*
- B) Project the linearized equation to the hyperbolic space and obtain  $\Delta^{s,u}$ .*
- C) Project the linearized equation on the center subspace and use the Algorithm to obtain  $\Delta^c$  and  $\delta$ .*
- D) Set  $K + \Delta^s + \Delta^u + \Delta^c \rightarrow K$  and  $\lambda + \delta \rightarrow \lambda$*

# Newton method for the projections

## Algorithm (Computation of the projections by a Newton method)

Consider given  $F, K, \omega$  and an approximate solution  $(\Pi^s, \Pi^{cu})$ . Perform the following calculations:

1. Compute  $Z(\theta) = DF \circ K(\theta)$
2. (2.1) Compute  $E^{cu}(\theta) = \Pi^{cu}(\theta + \omega)Z(\theta)\Pi^s(\theta)$   
 (2.2) Compute  $E^s(\theta) = \Pi^s(\theta + \omega)Z(\theta)\Pi^{cu}(\theta)$
3. (3.1) Compute  $N_s(\theta) = \Pi^s(\theta + \omega)Z(\theta)\Pi^s(\theta)$   
 (3.2) Compute  $N_{cu}(\theta) = \Pi^{cu}(\theta + \omega)Z(\theta)\Pi^{cu}(\theta)$
4. (4.1) Solve for  $\Delta_s^s$  satisfying

$$N_s(\theta)\Delta_s^s(\theta) - \Delta_s^s(\theta + \omega)N_{cu}(\theta) = E^s(\theta)$$

- (4.2) Solve for  $\Delta_{cu}^s$  satisfying

$$N_{cu}(\theta)\Delta_{cu}^s(\theta) - \Delta_{cu}^s(\theta + \omega)N_s(\theta) = -E^{cu}(\theta)$$



# Newton method for the projections

## Algorithm

5. (5.1) Compute  $\tilde{\Pi}^s(\theta) = \Pi^s(\theta) + \Delta_s^s(\theta) + \Delta_{cu}^s(\theta)$ .
- (5.2) Compute the SVD decomposition of  $\tilde{\Pi}^s(\theta)$ :  $\tilde{\Pi}^s(\theta) = U(\theta)\Sigma(\theta)V^\top(\theta)$ .
- (5.3) Set the values in  $\Sigma(\theta)$  equal to the closer integer (which will be either 0 or 1).
- (5.4) Recompute  $\bar{\Pi}^s(\theta) = U(\theta)\Sigma(\theta)V^\top(\theta)$ .
6. Set  $\bar{\Pi}^s \rightarrow \Pi^s$  and  $\text{Id} - \bar{\Pi}^s \rightarrow \Pi^{cu}$  and iterate the procedure.

# Solutions of iterative cohomology equations

## Algorithm (Solution of difference equations with non constant coefficient)

Given  $A(\theta)$ ,  $B(\theta)$  such that  $\|A^{-1}(\theta)\| \cdot \|B(\theta)\| \leq \kappa < 1$ , and  $\eta(\theta)$  perform the following operations:

1. Compute  $\Delta(\theta) = A^{-1}(\theta)\eta(\theta)$
2. Compute
  - (2.1)  $\tilde{\Delta}(\theta) = A^{-1}(\theta)\Delta(\theta + \omega)B(\theta) + \Delta(\theta)$
  - (2.2)  $\tilde{A}^{-1}(\theta) = A^{-1}(\theta)A^{-1}(\theta + \omega)$
  - (2.3)  $\tilde{B}(\theta) = B(\theta + \omega)B(\theta)$
3. Set
 
$$\begin{aligned}\tilde{\Delta} &\rightarrow \Delta \\ \tilde{A} &\rightarrow A \\ \tilde{B} &\rightarrow B \\ 2\omega &\rightarrow \omega\end{aligned}$$
4. Iterate points 2 – 3

# Algorithm (Solution of difference equations with non constant coeff (1D))

Given  $A(\theta)$ ,  $B(\theta)$  and  $\eta(\theta)$ . Perform the following instructions:

1. (1.1) Compute  $L(\theta) = \log(A(\theta)) - \log(B(\theta))$

(1.2) Compute  $\bar{L} = \int_{\mathbb{T}^\ell} L$

2. Solve for  $L_C$  satisfying

$$L_C(\theta) - L_C \circ T_\omega(\theta) = L(\theta) - \bar{L}$$

as well as having zero average.

3. (3.1) Compute  $C(\theta) = \exp(L_C(\theta))$

(3.2) Compute  $\nu = \exp(\bar{L})$

4. Compute  $\tilde{\eta}(\theta) = C(\theta + \omega)B^{-1}(\theta)\eta(\theta)$

5. Solve for  $W$  satisfying

$$\nu W(\theta) - W(\theta + \omega) = \tilde{\eta}(\theta)$$

6. Set  $\Delta(\theta) = C^{-1}(\theta)W(\theta)$

# Fast iteration of the cocycles

Given a function  $M : \mathbb{T}^\ell \rightarrow GL(d, \mathbb{R})$  and a vector  $\omega \in \mathbb{R}^\ell$ , we consider the **cocycle** over the rotation  $T_\omega$  associated to the matrix  $M$ . This is the function  $\mathcal{M} : \mathbb{Z} \times \mathbb{T}^\ell \rightarrow GL(d, \mathbb{R})$  defined by

$$\mathcal{M}(n, \theta) = \begin{cases} M(\theta + (n-1)\omega) \cdots M(\theta) & n \geq 1 \\ \text{Id} & n = 0 \\ M^{-1}(\theta + (n+1)\omega) \cdots M^{-1}(\theta) & n \leq -1 \end{cases} \quad (3)$$

**Remark** In our case  $M(\theta) = DF(K(\theta))$

## Algorithm

Given  $M(\theta)$ , compute

$$\tilde{M}(\theta) = M(\theta + \omega)M(\theta). \quad (4)$$

Set  $\tilde{M} \rightarrow M$ ,  $2\omega \rightarrow \omega$  and iterate the procedure.

**Remark** We can compute  $\mathcal{M}(2^k, \theta)$  in  $k$  operations.

# The QR method

## Algorithm (Fast computation of cocycles with QR)

Given  $M(\theta)$  and a QR decomposition of  $M(\theta)$ ,  $M(\theta) = Q(\theta)R(\theta)$ , perform the following operations:

1. Compute  $S(\theta) = R(\theta + \omega)Q(\theta)$
2. Compute pointwise a QR decomposition of  $S$ ,  $S(\theta) = \bar{Q}(\theta)\bar{R}(\theta)$ .
3. Compute  $\tilde{Q}(\theta) = Q(\theta + \omega)\bar{Q}(\theta)$ 
  - 3.1  $\tilde{R}(\theta) = \bar{R}(\theta)R(\theta + \omega)$
  - 3.2  $\tilde{M}(\theta) = \tilde{Q}(\theta)\tilde{R}(\theta)$
4. Set  $M \leftarrow \tilde{M}$ 
  - 4.1  $R \leftarrow \tilde{R}$
  - 4.2  $Q \leftarrow \tilde{Q}$
  - 4.3  $2\omega \leftarrow \omega$

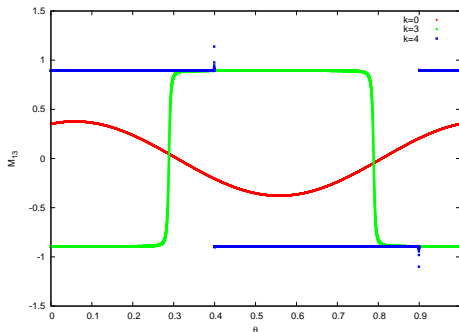
and iterate the procedure.

# Straddle the saddle

Consider a change of coordinates  $Q$  so that the cocycle

$$\tilde{M}(\theta) = Q(\theta + \omega)^{-1} M(\theta) Q(\theta)$$

is close to a constant.



**Figure:** The straddle the saddle phenomenon. We plot one of the components of the cocycle  $\mathcal{M}(2^k, \theta)$  for the values  $k = 0, 3, 4$ . The case  $k = 0$  was scaled by a factor 200.

- Consider we have computed  $\mathcal{M}(2^k, \theta)$ .
- Compute the QR decomposition of the cocycle  $\mathcal{M}(2^k, \theta)$ .
- Pick the column of Q ( $m(\theta)$ ) associated to the largest value in the diagonal of R.
- Shift it by an angle  $-2^k\omega$ .
- We have  $M(\theta)m(\theta) = \lambda_{\max}(\theta)m(\theta + \omega)$
- Therefore,  $\lambda_{\max}(\theta) = ([M(\theta)m(\theta)]^T [M(\theta)m(\theta)])^{1/2}$

# Primary KAM tori for the Froeschlé Map

4D exact symplectic map defined on the 2-cylinder  $\mathbb{R}^2 \times \mathbb{T}^2$ .

$$\bar{p}_1 = p_1 + \epsilon \left( \frac{\lambda_1}{2\pi} \sin(2\pi q_1) + \frac{\lambda_{12}}{2\pi} \sin(2\pi(q_1 + q_2)) \right)$$

$$\bar{p}_2 = p_2 + \epsilon \left( \frac{\lambda_2}{2\pi} \sin(2\pi q_2) + \frac{\lambda_{12}}{2\pi} \sin(2\pi(q_1 + q_2)) \right)$$

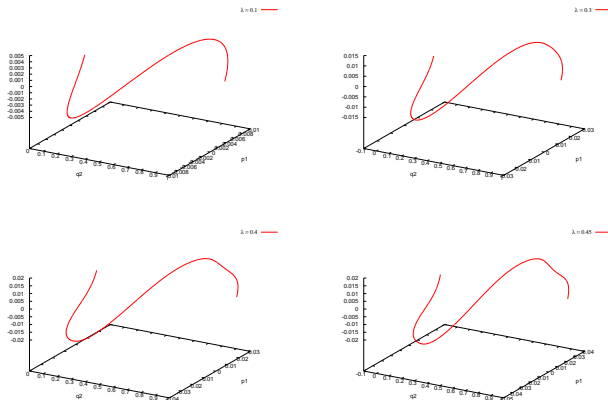
$$\bar{q}_1 = q_1 + \bar{p}_1 \quad (\text{mod } 1)$$

$$\bar{q}_2 = q_2 + \bar{p}_2 \quad (\text{mod } 1)$$

**Remark:** If  $\lambda_{12} = 0$ , the system consists of two uncoupled standard maps.



# Hyperbolic invariant tori of the Froeschlé Map



**Figure:** The  $q_1q_2p_1$  projection of the primary hyperbolic invariant tori of the Froeschlé map of frequency  $\omega_g$  for  $\lambda_{12} = 0.1, 0.3, 0.4, 0.45$  and  $\lambda_2 = 0.5, \lambda_1 = 0.01, \epsilon = 1$ . We have used  $N = 2^{11}$  Fourier modes.